Asymptotics of Orthogonal Polynomials Beyond the Scope of Szegő's Theorem*

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Received March 15, 2006

To the memory of Boris Yakovlevich Levin

ABSTRACT. First, we give a simple proof of a remarkable result due to Videnskii and Shirokov: let B be a Blaschke product with n zeros; then there exists an outer function ϕ , $\phi(0) = 1$, such that $\|(B\phi)'\| \leq Cn$, where C is an absolute constant. Then we apply this result to a certain problem of finding the asymptotics of orthogonal polynomials.

KEY WORDS: Orthogonal polynomial, extremal problem, Blaschke product, CMV matrix

1. Introduction and Main Results

Let us introduce the quantity investigated by Videnskii–Shirokov [6].

Let X be a Banach space of functions analytic in the open unit disk $\mathbb{D}$, let E be a finite collection of points in $\mathbb{D}$, and let B_E denote the Blaschke product vanishing precisely at the points of E . Characterization of zero sets of functions that belong to X is a difficult and important problem. It is closely related to the estimation of the quantity

$$\phi(E, X) := \inf\{\|f\| : f \in X, f|_E = 0, f(0) = B_E(0)\}.$$

The following sequence plays an important part in understanding the structure of zero sets of functions in X :

$$r_n(X) := \sup\{\phi(E, X) : \text{card } E = n\}.$$

There is very interesting research related to the asymptotics of $r_n(l_a^1)$, where l_a^1 stands for analytic functions with absolutely summable Taylor coefficients. Schäffer [5], Gluskin, Meyer, and Pajor [1], and Queffelec [4] proved that $r_n(l_a^1) \asymp \sqrt{n}$. Actually, Schäffer, motivated by Van der Waerden, studied the asymptotics of another quantity, which turns out to be equal to $r_n(l_a^1)$ (see [5], [1], and [2]). In [2], one can find an easy proof of the coincidence of $r_n(l_a^1)$ with this other quantity

$$k_n := \sup\{|\det T| \|T^{-1}\| : \|T\| \leq 1\},$$

where the supremum is taken over all invertible matrices of order n and over *all norms*! But if one fixes the norm of matrix T as acting in an n -dimensional Banach space, then one can consider an analog of k_n , or the following problem of finding the best estimate

$$\|T^{-1}\| \leq \Phi_n(\delta),$$

where we consider all invertible T in an n -dimensional Banach space A with $\|T\| \leq 1$ and spectral radius $r(T^{-1}) = \delta$. These estimates were considered in detail by Nikolski [2]. In [2], the following version of the problem is also considered: the condition $\|T\| \leq 1$ is replaced by

$$\|f(T)\| \leq C\|f\|$$

*The research of the first author was partially supported by the Austrian Science Fund FWF, project P16390-N04. The research of the second author was partially supported by NSF grant DMS-0200713. The research of the third author was partially supported by the Marie Curie Fund, contract MIF1-CT-2005-006966.

for every polynomial f . In other words, the operator T is assumed to satisfy functional calculus with respect to the function space (algebra) A . It is easy to see that for Banach algebras $A = X$ this brings us to estimates of yet another interesting quantity,

$$\text{cap}_X(E) := \inf\{\|f\|_X : f(0) = 1, f|_E = 0\}.$$

It is immediate that our quantities $\phi(E, X)$ and $\text{cap}_X(E)$ are essentially related to each other.

In [2], sharp estimates of $\text{cap}_{B_{p,q}^s}(E)$ are given. Here $B_{p,q}^s$ is the Besov class (see the definition below). Note that

$$B_{\infty,\infty}^s = \Lambda^s$$

for noninteger s , but only the strict inclusion

$$\Lambda^s \subset B_{\infty,\infty}^s \quad (1)$$

holds for integer s .

