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Entropy Extension*

A. E. Litvak, V. D. Milman, A. Pajor, and N. Tomczak-Jaegermann

Dedicated to the memory of B. Ya. Levin, an outstanding analyst. The second named author would like to note that he was Levin's student and Levin's scientific integrity and honesty have served him as an example for his entire life.

Abstract. We prove an "entropy extension-lifting theorem." It consists of two inequalities for the covering numbers of two symmetric convex bodies. The first inequality, which can be called an "entropy extension theorem," provides estimates in terms of entropy of sections and should be compared with the extension property of ℓ_∞. The second one, which can be called an "entropy lifting theorem," provides estimates in terms of entropies of projections.

Key words: metric entropy, entropy extension, entropy lifting, entropy decomposition, covering numbers

1. Introduction

One of important consequences of the Hahn–Banach theorem is the so-called "extension property of ℓ_∞." It states that, given a normed space X and a subspace $Y \subset X$, every linear operator $S\colon Y \to \ell_\infty$ can be extended to an operator $T\colon X \to \ell_\infty$ having the same norm as S. This theorem is used in the proofs of many results in Banach space theory and related fields. In particular, it was one of ingredients of the following result on covering numbers, obtained recently in [1]:

Let $0 < a < r < A$ and $1 \leqslant k < n$. Let $K, L \subset \mathbb{R}^n$ be symmetric convex bodies, and let $K \subset AL$. Let $E \subset \mathbb{R}^n$ be a k-codimensional subspace such that $K \cap E \subset aL$. Then

$$N(K, 2rL) \leqslant 2^k \left(\frac{A+r}{r-a}\right)^k.$$

Here, as usual, $N(K, L)$ denotes the covering number (see the definition below).

In a sense, the last result is a (weak) version of the extension theorem for entropy: *if we control the norm of the identity operator* (= *half diameter of the unit ball*) *in a subspace*, *then we control entropy in the entire space*. Note that if $K \cap E \subset aL$, then trivially $N(K \cap E, aL) \le 1$. However, why should the diameter play such a crucial role? Can one achieve a similar control of entropy in the entire space from the knowledge of entropy (rather than the diameter) in a subspace? Intuition does not support such a hope. However, quite surprisingly, this is possible. In the present paper, we prove a strong version of an extension theorem for entropy: *if we control entropy in a subspace*, *then we control entropy in the entire space*; see Theorem 3.1 below for the precise statement.

We also provide a version of the inverse statement in Theorem 4.1 below. In the last section, we discuss the nonsymmetric case.

2. Notation and Preliminaries

By a convex body we always mean a closed convex set with nonempty interior. By a symmetric convex body we mean a convex body centrally symmetric with respect to the origin.

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Let $K \subset \mathbb{R}^m$ be a convex body with origin in its interior. We denote the volume of K by $|K|$ and the polar of K by K^0; i.e.,

$$K^0 = \{x \mid \langle x, y \rangle \leqslant 1 \text{ for every } y \in K\}.$$

Let K and L be subsets of $\mathbb{R}^m$. We recall that the covering number $N(K, L)$ of K by L is defined as the minimal number N such that there exist vectors $x_1, \ldots, x_N$ in $\mathbb{R}^m$ satisfying

$$K \subset \bigcup_{i=1}^{N} (x_i + L).$$

We use the notation $N_A(K, L)$ if it is additionally required that $x_i \in A$, $1 \leqslant i \leqslant N$, where $A \subset \mathbb{R}^m$; and we let $\bar{N}(K, L) = N_K(K, L)$.

