An Erdős–Révész type law of the iterated logarithm for reflected fractional Brownian motion

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Abstract Let $B_H = \{B_H(t) : t \in \mathbb{R}\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. For the stationary storage process $Q_{B_H}(t) = \sup_{-\infty < s \leq t} (B_H(t) - B_H(s) - (t - s))$, $t \geq 0$, we provide a tractable criterion for assessing whether, for any positive, non-decreasing function f , $\mathbb{P}(Q_{B_H}(t) > f(t) \text{ i.o.})$ equals 0 or 1. Using this criterion we find that, for a family of functions $f_p(t)$, such that $z_p(t) = \mathbb{P}(\sup_{s \in [0, f_p(t)]} Q_{B_H}(s) > f_p(t))/f_p(t) = \mathcal{C}(t \log^{1-p} t)^{-1}$, for some $\mathcal{C} > 0$, $\mathbb{P}(Q_{B_H}(t) > f_p(t) \text{ i.o.}) = 1_{\{p \geq 0\}}$. Consequently, with $\xi_p(t) = \sup\{s : 0 \leq s \leq t, Q_{B_H}(s) \geq f_p(s)\}$, for $p \geq 0$, $\lim_{t \rightarrow \infty} \xi_p(t) = \infty$ and $\limsup_{t \rightarrow \infty} (\xi_p(t) - t) = 0$ a.s. Complementary, we prove an Erdős–Révész type law of the iterated logarithm lower bound on $\xi_p(t)$, i.e., $\liminf_{t \rightarrow \infty} (\xi_p(t) - t)/h_p(t) = -1$ a.s., $p > 1$; $\liminf_{t \rightarrow \infty} \log(\xi_p(t)/t)/(h_p(t)/t) = -1$ a.s., $p \in (0, 1]$, where $h_p(t) = (1/z_p(t))p \log \log t$.

Keywords Extremes of Gaussian fields · Storage processes · Fractional Brownian motion · Law of the iterated logarithm

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1 Introduction and main results

The analysis of properties of *reflected* stochastic processes, being developed in the context of classical *Skorokhod problems* and their applications to queueing theory, risk theory and financial mathematics, is an actively investigated field of applied probability. In this paper we analyze 0-1 properties of a class of such processes, that due to its importance in queueing theory (and dual risk theory) gained substantial interest; see, e.g., Norros (2004), Piterbarg (2001), Asmussen (2003), and Asmussen and Albrecher (2010) or novel works on γ -reflected Gaussian processes (Hashorva et al. 2013; Liu et al. 2015).

Consider a reflected (at 0) fractional Brownian motion with drift $Q_{B_H} = \{Q_{B_H}(t) : t \geq 0\}$, given by the following formula

$$Q_{B_H}(t) = B_H(t) - ct + \max \left(Q_{B_H}(0), - \inf_{s \in [0, t]} (B_H(s) - cs) \right), \quad (1)$$

where $c > 0$ and $B_H = \{B_H(t) : t \in \mathbb{R}\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, i.e., a centered Gaussian process with covariance function $\text{Cov}(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. We focus on the investigation of the long-time behavior of the unique stationary solution of (1), which has the following representation

$$Q_{B_H}(t) = \sup_{-\infty < s \leq t} (B_H(t) - B_H(s) - c(t-s)). \quad (2)$$

With no loss of generality in the reminder of this paper we assume that the drift parameter $c \equiv 1$. An important stimulus to analyze the distributional properties of Q_{B_H} and its functionals stems from the Gaussian fluid queueing theory, where the stationary buffer content process in a queue which is fed by B_H and emptied with constant rate $c = 1$ is described by (2); see e.g. Norros (2004). In particular, in the seminal paper by Hüsler and Piterbarg (1999) the exact asymptotics of one dimensional marginal distributions of Q_{B_H} was derived; see also Dieker (2005), Dębicki (2002), and Dębicki and Liu (2016) for results on more general Gaussian input processes.

The purpose of this paper is to investigate the asymptotic 0-1 behavior of the processes Q_{B_H} . Our first contribution is an analog of the classical finding of Watanabe (1970), where an asymptotic 0-1 type of behavior for centered stationary Gaussian processes was analyzed.

Theorem 1. *For all functions $f(t)$ that are positive and nondecreasing on some interval $[T, \infty)$, it follows that*

$$\mathbb{P}(Q_{B_H}(t) > f(t) \text{ i.o.}) = 0 \quad \text{or} \quad 1,$$

according as the integral

$$\mathcal{J}_f := \int_T^\infty \frac{1}{f(u)} \mathbb{P} \left(\sup_{t \in [0, f(u)]} Q_{B_H}(t) > f(u) \right) du$$

is finite or infinite.

The exact asymptotics, as u grows large, of the probability in $\mathcal{J}_f$ was found by Piterbarg (2001, Theorem 7). Namely, for any $T > 0$,

$$\mathbb{P} \left(\sup_{t \in [0, Tf(u)]} Q_{B_H}(t) > f(u) \right) = \sqrt{\pi} a^{\frac{2}{H}} b^{-\frac{1}{2}} \mathcal{H}_{B_H}^2 T(v_f(u))^{\frac{2}{H}-1} \Psi(v_f(u))(1+o(1)), \text{ as } u \rightarrow \infty, \quad (3)$$

where $v_f(u) = Af^{1-H}(u)$, $\Psi(u) = 1 - \Phi(u)$, Φ is the distribution function of the unit normal law and the constants $a, b, A, \mathcal{H}_{B_H}$ are given explicitly in Section 2. Since relation (3) also holds when $T = T(u) \rightarrow 0$, provided that $T(u)(f(u))^{(1-H)/H} \rightarrow \infty$, we have that for $H \in (0, \frac{1}{2})$, as $u \rightarrow \infty$,

$$\frac{1}{f(u)} \mathbb{P} \left(\sup_{t \in [0, f(u)]} Q_{B_H}(t) > f(u) \right) \sim \mathbb{P} \left(\sup_{t \in [0, 1]} Q_{B_H}(t) > f(u) \right).$$

Theorem 1 provides a tractable criterion for settling the dichotomy of $\mathbb{P}(Q_{B_H}(t) > f(t) \text{ i.o.})$. For instance, let $C_H = (2(1-H)^2 - H)/(2H(1-H))$ and

$$f_p(s) = \left(\frac{2}{A^2} (\log s + (1 + C_H - p) \log_2 s) \right)^{\frac{1}{2(1-H)}}, \quad p \in \mathbb{R}, H \in (0, 1). \quad (4)$$

One can check that, as $u \rightarrow \infty$,

$$\frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) = \frac{a^{\frac{2}{H}} b^{-\frac{1}{2}}}{\sqrt{2}} \mathcal{H}_{B_H}^2 A^{\frac{1}{1-H}} 2^{C_H} (u \log^{1-p} u)^{-1} (1 + o(1)). \quad (5)$$

Hence, for any $p \in \mathbb{R}$,

$$\mathbb{P}(Q_{B_H}(t) > f_p(t) \text{ i.o.}) = \begin{cases} 1 & \text{if } p \geq 0, \\ 0 & \text{if } p < 0. \end{cases}$$

Corollary 1. For any $H \in (0, 1)$,

$$\limsup_{t \rightarrow \infty} \frac{Q_{B_H}(t)}{(\log t)^{\frac{1}{2(1-H)}}} = \left(\frac{2}{A^2} \right)^{\frac{1}{2(1-H)}} \text{ a.s.}$$

This result extends findings of Zeevi and Glynn (2000, Theorem 1), where it was proven that the above convergence holds weakly as well as in L_p for all $p \in [1, \infty)$.