One result in [2], stated in a form convenient for us, is

$$r_n(B_{\infty,\infty}^s) \asymp n^s.$$

By our previous remark, this gives $r_n(\Lambda^s) \asymp n^s$ for noninteger s . The question for integer s seems to be subtler. In fact, the proof of this fact for integers given by Videnskii and Shirokov [6, Theorem 3.1] is quite involved.

We wish to give a very simple proof here. It follows the ideas of [2] and [6] but is rather short. We treat the case $s = 1$. Let $\mathbb{T} = \partial\mathbb{D}$.

Theorem 1.1. *Let B be a Blaschke product with n zeros in the disk $\mathbb{D}$. Then there exists an outer function ϕ satisfying the following properties:*

$$\|(B\phi)'\|_{L^\infty(\mathbb{T})} \leq Cn, \quad (2)$$

$$\phi(0) = 1, \quad (3)$$

$$\|\phi\|_{L^\infty(\mathbb{T})} \leq A. \quad (4)$$

Here C and A are absolute constants.

1.1. Classes $B_{p,q}^s$ and Λ^s . Let W_n be the Vallée-Poussin kernels defined by their Fourier coefficients as follows: $W_0(z) = 1 + z$, and if $n \geq 1$, then $\widehat{W_n}(2^n) = 1$, $\widehat{W_n}(k) = 0$ for $k \notin (2^{n-1}, 2^{n+1})$, and $\widehat{W_n}(k)$ is affine on $[2^{n-1}, 2^n]$ as well as on $[2^n, 2^{n+1}]$.

Now the class $B_{p,q}^s$ consists of analytic functions $f(z) = \sum \hat{f}(n)z^n$ in $\mathbb{D}$ such that the sequence $\{2^{ns}\|f \star W_n\|_{L^p(\mathbb{T})}\}$ belongs to l^q .

For integer $s = n$, the class Λ^s consists of all functions analytic in $\mathbb{D}$ for which the seminorm

$$\|f\|_{\Lambda^n} := \|f^{(n)}\|_{H^\infty}$$

is finite. If $s = n + \alpha$, $0 < \alpha < 1$, then $f \in \Lambda^s$ means that the previous seminorm is finite and, in addition, the seminorm

$$\|f^{(n)}\|_{\Lambda^\alpha} := \sup_{z, \zeta \in \mathbb{D}, z \neq \zeta} \frac{|f^{(n)}(z) - f^{(n)}(\zeta)|}{|z - \zeta|^\alpha}$$

is finite.

Acknowledgement. We are thankful to the referee for many remarks, which improved the exposition.

2. Proof of Theorem 1.1

Proof. Let $\tilde{B}$ be the Blaschke product with the same zeros $\{z_1, \dots, z_n\}$ as B but in $(1 + \frac{1}{n})\mathbb{D}$. If $R = 1 + \frac{1}{n}$, then

$$\tilde{B} = \prod_{k=1}^n \frac{\frac{z}{R} - \frac{z_k}{R}}{1 - \frac{1}{R^2} \bar{z}_k z}.$$

We can write

$$R^n \tilde{B} =: B\phi_0,$$

having this as a definition of ϕ_0 .

Thus

$$\phi_0(z) = \prod_{k=1}^n \frac{1 - \bar{z}_k z}{1 - \frac{1}{R^2} \bar{z}_k z}.$$

Obviously, if $z \in \mathbb{T}$, then $|\tilde{B}| < 1$ and hence

$$|\phi_0(z)| \leq \left(1 + \frac{1}{n}\right)^n \leq A.$$

So (4) is satisfied. Also $\phi_0(0) = 1$. To check (2), we write out the Cauchy formula for $\tilde{B}(\zeta_0)$ and $\zeta_0 \in \mathbb{T}$:

$$\tilde{B}(\zeta_0) = \frac{1}{2\pi i} \int_{(1+\frac{1}{n})\mathbb{T}} \frac{\tilde{B}(\zeta)}{\zeta - \zeta_0} d\zeta.$$