For a symmetric convex body $K \subset \mathbb{R}^m$ and $\varepsilon \in (0, 1)$, we need an upper bound for the covering number $N(K, \varepsilon K)$. The standard estimate is

$$N(K, \varepsilon K) \leqslant \bar{N}(K, \varepsilon K) \leqslant (1 + 2/\varepsilon)^m, \tag{2.1}$$

which follows by comparing volumes and which would be sufficient for our results. However, if the positions of centers are not important, we prefer to use a more sophisticated estimate that follows from a more general result by Rogers–Zong [5]; namely

$$N(K, \varepsilon K) \leqslant \theta_m (1 + 1/\varepsilon)^m, \tag{2.2}$$

where

$$\theta_m \leqslant \min\{2^m, m(\ln m + \ln(\ln m) + 5)\}.$$

In fact, the Rogers–Zong lemma implies that θ_m is bounded above by the so-called covering density of K (see [3], [4] for precise definitions and upper bounds), while the bound 2^m follows immediately from (2.1).

3. Entropy Extension-Lifting Theorem

The main result of this paper is the following "entropy extension-lifting theorem." It consists of two inequalities for entropies. The first inequality relates the entropy of K and L to the entropy of sections of small codimension and can be called an "entropy extension theorem," while the second inequality assumes information on entropies of projections of small corank and can be called an "entropy lifting theorem."

Theorem 3.1. *Let $0 < a < r < A$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$ such that $K \subset AL$. Let E be a subspace of $\mathbb{R}^n$, and let $P \colon \mathbb{R}^n \to \mathbb{R}^n$ be a projection with* $\ker P = E$.

(i) *If* $\operatorname{codim} E = k$, *then*

$$N(K, rL) \leqslant \theta_k \left(1 + \frac{A}{r - a}\right)^k N\left(K \cap E, \frac{a}{3}\, L\right).$$

(ii) *If* $\dim E = k$, *then*

$$N(K, rL) \leqslant \theta_k \left(\frac{2A + r}{r - a}\right)^k N\left(PK, \frac{a}{2}\, PL\right).$$

Note one special case of this theorem, namely, the case in which $N(K \cap E, (a/3)L \cap E) = 1$ (resp. $N(PK, (a/2)PL) = 1$). Taking $b = a/3$ and $R = r/3$ in the first part and $b = a/2$ and $R = r/2$ in the second part, we readily obtain the following consequence of Theorem 3.1 (the first part of which has been already mentioned in the Introduction).

Corollary 3.2. *Let $0 < b < R < A$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$ such that $K \subset AL$. Let E be a subspace of $\mathbb{R}^n$, and let $P \colon \mathbb{R}^n \to \mathbb{R}^n$ be a projection with* $\ker P = E$.

(i) *If* $\operatorname{codim} E = k$ *and* $K \cap E \subset bL \cap E$, *then*

$$N(K, 3RL) \leqslant \theta_k \left(1 + \frac{A}{3(R-b)}\right)^k.$$

(ii) *If* $\dim E = k$ *and* $PK \subset bPL$, *then*

$$N(K, 2RL) \leqslant \theta_k \left(\frac{A+R}{R-b}\right)^k.$$

This corollary was one of the main results on covering numbers in [1] (see Corollaries 1.6 and 1.7 there), which was essentially used in the proofs of other results in [1] and [2]. Actually, our present work has been inspired by this result.

Now we proceed to the proof of Theorem 3.1. First, we obtain a more general result estimating the entropy of sets in terms of the entropy of projections of these sets and the entropy of sections of related (but slightly more complicated) sets, in the spirit of the Rogers–Shephard lemma for volumes. We call it the "entropy decomposition lemma." It will imply Theorem 3.1.

Theorem 3.3. *Let* K, L_1, *and* L_2 *be subsets of* $\mathbb{R}^n$. *Let* E *be a subspace of* $\mathbb{R}^n$, *and let* $P\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a projection with* $\ker P = E$. *Then*

$$\begin{aligned} N(K, L_1 + L_2) &\leqslant \bar{N}(PK, PL_1) \max_{z \in K} N((K - L_1 - z) \cap E, L_2) \\ &\leqslant \bar{N}(PK, PL_1) N((K - K - L_1) \cap E, L_2) \end{aligned}$$

and

$$N(K, L_1 + L_2) \leqslant N(PK, PL_1) \max_{z \in \mathbb{R}^n} N((K - L_1 - z) \cap E, L_2).$$

Proof. We prove the first estimate; the proof of the second one goes along the same lines with obvious modifications.