Now consider the process $\xi_p = \{\xi_p(t) : t \geq 0\}$ defined as

$$\xi_p(t) = \sup\{s : 0 \leq s \leq t, Q_{B_H}(s) \geq f_p(s)\}.$$

Since $\mathcal{J}_{f_p} = \infty$ for $p \geq 0$, from Theorem 1 it follows that

$$\lim_{t \rightarrow \infty} \xi_p(t) = \infty \text{ a.s. and } \limsup_{t \rightarrow \infty} (\xi_p(t) - t) = 0 \text{ a.s.}$$

Let, cf. (5),

$$h_p(t) = p \left(\frac{1}{f_p(t)} \mathbb{P} \left(\sup_{s \in [0, f_p(t)]} Q_{B_H}(s) > f_p(t) \right) \right)^{-1} \log_2 t.$$

The second contribution of this paper is an Erdős–Révész type of law of the iterated logarithm for the process ξ_p . We refer to Shao (1992) for more background and references on Erdős–Révész type law of the iterated logarithm and a related result for centered stationary Gaussian processes; see also Dębicki and Kosiński (2016) for extensions to order statistics.

Theorem 2. *If $p > 1$, then*

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} = -1 \text{ a.s.}$$

If $p \in (0, 1]$, then

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} = -1 \text{ a.s.}$$

Now, let us complementary put $\eta_p = \{\eta_p(t) : t \geq 0\}$, where

$$\eta_p(t) = \inf\{s \geq t : Q_{B_H}(s) \geq f_p(s)\}.$$

Since

$$\mathbb{P}(\xi_p(t) - t \leq -x) = \mathbb{P} \left(\sup_{s \in [t-x, t]} \frac{Q_{B_H}(s)}{f_p(s)} < 1 \right)$$

and

$$\mathbb{P}(z - \eta_p(z) \leq -x) = \mathbb{P} \left(\sup_{s \in [z, z+x]} \frac{Q_{B_H}(s)}{f_p(s)} < 1 \right),$$

then it follows that

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} = \liminf_{z \rightarrow \infty} \frac{z - \eta_p(z)}{h_p(z)}. \quad (6)$$

Theorem 2 shows that for t big enough, there exists an s in $[t - h_p(t), t]$ (as well as in $[t, t + h_p(t)]$ by (6)) such that $Q_{B_H}(s) \geq f_p(s)$ and that the length $h_p(t)$ of the interval is the smallest possible. This shines new light on results, which are intrinsically connected with Gumbel limit theorems; see, e.g., Leadbetter et al. (1983), where the function $h_p(t)$ plays crucial role. We shall pursue this elsewhere.

The paper is organized as follows. In Section 2 we introduce some useful properties of storage processes fed by fractional Brownian motion. In Section 3 we provide a collection of basic results on how to interpret extremes of the storage process Q_{B_H} as extremes of a Gaussian field related to the fractional Brownian motion B_H . Furthermore, in Section 4 we prove lemmas, which constitute building blocks of the proofs of the main results.

2 Properties of the storage process

In this section we introduce some notation and state some properties of the supremum of the process Q_{B_H} as derived in Piterbarg (2001) and Hüsler and Piterbarg (2004b). We begin with the relation

$$\mathbb{P}\left(\sup_{t \in [0, T]} Q_{B_H}(t) > u\right) = \mathbb{P}\left(\sup_{\substack{s \in [0, T/u] \\ \tau \geq 0}} Z_u(s, \tau) > u^{1-H}\right), \quad \text{for any } T > 0, \quad (7)$$

where, with $v(\tau) = \tau^{-H} + \tau^{1-H}$,

$$Z_u(s, \tau) := \frac{B_H(u(\tau + s)) - B_H(su)}{\tau^H u^H v(\tau)}$$

is a Gaussian field. Note that the self-similarity property of B_H implies that the field Z_u has the same distribution for any u . Thus, we do not use u as an additional parameter in the following notation whenever it is not needed; let $Z(s, \tau) := Z_1(s, \tau)$. Furthermore, the field $Z(s, \tau)$ is stationary in s , but not in τ . The variance $\sigma_Z^2(\tau)$ of the field $Z(s, \tau)$ equals $v^{-2}(\tau)$ and $\sigma_Z(\tau)$ has a single maximum point at

$$\tau_0 = \frac{H}{1-H}.$$

Taylor expansion leads to

$$\sigma_Z(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau - \tau_0)^2 + O((\tau - \tau_0)^3),$$

as $\tau \rightarrow \tau_0$, where

$$A = \frac{1}{1-H} \left(\frac{H}{1-H}\right)^{-H} = v(\tau_0),$$

$$B = H \left(\frac{H}{1-H}\right)^{-H-2} = v''(\tau_0).$$

Let us define the correlation function of the process Z_u as follows

$$\begin{aligned} r_{u,u'}(s, \tau, s', \tau') &:= \mathbb{E}Z_u(s, \tau)Z_u(s', \tau')v(\tau)v(\tau') \\ &= \frac{|us - u's'|^{2H}}{2(u\tau u'\tau')^H} \left(\left|1 + \frac{u\tau}{(us - u's')}\right|^{2H} - \left|1 + \frac{(u\tau - u'\tau')}{(us - u's')}\right|^{2H} + \left|1 - \frac{u'\tau'}{(us - u's')}\right|^{2H} - 1 \right). \end{aligned} \quad (8)$$

By series expansion we find for any fixed $\tau_1 < \tau_0 < \tau_2$ and τ, τ' with $0 < \tau_1 < \tau, \tau' < \tau_2 < \infty$,

$$|r_{u,u'}(s, \tau, s', \tau')| \leq \frac{|us - u's'|^{2H}}{(u\tau u'\tau')^H} 2H|2H - 1||us - u's'|^{-2}(u\tau u'\tau'), \quad 2H \neq 1,$$

provided that $|\frac{u}{us-u's'}|$ and $|\frac{u'}{us-u's'}|$ are sufficiently small. For $2H = 1$, we have $r_{u,u'}(s, \tau, s', \tau') = 0$ since the increments of Brownian motion on disjoint intervals are independent. Therefore,

$$r^*(t) := \sup_{\substack{\left|\frac{us-u's'}{u}\right|, \left|\frac{us-u's'}{u'}\right| \geq t \\ \tau_1 < \tau, \tau' < \tau_2 \\ u, u', s, s' > 0}} |r_{u,u'}(s, \tau, s', \tau')| \leq \mathcal{K}t^{-\lambda}, \quad (9)$$

for $\lambda = 2 - 2H > 0$, t sufficiently large and some positive constant $\mathcal{K}$ depending only on H , τ_1 and τ_2 . Similarly, from (8) it follows that for any fixed M there exists $\delta \in (0, 1)$ such that

$$\begin{aligned} 0 < \delta &\leq \inf_{\substack{\left|\frac{us-u's'}{u}\right|, \left|\frac{us-u's'}{u'}\right| \leq M \\ |\tau-\tau^*|, |\tau'-\tau^*| \leq m}} r_{u,u'}(s, \tau, s', \tau') \\ &\leq \sup_{\substack{\left|\frac{us-u's'}{u}\right|, \left|\frac{us-u's'}{u'}\right| \leq M \\ |\tau-\tau^*|, |\tau'-\tau^*| \leq m}} r_{u,u'}(s, \tau, s', \tau') \leq 1 - \delta < 1, \end{aligned} \quad (10)$$

for sufficiently small m .