Hence

$$|\tilde{B}'(\zeta_0)| \leq \frac{1}{2\pi} \int_{(1+\frac{1}{n})\mathbb{T}} \frac{|\tilde{B}(\zeta)|}{|\zeta - \zeta_0|^2} dm(\zeta).$$

Therefore,

$$|\tilde{B}'(\zeta_0)| \leq C \int_{(1+\frac{1}{n})\mathbb{T}} \frac{1}{|\zeta - \zeta_0|^2} dm(\zeta) \leq Cn.$$

But then

$$|(B\phi_0)'(\zeta_0)| = \left(1 + \frac{1}{n}\right)^n |\tilde{B}'(\zeta_0)| \leq ACn. \quad \square$$

Remarks. 1. Replacing $R = 1 + \frac{1}{n}$ by $R = 1 + \frac{\varepsilon}{n}$, we can improve the estimate of ϕ_0 , $\|\phi_0\|_\infty \leq 1 + \varepsilon$, by paying the price in (2), where the constant now depends on ε ,

$$\|(B\phi)'\|_{L^\infty(\mathbb{T})} \leq C_\varepsilon n. \quad (5)$$

2. We have also proved that

$$\|B\phi_0\|_{\Lambda^s} \leq Cn^s.$$

For $s \in (0, 1)$, this readily follows from (2) and (4). For $s = 1$, we have just presented the proof. For $s > 1$, the proof goes exactly along the same lines, but we need to differentiate more times.

The second remark contains the claim of Theorem 3.1 in [6].

The first remark plays a crucial part in this paper, where we consider the asymptotics of polynomials orthogonal with respect to a measure μ of the following type:

$$d\mu := \frac{dm}{|\psi(e^{i\theta})|^2} + \sum_{k=1}^{\infty} \mu_k \delta_{z_k},$$

where the z_k form a Blaschke sequence.

In this situation, one should distinguish between Blaschke sequences inside $\mathbb{D}$ and outside $\mathbb{D}$. The second case is much more interesting, since the expected asymptotics of $L^2(\mu)$ -orthogonal polynomials is different from the classical asymptotics of Szegő, where all $\mu_k = 0$.

Orthogonalization with respect to the measures described above is an attempt to generalize the results due to Peherstorfer and Yuditskii [3], where the “point spectrum” was assumed to lie on the real line. This paper extends the strong asymptotic results for orthogonal polynomials that satisfy a Szegő condition on a real interval to the case in which the measure additionally has a denumerable set of mass points outside the interval. The authors of [3] assume that the mass points outside the interval accumulate only at the endpoints of the interval and that the Blaschke condition is satisfied. Under these conditions, strong asymptotic results are given for the orthonormal polynomials and

for their leading coefficients. The asymptotic behavior is stated in terms of the Szegő function for the absolutely continuous part and of a Blaschke product related to the extra mass points.

3. Orthogonal Polynomials and Smoothness of $B\phi$

Let μ be as above with

$$|z_k| > 1, \quad \sum_k (|z_k| - 1) < \infty,$$

and let $\{P_n\}$ denote the sequence of analytic polynomials orthonormal with respect to μ .

We write

$$P_n(z) = \tau_n z^n + \cdots + a_{0n},$$

and we wish to find the limit of τ_n . (We shall prove that it exists in many cases.)

We also consider rational functions of the form

$$R_{n,m}(z) = \eta_{n,m} z^n + \cdots + a_{0n} + \cdots + a_{m,n} z^m, \quad (6)$$

where $m = -n$ or $-(n-1)$. We wish them to be mutually orthogonal in $L^2(\mu)$ and be of norm 1 in this space. We intend to discuss the asymptotics of the “leading” coefficient $\eta_n := \eta_{n, -(n-1)}$.