Set

$$N_1 := \bar{N}(PK, PL_1).$$

Then, by definition, there exist $z_1, \ldots, z_{N_1} \in PK$ such that

$$PK \subset \bigcup_{i=1}^{N_1} (z_i + PL_1).$$

For every $x \in K$, take $i(x) \leqslant N_1$ and $y_x \in PL_1$ such that

$$Px = z_{i(x)} + y_x.$$

(If there exists more than one $i(x)$ (or y_x) with this property, take any of them and fix it in the subsequent argument.)

For $1 \leqslant i \leqslant N_1$, pick $\tilde{z}_i \in K$ such that $P\tilde{z}_i = z_i$, and for every $y \in PL_1$, pick $\tilde{y} \in L_1$ such that $P\tilde{y} = y$.

Now define

$$v(x) = \tilde{z}_{i(x)} + \tilde{y}_x \in \tilde{z}_{i(x)} + L_1$$

and

$$w(x) = x - v(x) = x - \tilde{z}_{i(x)} - \tilde{y}_x$$

for every $x \in K$. Denote

$$T_i := K - L_1 - \tilde{z}_i \quad \text{for } i \leqslant N_1.$$

Then $w(x) \in T_{i(x)}$ for every $x \in K$. Note also that $w(x) \in E$ for every $x \in K$, since

$$Pw(x) = Px - Pv(x) = Px - z_{i(x)} - y_x = 0.$$

Thus $w(x) \in T_{i(x)} \cap E$ and

$$x = w(x) + v(x) \in T_{i(x)} \cap E + \tilde{z}_{i(x)} + L_1$$

for every $x \in K$. It follows that

$$K \subset \bigcup_{i=1}^{N_1} (T_i \cap E + \tilde{z}_{i(x)} + L_1).$$

Since

$$N(T_i \cap E, L_2) \leqslant \max_{z \in K} N((K - L_1 - z) \cap E, L_2)$$

for every $i \leqslant N_1$, the result follows. □

Proof of Theorem 3.1. Let $\varepsilon := r - a$. To prove (i), first note that $N(K, rL) \leqslant N(K, (\varepsilon/A)K + aL)$, since $(1/A)K \subset L$. Thus, using Theorem 3.3 with $L_1 = (\varepsilon/A)K$ and $L_2 = aL$, we obtain

$$N(K, rL) \leqslant \bar{N}\left(PK, \frac{\varepsilon}{A}\, PK\right) N\left(\left(2 + \frac{\varepsilon}{A}\right) K \cap E, aL\right).$$

Now by the estimate (2.2) the first factor is bounded by $\theta_k(1 + A/\varepsilon)^k$, while the second factor is less than or equal to $N(3K \cap E, aL) = N(K \cap E, (a/3)L)$. This concludes the proof of (i).

To prove (ii), we use Theorem 3.3 with $L_1 = aL$ and $L_2 = \varepsilon L$ to obtain

$$N(K, rL) \leqslant \bar{N}(PK, aPL)\, N((2K + aL) \cap E, \varepsilon L).$$

To estimate the first factor, note the well-known general fact that $\bar{N}(K', L') \leqslant N(K', (1/2)L')$ for arbitrary sets K' and L', with L' symmetric. For the second factor, we use the estimate (2.2) to obtain

$$N((2K + aL) \cap E, \varepsilon L) \leqslant N((2A + a)L \cap E, \varepsilon L) \leqslant \theta_k \left(\frac{2A + r}{r - a}\right)^k.$$ □

4. Lower Bounds for Entropy

Here we prove a theorem which is, in a sense, the converse of Theorem 3.3.

Theorem 4.1. *Let $0 < t < 1$. Let K_1 and K_2 be subsets of $\mathbb{R}^n$, and let L_1 and L_2 be symmetric convex bodies in $\mathbb{R}^n$. Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a projection, and let $E = \ker P$. Then*

$$N(tK_1 + (1-t)K_2, (tL_1) \cap ((1-t)L_2)) \geqslant \bar{N}(PK_1, 2PL_1)\bar{N}(K_2 \cap E, 2L_2 \cap E).$$

Note that, taking $K_1 = K_2$ and, additionally, $L_1 = ((1-t)/t)L_2$, we have the following corollary.