2.1 Asymptotics

Due to the following lemma, while analyzing tail asymptotics of the supremum of Z , we can restrict the considered domain of (s, τ) to a strip with $|\tau - \tau_0| \leq \log v/v$.

Lemma 1 (Piterbarg (2001, Lemma 2 and 4)) *There exists a positive constant C such that for any $v, T > 0$,*

$$\mathbb{P} \left(\sup_{\substack{s \in [0, T] \\ |\tau - \tau_0| \geq \log v/v}} AZ(s, \tau) > v \right) \leq CT v^{2/H} \exp \left(-\frac{1}{2} v^2 - b \log^2 v \right), \quad (11)$$

where $b = B/(2A)$. Furthermore, for any $T > 0$, with $a = 1/(2\tau_0^{2H})$, as $v \rightarrow \infty$,

$$\mathbb{P} \left(\sup_{\substack{s \in [0, T] \\ |\tau - \tau_0| \leq \log v/v}} AZ(s, \tau) > v \right) = \sqrt{\pi} a^{\frac{2}{H}} b^{-\frac{1}{2}} \mathcal{H}_{BH}^2 T v^{\frac{2}{H}-1} \Psi(v) (1 + o(1)),$$

where

$$\mathcal{H}_{BH} = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \exp \left(\sup_{t \in [0, T]} \left(\sqrt{2} B_H(t) - t^{2H} \right) \right) \in (0, \infty),$$

is the so-called Pickands' constant. This holds also for $T = v^{-1/H'}$, with $1 > H' > H$.

Hüsler and Piterbarg (2004a, Corollary 2) showed that the above actually holds true for T depending on v such that $v^{-1/H'} < T < \exp(cv^2)$, for any $H' \in (H, 1)$ and $c \in (0, \frac{1}{2})$.

2.2 Discretization

Let $\tau^*(v) = \log v/v$ and $J(v) = \{\tau : |\tau - \tau_0| \leq \tau^*(v)\}$. For a fixed $T, \theta > 0$ and some $v > 0$, let us define a discretization of the set $[0, T] \times J(v)$ as follows

$$\begin{aligned} s_l &= lq(v), \quad 0 \leq l \leq L, \quad L = [T/q(v)], \quad q(v) = \theta v^{-\frac{1}{H}}, \\ \tau_n &= \tau_0 + nq(v), \quad 0 \leq |n| \leq N, \quad N = [\tau^*(v)/q(v)]. \end{aligned}$$

Along the same lines as in Hüsler and Piterbarg (2004b, Lemma 6) we get the following lemma.

Lemma 2 *There exist positive constants $K_1, K_2, v_0 > 0$, such that, for any $\theta > 0$ and $v \geq v_0$,*

$$\mathbb{P} \left(\max_{\substack{0 \leq l \leq L \\ 0 \leq |n| \leq N}} AZ(s_l, \tau_n) \leq v - \frac{\theta^{\frac{H}{2}}}{v}, \sup_{\substack{s \in [0, T] \\ \tau \in J(v)}} AZ(s, \tau) > v \right) \leq K_1 v^{\frac{2}{H}-1} \Psi(v) \theta^{\frac{H}{2}} \exp(-\theta^{-H}/K_2).$$

Finally, it is possible to approximate tail asymptotics of the supremum of Z on the strip $[0, T] \times J(v)$ by maximum taken over discrete time points. The proof of the following lemma follows line-by-line the same as the proof of Piterbarg (2001, Lemma 4) and thus we omit it. Similar result can be found in, e.g., Hüsler and Piterbarg (2004b, Lemma 7).

Lemma 3 *For any $T, \theta > 0$, as $v \rightarrow \infty$,*

$$\mathbb{P} \left(\max_{\substack{0 \leq l \leq L \\ 0 \leq |n| \leq N}} AZ(s_l, \tau_n) > v \right) = \sqrt{\pi} a^{\frac{2}{H}} b^{-\frac{1}{2}} (\mathcal{H}_{BH}^\theta)^2 T v^{\frac{2}{H}-1} \Psi(v) (1 + o(1)),$$

where $\mathcal{H}_{BH}^\theta = \lim_{S \rightarrow \infty} S^{-1} \mathbb{E} \exp \left(\sup_{t \in \theta \mathbb{Z} \cap [0, S]} \left(\sqrt{2} B_H(t) - t^{2H} \right) \right)$.

It follows easily that $\mathcal{H}_{BH}^\theta \rightarrow \mathcal{H}_{BH}$ as $\theta \rightarrow 0$, so that the above asymptotics is the same as in Lemma 1 when the discretization parameter θ decreases to zero so that the number of discretization points grows to infinity.

3 Auxiliary lemmas

We begin with some auxiliary lemmas that are later needed in the proofs. The first lemma is a slightly modified version of Leadbetter et al. (1983, Theorem 4.2.1).

Lemma 4 (Berman's inequality) *Suppose that $\xi_1, \dots, \xi_n$ are normal random variables with correlation matrix $\Lambda^1 = (\Lambda_{i,j}^1)$ and $\eta_1, \dots, \eta_n$ similarly with correlation matrix $\Lambda^0 = (\Lambda_{i,j}^0)$. Let $\sigma(\xi_i) = \sigma(\eta_i) \in (0, 1]$, $\rho_{i,j} = \max(|\Lambda_{i,j}^1|, |\Lambda_{i,j}^0|)$ and u_i be real numbers, $i = 1, \dots, n$. Then,*

$$\begin{aligned} \mathbb{P} \left(\bigcap_{j=1}^n \{\xi_j \leq u_j\} \right) - \mathbb{P} \left(\bigcap_{j=1}^n \{\eta_j \leq u_j\} \right) \\ \leq \frac{1}{2\pi} \sum_{1 \leq i < j \leq n} \left(\Lambda_{i,j}^1 - \Lambda_{i,j}^0 \right)^+ (1 - \rho_{i,j}^2)^{-\frac{1}{2}} \exp \left(-\frac{u_i^2 + u_j^2}{2(1 + \rho_{i,j})} \right). \end{aligned}$$

The following lemma is a general form of the Borel-Cantelli lemma; cf. (Spitzer 1964).

Lemma 5 (Borel-Cantelli lemma) *Consider a sequence of events $\{E_k\}_{k=0}^\infty$. If*

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k) < \infty,$$

then $\mathbb{P}(E_n \text{ i.o.}) = 0$. Whereas, if

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k) = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\sum_{1 \leq k \neq t \leq n} \mathbb{P}(E_k E_t)}{(\sum_{k=1}^n \mathbb{P}(E_k))^2} \leq 1,$$

then $\mathbb{P}(E_n \text{ i.o.}) = 1$.