Here are our two main results. We have the convention that τ_n and $\eta_n = \eta_{n, -(n-1)}$ are all positive and that $B(0)$ and $\psi(0)$ are also positive. The first theorem deals with the perturbation of so-called CMV matrices.

Theorem 3.1. *For any $\{z_k\}$ satisfying the Blaschke condition as above, for any summable nonnegative $\{\mu_k\}$, and for ψ bounded away from 0, one has*

$$\lim_{n \rightarrow \infty} \eta_n = B(0)\psi(0).$$

The asymptotics of τ_n seems to be subtler. For certain geometric configurations of $\{z_k\}$, no extra assumptions on $\{\mu_k\}$ is needed. For example, if $\{z_k\}$ converge in a Stolz star to only a closed set $E \subset \mathbb{T}$ satisfying $m(E) = 0$ and $\sum l_n \log \frac{1}{l_n} < \infty$ (here l_n stand for the lengths of complementary intervals of E ; such sets are called Carleson subsets of $\mathbb{T}$), then we do not need any extra assumptions on $\{\mu_k\}$. The case of real $\{z_k\}$ considered in [3] is a particular case of such a geometry. But if $\{z_k\}$ is an arbitrary Blaschke sequence outside $\mathbb{D}$, we still need one assumption on $\{\mu_k\}$ saying that the series $\sum_k \mu_k$ converges at a certain rate. Here is this assumption:

$$\sum_{1 < |z_k| < 1 + \frac{1}{n}} \mu_k \leq C_A \frac{1}{(\log n)^A}, \quad \forall A. \quad (7)$$

Theorem 3.2. *For any $\{z_k\}$ satisfying the Blaschke condition as above, for any summable nonnegative $\{\mu_k\}$ satisfying the extra condition (7), and for ψ bounded away from 0 and from ∞ , one has*

$$\lim_{n \rightarrow \infty} \tau_n = B(0)\psi(0).$$

4. Proof of Theorem 3.1

Lemma 4.1. *The solution of the extremal problem*

$$\text{find } \eta_n = \sup\{|\eta| : R(z) = \eta z^n + \cdots + a z^{-(n-1)}, \|R\|_{L^2(\mu)} \leq 1\}$$

is attained at the function $R = R_n := R_{n, -(n-1)}$ in (6).

Proof. Consider another extremal problem: $\inf\{\|Q\|_{L^2(\mu)} : Q(z) = z^n + \cdots + a z^{-(n-1)}\}$. Obviously, these are $L^2(\mu)$ -orthogonal functions. Our extremal functions R_n are just $Q/\|Q\|_{L^2(\mu)}$, and so $\inf\|Q\|_{L^2(\mu)}$ means $\sup|\eta|$. $\square$

4.1. Lower Bound for η_n . Let $\zeta_k = 1/\bar{z}_k$, let B be the Blaschke product with zeros $\{\zeta_k\}_{k=1}^\infty$, and let B_n be the partial Blaschke product constructed for

$$\zeta_k : |\zeta_k| < 1 - \frac{1}{k_n},$$

where $k_n := [\varepsilon_n n]$, $\varepsilon_n \rightarrow 0$.

We choose ε_n later.

Without loss of generality, we assume that

$$\sum_k (1 - |\zeta_k|) < 1.$$

The number of zeros of B_n is at most k_n .

We take $l_n := [A_n \varepsilon_n n]$, where A_n grows to infinity very slowly; namely, $A_n \varepsilon_n \rightarrow 0$. Put

$$R := 1 + \frac{1}{l_n}.$$

Let $\tilde{B}$ be the Blaschke product with the same zeros as B_n but in $R\mathbb{D}$. Then we introduce ϕ by the formula (as in Theorem 1.1)

$$R^{k_n} \tilde{B} =: B\phi, \quad (8)$$

or

$$\phi := \prod_{k=1}^{k_n} \frac{1 - \bar{\zeta}_k z}{1 - \frac{1}{R^2} \bar{\zeta}_k z}.$$