Corollary 4.2. *Let $0 < t < 1$. Let K be a convex body in $\mathbb{R}^n$, and let L, L_1, and L_2 be symmetric convex bodies in $\mathbb{R}^n$. Let $P \colon \mathbb{R}^n \to \mathbb{R}^n$ be a projection, and let $E = \ker P$. Then*

$$N(K, (tL_1) \cap ((1-t)L_2)) \geqslant \bar{N}(PK, 2PL_1)\bar{N}(K \cap E, 2L_2 \cap E).$$

and

$$N(K, L) \geqslant \bar{N}(tPK, 2PL)\bar{N}((1-t)K \cap E, 2L \cap E).$$

In the proof, we use the notion of packing numbers. Recall that for K and L in $\mathbb{R}^n$ the packing number $P(K, L)$ of K by L is defined as the maximal number M such that there exist vectors x_1, …, $x_M \in K$ satisfying

$$(x_i + L) \cap (x_j + L) = \varnothing \quad \text{for any } i \neq j.$$

In other words, $x_i - x_j \notin L_0 := L - L$. Such a set of points is said to be L_0-separated. It is well known (and easy to check) that if L is a symmetric convex body (so that $L - L = 2L$), then

$$\bar{N}(K, 2L) \leqslant P(K, L) \leqslant N(K, L).$$

Proof of Theorem 4.1. Let $N_1 = P(PK_1, PL_1) \geqslant \bar{N}(PK_1, 2PL_1)$. Then there exist $z_1, \ldots, z_{N_1} \in PK_1$ such that $z_i - z_j \notin 2PL_1$ whenever $i \neq j$. For $1 \leqslant i \leqslant N_1$, pick $\tilde{z}_i \in K_1$ such that $P\tilde{z}_i = z_i$.

Let $N_2 = P(K_2 \cap E, L_2 \cap E) \geqslant \bar{N}(K_2 \cap E, 2L_2 \cap E)$. Then there exist $w_1, \ldots, w_{N_2} \in K_2 \cap E$ such that $w_k - w_\ell \notin 2L_2$ whenever $k \neq \ell$.

For any $i \leqslant N_1$ and $k \leqslant N_2$, let $x_{i,k} := t\tilde{z}_i + (1-t)w_k$ and consider the set

$$\mathscr{A} = \{x_{i,k}\}_{i \leqslant N_1, k \leqslant N_2} \subset tK_1 + (1-t)K_2.$$

We claim that $x_{i,k} - x_{j,\ell} \notin (2tL_1) \cap (2(1-t)L_2)$ if the pair (i,k) is different from (j,ℓ). Indeed, if $i \neq j$, then $P(x_{i,k} - x_{j,\ell}) = t(z_i - z_j) \notin 2tPL_1$, and hence $x_{i,k} - x_{j,\ell} \notin 2tL_1$. If $i = j$, then $k \neq \ell$ and $x_{i,k} - x_{j,\ell} = (1-t)(w_k - w_\ell) \notin 2(1-t)L_2$. Thus $\mathscr{A}$ is $(2tL_1) \cap (2(1-t)L_2)$-separated, which implies that

$$\begin{aligned} N(tK_1 + (1-t)K_2, (tL_1) \cap ((1-t)L_2)) &\geqslant P(tK_1 + (1-t)K_2, (tL_1) \cap ((1-t)L_2)) \\ &\geqslant N_1 N_2 \geqslant \bar{N}(PK_1, 2PL_1)\bar{N}(K_2 \cap E, 2L_2 \cap E). \end{aligned}$$

This concludes the proof. □

5. Additional Observations

In this section, we extend the theorem in [LPT], which was mentioned in the introduction and also as the first part of Corollary 3.2, to the case of nonsymmetric bodies. To keep the present paper self-contained, we use the statement of Corollary 3.2.