Lemma 6 *For any $\varepsilon \in (0, 1)$, there exist positive constants K and ρ depending only on ε, H, p and λ such that*

$$\begin{aligned} \mathbb{P} \left(\sup_{s < t \leq T} \frac{Q_{B_H}(t)}{f_p(t)} \leq 1 \right) \leq \exp \left(-\frac{(1-\varepsilon)}{(1+\varepsilon)} \int_{S+f_p(S)}^T \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \right) \\ + K S^{-\rho}, \end{aligned}$$

for any $T - f_p(S) \geq S \geq K$, with $f_p(T)/f_p(S) \leq C$ and C being some universal positive constant.

Proof Let $\varepsilon \in (0, 1)$ be some positive constant. For the reminder of the proof let K and ρ be two positive constants depending only on ε, H, p and λ that may differ from line to line. For any $k \geq 0$ put $s_0 = S, y_0 = f_p(s_0), t_0 = s_0 + y_0, x_0 = f_p(t_0)$ and

$$\begin{aligned} s_k &= t_{k-1} + \varepsilon x_{k-1}, & y_k &= f_p(s_k), & t_k &= s_k + y_k, & x_k &= f_p(t_k), \\ I_k &= (s_k, t_k], & v_k &= A x_k^{1-H} = v_{f_p}(t_k), & \tilde{I}_k &= \frac{I_k}{x_k} = (\tilde{s}_k, \tilde{t}_k], & |\tilde{I}_k| &= \frac{y_k}{x_k}. \end{aligned} \quad (12)$$

From this construction, it is easy to see that the intervals I_k are disjoint. Furthermore, $\delta(I_k, I_{k+1}) = \varepsilon x_k$, and $1 - \varepsilon \leq y_k/x_k \leq 1$, for any $k \geq 0$ and sufficiently large S .

Note that, for any $k \geq 0$, $|I_k| \sim f_p(S)$ as S grows large, therefore if $T(S, \varepsilon)$ is the smallest number of intervals $\{I_k\}$ needed to cover $[S, T]$, then $T(S, \varepsilon) \leq [(T - S)/(f_p(S)(1 + \varepsilon))]$. Moreover, since $f_p(T)/f_p(S)$ is bounded by the constant $\mathcal{C} > 0$ not depending on S and ε , it follows that, $x_k/x_t \leq \mathcal{C}$ for any $0 \leq t < k \leq T(S, \varepsilon)$.

Now let us introduce a discretization of the set $\tilde{I}_k \times J(v_k)$ as in Section 2.2. That is, for some $\theta > 0$, define grid points

$$\begin{aligned} s_{k,l} &= \tilde{s}_k + lq_k, \quad 0 \leq l \leq L_k, \quad L_k = [(1 - \varepsilon)/q_k], \quad q_k = \theta v_k^{-\frac{1}{H}}, \\ \tau_{k,n} &= \tau_0 + nq_k, \quad 0 \leq |n| \leq N_k, \quad N_k = \lceil \tau^*(v_k)/q_k \rceil. \end{aligned}$$

Since f_p is an increasing function, it easily follows that,

$$\begin{aligned} \mathbb{P} \left(\sup_{S < t \leq T} \frac{Q_{B_H}(t)}{f_p(t)} \leq 1 \right) &\leq \mathbb{P} \left(\bigcap_{k=0}^{T(S, \varepsilon)} \left\{ \sup_{t \in I_k} Q_{B_H}(t) \leq x_k \right\} \right) \\ &\leq \mathbb{P} \left(\bigcap_{k=0}^{T(S, \varepsilon)} \left\{ \sup_{\substack{s \in I_k/x_k \\ \tau \in J(v_k)}} AZ_{x_k}(s, \tau) \leq v_k \right\} \right) \\ &\leq \mathbb{P} \left(\bigcap_{k=0}^{T(S, \varepsilon)} \left\{ \max_{\substack{0 \leq l \leq L_k \\ 0 \leq |n| \leq N_k}} AZ_{x_k}(s_{k,l}, \tau_{k,n}) \leq v_k \right\} \right) \\ &\leq \prod_{k=0}^{T(S, \varepsilon)} \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_k \\ 0 \leq |n| \leq N_k}} AZ_{x_k}(s_{k,l}, \tau_{k,n}) \leq v_k \right) \\ &\quad + \sum_{0 \leq t < k \leq T(S, \varepsilon)} C_{k,t} =: P_1 + P_2, \end{aligned}$$

where the last inequality follows from Berman's inequality with

$$C_{k,t} = \sum_{\substack{0 \leq l \leq L_k \\ 0 \leq p \leq L_t}} \sum_{\substack{|n| \leq N_k \\ |m| \leq N_t}} \frac{|r_{x_k, x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})|}{\sqrt{1 - r_{x_k, x_t}^2(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})}} \exp \left(- \frac{\frac{1}{2}(v_k^2 + v_t^2)}{1 + |r_{x_k, x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})|} \right).$$

Estimate of P_1 .

Note that we can use the fact that Z_{x_k} has the same distribution as $Z_1 \equiv Z$ for any x_k . Since the process Z is stationary with respect to the first variable, from Lemma 3, for any $\varepsilon \in (0, 1)$, sufficiently large S and small θ ,

$$\begin{aligned} P_1 &\leq \exp \left(- \sum_{k=0}^{T(S, \varepsilon)} \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_k \\ 0 \leq |n| \leq N_k}} AZ_{x_k}(s_{k,l}, \tau_{k,n}) > v_k \right) \right) \\ &\leq \exp \left(- \left(1 - \frac{\varepsilon}{4}\right) \sum_{k=0}^{T(S, \varepsilon)} \mathbb{P} \left(\sup_{(s, \tau) \in I_k \times J(v_k)} AZ(s, \tau) > v_k \right) \right) \end{aligned}$$

Then, by (7) combined with (3),

$$\begin{aligned}
 P_1 &\leq \exp \left(- \left(1 - \frac{\varepsilon}{2} \right) \sum_{k=0}^{T(S,\varepsilon)} \mathbb{P} \left(\sup_{\substack{s \in \tilde{I}_k \\ \tau \geq 0}} AZ(s, \tau) > v_k \right) \right) \\
 &= \exp \left(- \left(1 - \frac{\varepsilon}{2} \right) \sum_{k=0}^{T(S,\varepsilon)} \mathbb{P} \left(\sup_{t \in [0, \frac{y_k}{x_k} f_p(t_k)]} Q_{B_H}(t) > f_p(t_k) \right) \right) \\
 &\leq \exp \left(- (1 - \varepsilon) \sum_{k=0}^{T(S,\varepsilon)} \mathbb{P} \left(\sup_{t \in [0, f_p(t_k)]} Q_{B_H}(t) > f_p(t_k) \right) \frac{f_p(s_k)}{f_p(t_k)} \right) \\
 &\leq \exp \left(- \frac{1 - \varepsilon}{1 + \varepsilon} \int_{S + f_p(S)}^T \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \right).
 \end{aligned}$$

Estimate of P_2 .