In Theorem 1.1, we proved that

$$\|B\phi\|_{\Lambda^1} \leq Cl_n.$$

It follows from (1) that

$$\|B\phi\|_{B_{\infty,\infty}^1} \leq Cl_n.$$

Let 2^k be the largest number of this form smaller than n . Consider $W_k, W_{k+1}, \dots$ (We have introduced Vallée-Poussin kernels in Sec. 1.) We have just noted that

$$\|W_j \star B\phi\|_\infty \leq Cl_n 2^{-j}, \quad j \geq k. \quad (9)$$

So (9) gives

$$\left\| \sum_{j \geq k} W_j \star B\phi \right\|_\infty \leq C \frac{l_n}{n}. \quad (10)$$

Therefore, if V_k is the modified Vallée-Poussin kernel,

$$\widehat{V}_k(j) = 1, \quad j \in [-2^{k-1}, 2^{k-1}], \quad \widehat{V}_k(j) = 1, \quad |j| \geq 2^k,$$

and $\widehat{V}_k(j)$ is affine otherwise, then from (10) we conclude that

$$\|V_k \star (B\phi) - B\phi\|_\infty \leq Cl_n/n. \quad (11)$$

Consider the following modified Vallée-Poussin kernel VP_n :

$$\widehat{VP}_n(j) = 1, \quad j \in [-n, n], \quad \widehat{VP}_n(j) = 1, \quad |j| \geq n,$$

and $\widehat{VP}_n(j)$ is affine otherwise. It is easy to see that we have proved (using $V_k \star VP_n = V_k$) that

$$\|VP_n \star (B\phi) - B\phi\|_\infty \leq 4Cl_n/n \leq C'\delta_n, \quad (12)$$

where $\delta_n \rightarrow 0$.

Note the following upper bound for ϕ (keeping in mind that $l_n \approx A_n k_n$):

$$|(B_n \phi)(z)| = (1 + 1/l_n)^{k_n} |\tilde{B}(\zeta)| \leq e^{C/A_n} =: 1 + \delta'_n, \quad \delta'_n \rightarrow 0. \quad (13)$$

Consider $G_n = VP_n \star (B\phi)$ and $r_n = G_n/z^n$. Clearly (since $\phi(0) = 1$)

$$|G_n(0) - B(0)\phi(0)| = |G_n(0) - B(0)| \leq \delta_n. \quad (14)$$

The rational function r_n is of the type we wish but only after an application of the symmetry $n \rightarrow -n$ on $\mathbb{Z}$.

Let us estimate the norm

$$\|r_n\|^2 := \int_{\mathbb{T}} |r_n|^2 dm + \sum_{k=1}^{\infty} \mu_k |r_n(\zeta_k)|^2.$$

From (12) and (13), we have $|r_n| \leq 1 + C(\delta_n + \delta'_n)$ on $\mathbb{T}$. So $\int_{\mathbb{T}} |r_n|^2 dm \leq 1 + C\delta''_n$, $\delta''_n \rightarrow 0$. Now we split the sum into two, $\alpha_1 := \sum_{k=1}^{k_n} \mu_k |r_n(\zeta_k)|^2$ and $\alpha_2 := \sum_{k>k_n} \mu_k |r_n(\zeta_k)|^2$.

To estimate α_1 , note that (12) says that $G_n - B\phi$ is small on $\mathbb{T}$, and the construction of G_n says that $(G_n - B\phi)(z)$ has zero of multiplicity n at the origin. This is an analytic function in $\mathbb{D}$, and so

$$|G_n(z) - B\phi(z)| \leq C\delta_n |z|^n, \quad (15)$$

where $\delta_n \rightarrow 0$, by the classical Schwarz lemma.