First, we extend it to the case in which K is not a symmetric body. We need the following simple lemma.

Lemma 5.1. *Let $a > 0$ and $1 \leqslant k \leqslant n$. Let K be a convex body in $\mathbb{R}^n$, and let L be a symmetric convex body in $\mathbb{R}^n$. Let E be a k-codimensional subspace of $\mathbb{R}^n$. Assume that $2a$ is the maximal diameter of $K \cap (E - z)$ over all choices of $z \in \mathbb{R}^n$; that is,*

$$\forall\, x \in \mathbb{R}^n \;\; \exists\, y \in E \quad \text{such that } (x + K) \cap E - y \subset aL.$$

Then

$$(K - K) \cap E \subset 2aL.$$

Proof. Let $z \in (K - K) \cap E$. Then $z = v - w$, where v, $w \in K$. Write $v = v_1 + v_2$ and $w = w_1 + w_2$, where v_1, $w_1 \in E^\perp$ and v_2, $w_2 \in E$. Since $z \in E$, we have $v_1 = w_1$.

By the assumptions of the lemma, there exists a $y \in E$ such that

$$(K - v_1) \cap E - y \subset aL.$$

Therefore, $v_2 - y \subset aL$ and $w_2 - y \subset aL$, which implies that

$$z = v - w = (v_2 - y) - (w_2 - y) \subset 2aL. \qquad \square$$

Combining Lemma 5.1 with Corollary 3.2 (applied to $K - K$), we readily obtain the following theorem.

Theorem 5.2. *Let $0 < a < A$ and $1 \leqslant k \leqslant n$. Let K be a convex body in $\mathbb{R}^n$, and let L be a symmetric convex body in $\mathbb{R}^n$ such that $K \subset AL$. Let E be a k-codimensional subspace of $\mathbb{R}^n$. Assume that $2a$ is the maximal diameter of $K \cap (E - x)$ over all choices of $x \in \mathbb{R}^n$. Then*

$$N(K - K, 3rL) \leqslant \theta_k \left(1 + \frac{2A}{3(r - 2a)}\right)^k$$

for every $r > 2a$.

Now we consider the case in which K is symmetric and L is not. First, note that the conclusion of Corollary 3.2 holds in this case with L replaced by $L \cap -L$. Indeed, if $K = -K$ is such that $K \subset RL$ and $K \cap E \subset aL \cap E$, then $-K \subset RL$ and $-K \cap E \subset aL \cap E$, which implies that $K \subset R(L \cap -L)$ and $K \cap E \subset a(L \cap -L) \cap E$. Therefore, optimizing over all shifts of L, i.e., over all choices of the center of L, we can extend Corollary 3.2 in the following way.

Theorem 5.3. *Let $0 < a < A$ and $1 \leqslant k \leqslant n$. Let K be a symmetric convex body in $\mathbb{R}^n$, and let L be a convex body in $\mathbb{R}^n$. Let E be a k-codimensional subspace of $\mathbb{R}^n$. Assume that there exists a $z \in \mathbb{R}^n$ satisfying*

$$K \subset A(L - z) \quad \text{and} \quad K \cap E \subset a(L - z).$$

Then

$$N(K, 3r\bar{L}) \leqslant \theta_k \left(1 + \frac{A}{3(r-a)}\right)^k$$

for every $r > a$, *where* $\bar{L} = (L - z) \cap (-L + z)$.

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Department of Mathematical and Statistical Sciences,
University of Alberta, Edmonton, Alberta, Canada
e-mail: alexandr@math.ualberta.ca

Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel
e-mail: milman@math.tau.ac.il

Équipe d'Analyse et Mathématiques Appliquées,
Université de Marne-la-Vallée, France
e-mail: Alain.Pajor@univ-mlv.fr

Department of Mathematical and Statistical Sciences,
University of Alberta, Edmonton, Alberta, Canada
e-mail: nicole.tomczak@ualberta.ca