For any $0 \leq t < k \leq T(S, \varepsilon)$, $0 \leq l \leq L_k$, $0 \leq p \leq L_t$, we have

$$\begin{aligned}
 x_k s_{k,l} - x_t s_{t,p} &= (s_k + x_k l q_k) - (s_t + x_t p q_t) \\
 &= \sum_{i=t}^{k-1} (y_i + \varepsilon x_i) + x_k l q_k - x_t p q_t \geq \sum_{i=t}^{k-1} (y_i + \varepsilon x_i) - x_t (1 - \varepsilon) \\
 &\geq (y_t + \varepsilon x_t)(k - t) - x_t (1 - \varepsilon) \geq x_t (k - t) \varepsilon,
 \end{aligned}$$

where the last inequality holds provided that $k - t \geq s_0$ with s_0 sufficiently large. Therefore, c.f. (9),

$$\begin{aligned}
 r_{k,t}^* &:= \sup_{\substack{0 \leq l \leq L_k, 0 \leq p \leq L_t \\ |n| \leq N_k, |m| \leq N_t}} |r_{x_k, x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})| \leq r^*((k - t)\varepsilon) \leq \mathcal{K}(k - t)^{-\lambda} \\
 &\leq \min(1, \lambda)/4.
 \end{aligned}$$

Moreover, from (10) it follows that, for any $0 \leq k - t \leq s_0$, there exists a constant $\zeta \in (0, 1)$ depending only on ε such that for sufficiently large S ,

$$\sup_{\substack{0 \leq l \leq L_k, 0 \leq p \leq L_t \\ |n| \leq N_k, |m| \leq N_t}} |r_{x_k, x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})| \leq \zeta < 1.$$

Finally, recall that $N_k \leq L_k \leq \theta^{-1} v_k^{\frac{1}{H}}$ and $\exp(-v_k^2/2) = (t_k \log^{(1+C_H-p)} t_k)^{-1}$, c.f. (4), (12), so that

$$\begin{aligned}
 P_2 &\leq \frac{4}{\sqrt{1-\zeta^2}} \sum_{0 \leq t < k \leq T(S, \varepsilon)} L_k L_t N_k N_t r_{k,t}^* \exp\left(-\frac{v_k^2 + v_t^2}{2(1+r_{k,t}^*)}\right) \\
 &\leq K \left(\sum_{\substack{0 < k-t \leq s_0 \\ 0 \leq t < k \leq T(S, \varepsilon)}} + \sum_{\substack{k-t > s_0 \\ 0 \leq t < k \leq T(S, \varepsilon)}} \right) (\cdot) \\
 &\leq K \left(\sum_{k=0}^{\infty} v_k^{\frac{4}{H}} \exp\left(-\frac{v_k^2}{1+\zeta}\right) + \sum_{\substack{k-t > s_0 \\ 0 \leq t < k \leq T(S, \varepsilon)}} v_k^{\frac{2}{H}} v_t^{\frac{2}{H}} (k-t)^{-\lambda} \exp\left(-\frac{v_k^2 + v_t^2}{2(1+\frac{\lambda}{4})}\right) \right) \\
 &\leq K \left(\sum_{k=0}^{\infty} t_k^{-\frac{2}{1+\sqrt{\zeta}}} + \sum_{\substack{k-t > s_0 \\ 0 \leq t < k \leq T(S, \varepsilon)}} t_k^{-\frac{1}{1+\frac{\lambda}{2}}} t_t^{-\frac{1}{1+\frac{\lambda}{2}}} (k-t)^{-\lambda} \right) \\
 &\leq K \left(\sum_{k=[S]}^{\infty} k^{-\frac{2}{1+\sqrt{\zeta}}} + \sum_{[S] \leq t < k \leq \infty} k^{-\frac{1}{1+\frac{\lambda}{2}}} t^{-\frac{1}{1+\frac{\lambda}{2}}} (k-t)^{-\lambda} \right) \\
 &\leq K S^{-\rho},
 \end{aligned}$$

where the last inequality follows from basic algebra. $\square$

Let $S > 0$ be any fixed number, $a_0 = S$, $y_0 = f_p(a_0)$ and $b_0 = a_0 + y_0$. For $i > 0$, define

$$\begin{aligned}
 a_i &= b_{i-1}, \quad y_i = f_p(a_i), \quad b_i = a_i + y_i, \quad M_i = (a_i, b_i], \quad v_i = A y_i^{1-H}, \\
 \tilde{M}_i &= \frac{M_i}{y_i} = (\tilde{a}_i, \tilde{b}_i].
 \end{aligned} \tag{13}$$

From this construction it is easy to see that the intervals M_i are disjoint, $\cup_{j=0}^i M_j = (S, b_i]$ and $|\tilde{M}_i| = 1$. Now let us introduce a discretization of the set $\tilde{M}_i \times J(v_i)$ as in Section 2.2. That is, for some $\theta > 0$, define grid points

$$\begin{aligned}
 s_{i,l} &= \tilde{a}_i + l q_i, \quad 0 \leq l \leq L_i, \quad L_i = [1/q_i], \quad q_i = \theta v_i^{-\frac{1}{H}}, \\
 \tau_{i,n} &= \tau_0 + n q_i, \quad 0 \leq |n| \leq N_i, \quad N_i = [\tau^*(v_i)/q_i].
 \end{aligned} \tag{14}$$

With the above notation, we have the following lemma.

Lemma 7 For any $\varepsilon \in (0, 1)$ there exist positive constants K and ρ depending only on ε , H , p and λ such that, with $\theta_i = v_i^{-4/H}$,

$$\begin{aligned} & \mathbb{P} \left(\bigcap_{i=0}^{[(T-S)/f_p(S)]} \left\{ \max_{\substack{0 \leq l \leq L_i \\ 0 \leq |n| \leq N_i}} AZ_{y_i}(s_{i,l}, \tau_{i,n}) \leq v_i - \frac{\theta_i^{\frac{H}{2}}}{v_i} \right\} \right) \\ & \geq \frac{1}{4} \exp \left(-(1 + \varepsilon) \int_S^T \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \right) - K S^{-\rho}, \end{aligned}$$

for any $T - f_p(S) \geq S \geq K$, with $f_p(T)/f_p(S) \leq C$ and C being some universal positive constant.

Proof Put $\hat{v}_i = v_i - \theta_i^{\frac{H}{2}}/v_i$ and $I = [(T - S)/f_p(S)]$. Similarly as in the proof of Lemma 6 we find that Berman's inequality implies

$$\begin{aligned} & \mathbb{P} \left(\bigcap_{i=0}^I \left\{ \max_{\substack{0 \leq l \leq L_i \\ 0 \leq |n| \leq N_i}} AZ_{y_i}(s_{i,l}, \tau_{i,n}) \leq v_i - \frac{\theta_i^{\frac{H}{2}}}{v_i} \right\} \right) \\ & \geq \prod_{i=0}^I \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_i \\ 0 \leq |n| \leq N_i}} AZ_{y_i}(s_{i,l}, \tau_{i,n}) \leq v_i - \frac{\theta_i^{\frac{H}{2}}}{v_i} \right) - \sum_{0 \leq i < j \leq I} D_{i,j} =: P'_1 + P'_2, \end{aligned}$$

where

$$\begin{aligned} D_{i,j} &= \frac{1}{2\pi} \sum_{\substack{0 \leq l \leq L_i \\ 0 \leq p \leq L_j}} \sum_{\substack{|n| \leq N_i \\ |m| \leq N_j}} \frac{(\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m}))^+}{\sqrt{1 - \tilde{r}_{y_i, y_j}^2(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m})}} \\ & \times \exp \left(-\frac{\frac{1}{2}(\hat{v}_i^2 + \hat{v}_j^2)}{1 + |\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m})|} \right), \end{aligned}$$

with

$$\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m}) = -r_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m}).$$

Estimate of P'_1 .