In particular,

$$|(VP_n \star B\phi)(\zeta_k) - B\phi(\zeta_k)| \leq C\delta_n |\zeta_k|^n. \quad (16)$$

But $B(\zeta_k) = 0$, $k = 1, \dots, k_n$, and so we rewrite

$$|G_n(\zeta_k)| = |(VP_n \star B\phi)(\zeta_k)| \leq C\delta_n |\zeta_k|^n, \quad k = 1, \dots, k_n. \quad (17)$$

Finally, using the relation $r_n(z) = G_n(z)/z^n$, we obtain

$$|r_n(\zeta_k)| \leq C\delta_n, \quad k = 1, \dots, k_n. \quad (18)$$

Therefore,

$$|\alpha_1| \leq C\delta_n \sum_k \mu_k \leq C\delta_n. \quad (19)$$

To estimate α_2 , we start from (16) to write

$$|r_n(\zeta_k)| \leq C\delta_n + |(B\phi)(\zeta_k)|/|\zeta_k|^n, \quad k > k_n. \quad (20)$$

But if $k > k_n$, then $1 - |\zeta_k| < C/(\varepsilon_n n)$, and so

$$\frac{1}{|\zeta_k|^n} \leq e^{C/\varepsilon_n}.$$

In particular,

$$|\alpha_2| \leq \sum_{k>k_n} \mu_k |r_n(\zeta_k)|^2 \leq C\delta_n + e^{C/\varepsilon_n} \sum_{k>k_n} \mu_k.$$

We can always choose ε_n to decrease so slowly that $\sum_{k>k_n} \mu_k$ kills the growth of e^{C/ε_n} and the last expression

$$\gamma_n := e^{C/\varepsilon_n} \sum_{k>k_n} \mu_k$$

tends to zero.

Summarizing, we see that

$$\|r_n\| \leq 1 + \text{small quantity} \quad (21)$$

and

$$r_n(z) = \rho z^{-n} + \dots + az^{n-1}, \quad |\rho| \geq B(0) - \text{small quantity}. \quad (22)$$

Let us proceed to the reciprocal variable. Consider $R(z) := r_n^*(z) := \overline{r(1/\bar{z})} = \bar{\rho}z^n + \dots + \bar{a}z^{-(n-1)}$. Then we have just proved that $|\rho| \geq B(0)\phi(0) - \text{small quantity}$ and

$$\|R\|_{L^2(\mu)} \leq 1 + \text{small quantity} \quad (23)$$

This completes the proof of the estimate

$$\liminf_{n \rightarrow \infty} |\eta_n| \geq B(0). \quad (24)$$

This proves the lower bound for η_n for the case $\psi = 1$. For nontrivial ψ bounded away from zero, we approximate it from below by smooth ψ' and reproduce the above argument for $B\phi/\psi$ instead of $B\phi$.

5. Lower Bound for τ_n

Let ζ_n , B , B_n , and k_n be the same as at the beginning of Section 4.1.

We choose ε_n later.

Without the loss of generality, we assume that

$$\sum_k (1 - |\zeta_k|) < 1.$$

The number of zeros of B_n is at most k_n .

Choose $l_n := [A_n \varepsilon_n n]$, where A_n grows to infinity very slowly; namely, $A_n \varepsilon_n \rightarrow 0$. Put

$$R := 1 + \frac{1}{l_n}.$$

Let $\tilde{B}$ be the Blaschke product with the same zeros as B_n but in $R\mathbb{D}$. Then we introduce ϕ by formula (8), as in Sec. 4.1.

We retain all notation in the previous section. Let T_n denote the Taylor polynomial of degree n of $B_n\phi$. Our goal is very simple; we wish to repeat all the reasoning of the previous section, but instead of estimating $|G_n(z) - B\phi(z)| = |(VP_n \star B\phi)(z) - B\phi(z)|$ we wish to estimate $|T_n(z) - B\phi(z)| = |(D_n \star B\phi)(z) - B\phi(z)|$. Here D_n stands for the Dirichlet kernel. In other words, we just wish to replace the kernel VP_n by D_n .

Let $\zeta_0 \in \mathbb{T}$. One can write out the Cauchy formula for $B\phi(\zeta_0)$ over $R\mathbb{T}$, $R = 1 + 1/l_n$:

$$(B_n\phi - T_n)(\zeta_0) = \frac{1}{2\pi i} \int_{(1+1/l_n)\mathbb{T}} \frac{\zeta_0^n (B_n\phi)(\zeta)}{(\zeta - \zeta_0)\zeta^n} d\zeta.$$

We have seen in (13) that

$$|B_n\phi(\zeta)| = \left(1 + \frac{1}{l_n}\right)^{k_n} |\tilde{B}(z)| \leq e^{C/A_n} =: 1 + \kappa_n, \quad \kappa_n \rightarrow 0. \quad (25)$$

In particular, substituting this into our Cauchy formula, we obtain

$$\|B_n\phi - T_n\|_\infty \leq \frac{C}{(1 + \frac{1}{l_n})^n} \int_{(1+1/l_n)\mathbb{T}} \frac{|d\zeta|}{|\zeta - \zeta_0|},$$

or

$$\|B_n\phi - T_n\|_\infty \leq C \log l_n e^{-1/(A_n \varepsilon_n)} \leq C' \log n e^{-1/(A_n \varepsilon_n)}.$$

We denote $\delta_n := \log n e^{-1/(A_n \varepsilon_n)}$. Unlike δ_n in the previous section, this one does not necessarily go to zero. But we require that it does go to zero, thus putting some restriction on ε_n and A_n . So from now on we *assume* that

$$\delta_n := \log n e^{-1/(A_n \varepsilon_n)} \rightarrow 0 \quad (26)$$

and

$$\|B_n\phi - T_n\|_\infty \leq C\delta_n \rightarrow 0. \quad (27)$$

We also introduce

$$p_n(z) := T_n(z)/z^n, \quad P_n(z) := \overline{p_n(1/\bar{z})} = z^n \overline{T(1/\bar{z})}.$$

Then consider the norm

$$\|p_n\| := \int_{\mathbb{T}} |p_n|^2 dm + \sum_k \mu_k |p_n(\zeta_k)|^2.$$

Of course, we wish to prove that

$$\|p_n\| \leq 1 + \text{small quantity} \quad (28)$$

and

$$|T_n(0)| \geq B(0) - \text{small quantity.} \quad (29)$$

Relation (29) follows from (27) and $\phi(0) = 1$.

To prove (28), we note that (25) implies

$$\int_{\mathbb{T}} |p_n|^2 dm \leq 1 + C(\kappa_n + \delta_n) \leq 1 + \text{small quantity.}$$

We split the sum $\sum_k \mu_k |p_n(\zeta_k)|^2$ into two, $\beta_1 := \sum_{k=1}^{k_n} \mu_k |p_n(\zeta_k)|^2$ and $\beta_2 := \sum_{k>k_n} \mu_k |p_n(\zeta_k)|^2$.

To estimate β_1 , we note (27) and the fact that $B\phi - T_n$ is an analytic function in $\mathbb{D}$ with zero of multiplicity n at the origin. So, by the Schwarz lemma,

$$|T_n(z) - B\phi(z)| \leq C\delta_n |z|^n, \quad (30)$$

where $\delta_n \rightarrow 0$.