By Lemma 1 the correction term $\theta_i^{\frac{H}{2}}/v_i$ does not change the order of the asymptotics of the tail of Z . Furthermore, the tail asymptotics of the supremum on the strip

$(s, \tau) \in \tilde{M}_i \times J(v_i)$ are of the same order if $\tau \geq 0$. Hence, for every $\varepsilon > 0$, following the same lines of reasoning as in the estimation of P_1 in Lemma 6,

$$\begin{aligned} P'_1 &\geq \prod_{i=0}^I \left(1 - \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_i \\ 0 \leq |n| \leq N_i}} AZ_{y_i}(s_{i,l}, \tau_{i,n}) > \hat{v}_i \right) \right) \\ &\geq \frac{1}{4} \exp \left(- \sum_{i=0}^I \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_i \\ 0 \leq |n| \leq N_i}} AZ_{y_i}(s_{i,l}, \tau_{i,n}) > \hat{v}_i \right) \right) \\ &\geq \frac{1}{4} \exp \left(- \sum_{i=0}^I \mathbb{P} \left(\sup_{\substack{s \in \tilde{M}_i \\ \tau \in J(v_i)}} AZ(s, \tau) > v_i - \frac{\theta_i^{\frac{H}{2}}}{v_i} \right) \right) \\ &\geq \frac{1}{4} \exp \left(-(1 + \varepsilon) \sum_{i=0}^I \mathbb{P} \left(\sup_{\substack{s \in \tilde{M}_i \\ \tau \geq 0}} AZ(s, \tau) > v_i \right) \right) \\ &= \frac{1}{4} \exp \left(-(1 + \varepsilon) \sum_{i=0}^I \mathbb{P} \left(\sup_{t \in [0, f_p(a_i)]} Q_{B_H}(t) > f_p(a_i) \right) \right) \\ &\geq \frac{1}{4} \exp \left(-(1 + \varepsilon) \int_S^T \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \right), \end{aligned}$$

provided that S is sufficiently large.

Estimate of P'_2 .

Clearly, for $j \geq i + 2$ and any $0 \leq l \leq L_i, 0 \leq p \leq L_j$; c.f. (13),

$$y_j s_{j,p} - y_i s_{i,l} = a_j + y_j p q_j - (a_i + y_i l q_i) \geq (j - i - 1) y_i,$$

so that by (9), for any $0 \leq i < j \leq I$,

$$r_{i,j}^* := \sup_{\substack{0 \leq l \leq L_i, 0 \leq p \leq L_j \\ |n| \leq N_i, |m| \leq N_j}} |\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m})| \leq r^*(j - i - 1) \leq r^*(1) < 1. \quad (15)$$

On the other hand, by (10), there exist positive constants s_0 , such that for sufficiently large S ,

$$(\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m}))^+ = 0, \quad \text{if } j = i + 1, \quad |y_j s_{j,p} - y_i s_{i,l}| / y_i \leq s_0, \quad (16)$$

$$|\tilde{r}_{y_i, y_j}(s_{i,l}, \tau_{i,n}, s_{j,p}, \tau_{j,m})| \leq r^*(s_0) < 1, \quad \text{if } j = i + 1, \quad |y_j s_{j,p} - y_i s_{i,l}| / y_j > s_0 \quad (17)$$

Therefore, by (15)–(16) we obtain

$$P'_2 \leq \sum_{\substack{0 \leq i \leq I-1 \\ j=i+1}} \sum_{\substack{0 \leq l \leq L_i \\ 0 \leq p \leq L_j}} \sum_{\substack{|n| \leq N_i \\ |m| \leq N_j}} \frac{1}{\sqrt{1-r^*(s_0)}} \exp \left(-\frac{\frac{1}{2}(\hat{v}_i^2 + \hat{v}_j^2)}{1+r^*(s_0)} \right) \\ + \sum_{\substack{0 \leq i \leq I-2 \\ i+2 \leq j \leq I}} \sum_{\substack{0 \leq l \leq L_i \\ 0 \leq p \leq L_j}} \sum_{\substack{|n| \leq N_i \\ |m| \leq N_j}} \frac{r^*(j-i-1)}{\sqrt{1-r^*(1)}} \exp \left(-\frac{\frac{1}{2}(\hat{v}_i^2 + \hat{v}_j^2)}{1+r^*(j-i-1)} \right).$$

Completely similarly to the estimation of P_2 in the proof of Lemma 6, we can get that there exist positive constants K and ρ such that, for sufficiently large S ,

$$P'_2 \leq K S^{-\rho}.$$

□

The next lemma is a straightforward modification of Watanabe (1970, Lemma 3.1 and Lemma 4.1), see also Qualls and Watanabe (1971, Lemma 1.4).

Lemma 8 *If Theorem 1 is true under the additional condition, that for large t ,*

$$\left(\frac{2}{A^2} \log t \right)^{1/2(1-H)} \leq f(t) \leq \left(\frac{3}{A^2} \log t \right)^{1/2(1-H)}, \quad (18)$$

it is true without the additional condition.

4 Proof of the main results

Proof of Theorem 1 Note that the case $\mathcal{I}_f < \infty$ is straightforward and does not need any additional knowledge on the process Q_{B_H} apart from the stationarity property. Indeed, consider the sequence of intervals M_i as in Lemma 7. Then, for any $\varepsilon > 0$ and sufficiently large T ,

$$\sum_{k=[T]+1}^{\infty} \mathbb{P} \left(\sup_{t \in M_k} Q_{B_H}(t) > f(a_k) \right) = \sum_{k=[T]}^{\infty} \mathbb{P} \left(\sup_{t \in [0, f(b_k)]} Q_{B_H}(t) > f(b_k) \right) \leq \mathcal{I}_f < \infty,$$

and the Borel-Cantelli lemma completes this part of the proof since f is an increasing function.

Now let f be an increasing function such that $\mathcal{I}_f \equiv \infty$. Using the same notation as in Lemma 6 with f instead of f_p , we find that, for any $S, \varepsilon, \theta > 0$,

$$\mathbb{P} (Q_{B_H}(s) > f(s) \text{ i.o.}) \geq \mathbb{P} \left(\left\{ \sup_{t \in I_k} Q_{B_H}(t) > f(t_k) \right\} \text{ i.o.} \right) \\ \geq \mathbb{P} \left(\left\{ \max_{\substack{0 \leq l \leq L_k \\ 0 \leq |n| \leq N_k}} AZ_{x_k}(s_{k,l}, \tau_{k,n}) > v_k \right\} \text{ i.o.} \right).$$

Let

$$E_k = \left\{ \max_{\substack{0 \leq l \leq L_k \\ 0 \leq |n| \leq N_k}} AZ_{x_k}(s_{k,l}, \tau_{k,n}) \leq v_k \right\}.$$

For sufficiently large S and θ ; c.f. estimation of P_1 , we get

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k^c) \geq \frac{(1-\varepsilon)}{(1+\varepsilon)} \int_{S+f(S)}^{\infty} \frac{1}{f(u)} \mathbb{P}\left(\sup_{t \in [0, f(u)]} Q_{B_H}(t) > f(u)\right) du = \infty. \quad (19)$$

Note that

$$1 - \mathbb{P}(E_i^c \text{ i.o.}) = \lim_{m \rightarrow \infty} \prod_{k=m}^{\infty} \mathbb{P}(E_k) + \lim_{m \rightarrow \infty} \left(\mathbb{P}\left(\bigcap_{k=m}^{\infty} E_k\right) - \prod_{k=m}^{\infty} \mathbb{P}(E_k) \right).$$

The first limit equals to zero as a consequence of (19). The second limit equals to zero because of the asymptotic independence of the events E_k . Indeed, there exist positive constants K and ρ , depending only on H, ε, λ , such that for any $n > m$,

$$A_{m,n} = \left| \mathbb{P}\left(\bigcap_{k=m}^n E_k\right) - \prod_{k=m}^n \mathbb{P}(E_k) \right| \leq K(S+m)^{-\rho},$$

by the same calculations as in the estimate of P_2 in Lemma 6 after realizing that, by Lemma 8, we might restrict ourselves to the case when (18) holds. Therefore $\mathbb{P}(E_i^c \text{ i.o.}) = 1$, which completes the proof. $\square$

Proof of Theorem 2 In order to make the proof more transparent we divide it on several steps. $\square$

Step 1. Let $p > 1$. Then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq -(1+2\varepsilon)^2 \quad \text{a.s.}$$

Proof Let $\{T_k : k \geq 1\}$ be a sequence such that $T_k \rightarrow \infty$, as $k \rightarrow \infty$. Put $S_k = T_k - (1+2\varepsilon)^2 h_p(T_k)$. Since $h_p(t) = O(t \log^{1-p} t \log_2 t)$, then, for $p > 1$, $S_k \sim T_k$, as $k \rightarrow \infty$, and from Lemma 6 it follows that

$$\begin{aligned} \mathbb{P}\left(\frac{\xi_p(T_k) - T_k}{h_p(T_k)} \leq -(1+2\varepsilon)^2\right) &= \mathbb{P}(\xi_p(T_k) \leq S_k) = \mathbb{P}\left(\sup_{S_k < t \leq T_k} \frac{Q_{B_H}(t)}{f_p(t)} < 1\right) \\ &\leq \exp\left(-\frac{(1-\varepsilon)}{(1+\varepsilon)} \int_{S_k+f_p(S_k)}^{T_k} \frac{1}{f_p(u)} \mathbb{P}\left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u)\right) du\right) + 2KT_k^{-\rho}. \end{aligned}$$

Moreover, as $k \rightarrow \infty$,

$$\begin{aligned} & \int_{S_k + f_p(S_k)}^{T_k} \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \\ & \sim (1 + 2\varepsilon)^2 h_p(T_k) \frac{1}{f_p(T_k)} \mathbb{P} \left(\sup_{t \in [0, f_p(T_k)]} Q_{B_H}(t) > f_p(T_k) \right) \\ & = (1 + 2\varepsilon)^2 p \log_2 T_k. \end{aligned} \quad (20)$$

Now take $T_k = \exp(k^{1/p})$. Then,

$$\sum_{k=0}^{\infty} \mathbb{P}(\xi_p(T_k) \leq S_k) \leq 2K \sum_{k=0}^{\infty} k^{-(1+\varepsilon/2)} < \infty.$$

Hence, by the Borel-Cantelli lemma, we have

$$\liminf_{k \rightarrow \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.} \quad (21)$$

Since $\xi_p(t)$ is a non-decreasing random function of t , for every $T_k \leq t \leq T_{k+1}$, we have

$$\frac{\xi_p(t) - t}{h_p(t)} \geq \frac{\xi_p(T_k) - T_k}{h_p(T_k)} - \frac{T_{k+1} - T_k}{h_p(T_k)}.$$

For $p > 1$ elementary calculus implies

$$\lim_{k \rightarrow \infty} \frac{T_{k+1} - T_k}{h_p(T_k)} = 0,$$

so that

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq \liminf_{k \rightarrow \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \quad \text{a.s.},$$

which completes the proof of this step. $\square$

Step 2. Let $p > 1$. Then, for every $\varepsilon \in (0, 1)$,

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \leq -(1 - \varepsilon) \quad \text{a.s.}$$

Proof As in the proof of the lower bound (Step 1), we put

$$T_k = \exp(k^{(1+\varepsilon^2)/p}), \quad S_k = T_k - (1 - \varepsilon)h_p(T_k), \quad k \geq 1.$$

Let

$$B_k = \{\xi_p(T_k) \leq S_k\} = \left\{ \sup_{S_k < t \leq T_k} \frac{Q_{B_H}(t)}{f_p(t)} < 1 \right\}.$$

It suffices to show $\mathbb{P}(B_n \text{ i.o.}) = 1$, that is

$$\lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{k=m}^{\infty} B_k \right) = 1. \quad (22)$$

Let

$$a_0^k = S_k, \quad y_0^k = f_p(a_0^k), \quad b_0^k = a_0^k + y_0^k,$$

$$a_i^k = b_{i-1}^k, \quad y_i^k = f_p(a_i^k), \quad b_i^k = a_i^k + y_i^k, \quad M_i^k = (a_i^k, b_i^k], \quad v_i^k = A(y_i^k)^{1-H},$$

$$\tilde{M}_i^k = \frac{M_i^k}{y_i^k} = (\tilde{a}_i^k, \tilde{b}_i^k].$$

Define J_k to be the biggest number such that $b_{J_k-1}^k \leq T_k$ and $b_{J_k}^k > T_k$. In what follows let $b_{J_k}^k$ be redefined to T_k . Note that $J_k \leq [(T_k - S_k)/f_p(S_k)]$.