In particular,

$$|T_n(\zeta_k) - B\phi(\zeta_k)| \leq C\delta_n |\zeta_k|^n. \quad (31)$$

But $B(\zeta_k) = 0$, $k = 1, \dots, k_n$, and so we rewrite

$$|T_n(\zeta_k)| \leq C\delta_n |\zeta_k|^n, \quad k = 1, \dots, k_n. \quad (32)$$

Finally, using the relation $p_n(z) = T_n(z)/z^n$, we obtain

$$|p_n(\zeta_k)| \leq C\delta_n, \quad k = 1, \dots, k_n. \quad (33)$$

Therefore,

$$|\alpha_1| \leq C\delta_n \sum_k \mu_k \leq C\delta_n. \quad (34)$$

To estimate β_2 , we start from (31) to write

$$|p_n(\zeta_k)| \leq C\delta_n + |(B\phi)(\zeta_k)|/|\zeta_k|^n, \quad k > k_n. \quad (35)$$

But if $k > k_n$, then $1 - |\zeta_k| < \frac{C}{\varepsilon_n n}$, and so

$$\frac{1}{|\zeta_k|^n} \leq e^{C/\varepsilon_n}.$$

In particular,

$$|\beta_2| \leq \sum_{k>k_n} \mu_k |p_n(\zeta_k)|^2 \leq C\delta_n + e^{C/\varepsilon_n} \sum_{k>k_n} \mu_k.$$

We cannot always choose ε_n to decrease so slowly that $\sum_{k>k_n} \mu_k$ kills the growth of e^{C/ε_n} and the last expression

$$\gamma_n := e^{C/\varepsilon_n} \sum_{k>k_n} \mu_k \quad (36)$$

tends to zero. The reason is in the fact that we have condition (26) that makes ε_n tend to zero at a *certain* rate. But combining (26) with (36), we see that they are reconcilable if μ_k tend to zero sufficiently rapidly.

It is easy to conclude that (7) allows us to have both assumptions (26) and (36). Therefore, we obtain (28) and (29). Passing to the reciprocal variable, that is, from $p_n(z)$ to $P_n(z)$, proves that

$$\liminf_{n \rightarrow \infty} |\tau_n| \geq B(0).$$

If ψ is bounded away from zero, we approximate it from below by a smooth function, and assuming now that $1/\psi$ is smooth, we can reproduce our approximation arguments with $B\phi/\psi$ instead of $B\phi$.

6. Upper Bounds for τ_n and η_n

This is a simple estimate. For η_n , it is still slightly more difficult, so we show it in this case. We wish to prove that

$$\limsup_{n \rightarrow \infty} |\eta_n| \leq B(0)\psi(0). \quad (37)$$

Let $\psi_*(z) := \overline{\psi(1/\bar{z})}$ and $B_*(z) := \overline{B(1/\bar{z})}$. Let B_*^k be a finite Blaschke product that is a subproduct of B_* .

Consider the integral

$$\int_{\mathbb{T}} \frac{R_n(z)}{\psi_*(z)} \overline{z^n B_*^k(z)} dm(z),$$

whose absolute value evidently does not exceed one. Let us calculate the integral according to Cauchy theorem in the exterior of $\mathbb{D}$. We can do that, since R_n/z^{n+1} is analytic in the exterior of the unit disk and zero at infinity. We have

$$\int_{\mathbb{T}} \frac{R_n(z)}{\psi_*(z) z^{n+1} B_*^k(z)} dz = \frac{\eta_n}{B(0)\psi(0)} - \sum \frac{R_n(z_i)}{(B_*^k)'(z_i) \psi_*(z_i) z_i^{n+1}}.$$

For the last sum, we have the estimate

$$\left| \sum \frac{R_n(z_i)}{(B_*^k)'(z_i) \psi_*(z_i) z_i^{n+1}} \right|^2 \leq \sum \frac{1}{|(B_*^k)'(z_i) \psi_*(z_i) z_i^{n+1}|^2 \mu_i}$$

(since R_n is normalized in $L^2(\mu)$). The number of terms in the sum is finite, and so the sum goes to zero as n tends to infinity.

Thus

$$\limsup_{n \rightarrow \infty} |\eta_n| \leq B_*^k(0)\psi(0).$$

Since B_*^k is arbitrary, we have proved (37).

Of course, the reader can see that the proof for τ_n is exactly the same.

The proof of both theorems is complete.

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