Since f_p is an increasing function,

$$B_k = \bigcap_{i=0}^{J_k} \left\{ \sup_{t \in M_i^k} \frac{Q_{B_H}(t)}{f_p(t)} < 1 \right\} \supset \bigcap_{i=0}^{J_k} \left\{ \sup_{t \in M_i^k} Q_{B_H}(t) < y_i^k \right\}$$

$$= \bigcap_{i=0}^{J_k} \left\{ \sup_{\substack{s \in \tilde{M}_i^k \\ \tau \geq 0}} AZ_{y_i^k}(s, \tau) < v_i^k \right\}.$$

Analogously to (14), define a discretization of the set $\tilde{M}_i^k \times J(v_i^k)$ as follows

$$s_{i,l}^k = \tilde{a}_i^k + l q_i^k, \quad 0 \leq l \leq L_i^k, \quad L_i^k = [1/q_i^k], \quad q_i^k = \theta_i^k (v_i^k)^{-\frac{1}{H}}, \quad \theta_i^k = (v_i^k)^{-\frac{4}{H}}$$

$$\tau_{i,n}^k = \tau_0 + n q_i^k, \quad 0 \leq |n| \leq N_i^k, \quad N_i^k = [\tau^*(v_i^k)/q_i^k].$$

Finally, let

$$A_k = \bigcap_{i=0}^{J_k} \left\{ \max_{\substack{0 \leq l \leq L_i^k \\ 0 \leq |n| \leq N_i^k}} AZ_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \leq v_i^k - \frac{(\theta_i^k)^{\frac{H}{2}}}{v_i^k} \right\}.$$

Observe that

$$\mathbb{P} \left(\bigcup_{k=m}^{\infty} A_k \right) \leq \mathbb{P} \left(\bigcup_{k=m}^{\infty} B_k \right) + \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c).$$

Furthermore,

$$\begin{aligned}
 \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c) &\leq \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_i^k \\ 0 \leq |n| \leq N_i^k}} AZ_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \leq v_i^k - \frac{(\theta_i^k)^{\frac{H}{2}}}{v_i^k}, \sup_{\substack{s \in M_i^k \\ \tau \geq 0}} AZ_{y_i^k}(s, \tau) \geq v_i^k \right) \\
 &\leq \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \mathbb{P} \left(\max_{\substack{0 \leq l \leq L_i^k \\ 0 \leq |n| \leq N_i^k}} AZ_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \leq v_i^k - \frac{(\theta_i^k)^{\frac{H}{2}}}{v_i^k}, \sup_{\substack{s \in M_i^k \\ \tau \in J(v_i^k)}} AZ_{y_i^k}(s, \tau) \geq v_i^k \right) \\
 &\quad + \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \mathbb{P} \left(\sup_{\substack{s \in M_i^k \\ \tau \notin J(v_i^k)}} AZ_{y_i^k}(s, \tau) \geq v_i^k \right). \tag{23}
 \end{aligned}$$

By Lemma 2, for sufficiently large m and some $K_1, K_2 > 0$, the first sum is bounded from above by

$$\begin{aligned}
 &K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} (v_i^k)^{\frac{2(1-2H)}{H}} \exp \left(-\frac{(v_i^k)^2}{2} - \frac{(v_i^k)^4}{K_1} \right) \\
 &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \frac{(\log a_i^k)^{\frac{1-2H}{H}}}{a_i^k (\log a_i^k)^{1+C_H-p}} \exp \left(-\frac{\log^2(a_i^k)}{K_2} \right) \\
 &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} (S_k + i f_p(S_k))^{-2} \leq K \sum_{k=m}^{\infty} (S_k)^{-1} \leq K m^{-4}.
 \end{aligned}$$

Note that by (11), for sufficiently large m , the term in (23) is bounded from above by

$$\begin{aligned}
 &K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} (v_i^k)^{\frac{2}{H}} \exp \left(-\frac{1}{2} (v_i^k)^2 - b \log^2 v_i^k \right) \\
 &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \frac{1}{a_i^k \log^{1+p} a_i^k} \leq K \sum_{k=m}^{\infty} \sum_{i=S_k}^{\infty} \frac{1}{i \log^{1+p} i} \\
 &\leq K \sum_{k=m}^{\infty} (\log(S_k))^{-p} \leq K \sum_{k=m}^{\infty} k^{-(1+\varepsilon)^2} \leq K m^{-\varepsilon}.
 \end{aligned}$$

Therefore

$$\lim_{m \rightarrow \infty} \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c) = 0$$

and

$$\lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{k=m}^{\infty} B_k \right) \geq \lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{k=m}^{\infty} A_k \right).$$

In order to complete the proof of (22) we only need to show that

$$\mathbb{P}(A_n \text{ i.o.}) = 1. \quad (24)$$

Similarly to (20), we have

$$\int_{S_k}^{T_k} \frac{1}{f_p(u)} \mathbb{P} \left(\sup_{t \in [0, f_p(u)]} Q_{B_H}(t) > f_p(u) \right) du \sim (1 - \varepsilon) p \log_2 T_k.$$

Now from Lemma 7 it follows that

$$\mathbb{P}(A_k) \geq \frac{1}{4} \exp \left(-(1 - \varepsilon^2) p \log_2 T_k \right) - K S_k^{-\rho} \geq \frac{1}{8} k^{-(1-\varepsilon^4)},$$

for every k sufficiently large. Hence,

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty. \quad (25)$$

Applying Berman's inequality, we get for $t < k$

$$\mathbb{P}(A_k A_t) \leq \mathbb{P}(A_k) \mathbb{P}(A_t) + Q_{k,t}, \quad (26)$$

where

$$Q_{k,t} = \sum_{\substack{0 \leq i \leq J_k \\ 0 \leq j \leq J_t}} \sum_{\substack{0 \leq l \leq L_i^k \\ 0 \leq p \leq L_j^t}} \sum_{\substack{|n| \leq N_i^k \\ |m| \leq N_j^t}} \frac{|r_{y_i^k, y_j^t}(s_{i,l}^k, \tau_{i,n}^k, s_{j,p}^t, \tau_{j,m}^t)|}{\sqrt{1 - r_{y_i^k, y_j^t}^2(s_{i,l}^k, \tau_{i,n}^k, s_{j,p}^t, \tau_{j,m}^t)}} \\ \times \exp \left(- \frac{(v_i^k - (v_i^k)^{-3})^2 + (v_j^t - (v_j^t)^{-3})^2}{2(1 + |r_{y_i^k, y_j^t}(s_{i,l}^k, \tau_{i,n}^k, s_{j,p}^t, \tau_{j,m}^t)|)} \right).$$

For any $0 \leq i \leq J_k$, $0 \leq j \leq J_t$, $0 \leq l \leq L_i^k$, $0 \leq p \leq L_j^t$, and $t < k$,

$$y_i^k s_{i,l}^k - y_j^t s_{j,p}^t = a_i^k + y_i^k l q_i^k - (a_j^t + y_j^t p q_j^t) \geq S_k - T_t \geq S_k - T_{k-1} \geq \frac{1}{2} (T_k - T_{k-1}),$$

where the last inequality holds for k large enough, since

$$\frac{S_{k+1} - T_k}{T_{k+1} - T_k} \sim 1, \quad \text{as } k \rightarrow \infty.$$

Thus, for sufficiently large k and every $0 \leq t < k$, c.f. (9),

$$\sup_{\substack{0 \leq i \leq J_k \\ 0 \leq j \leq J_t \\ 0 \leq l \leq L_i^k, 0 \leq p \leq L_j^t \\ |n| \leq N_i^k, |m| \leq N_j^t}} |r_{y_i^k, y_j^t}(s_{i,l}^k, \tau_{i,n}^k, s_{j,p}^t, \tau_{j,m}^t)| \leq \sup_{\substack{0 \leq i \leq J_k \\ 0 \leq j \leq J_t}} r^* \left(\frac{T_k - T_{k-1}}{2\sqrt{y_i^k y_j^k}} \right) \\ \leq \mathcal{K} \left(\frac{T_k - T_{k-1}}{2f_p(T_k)} \right)^{-\lambda} \leq \mathcal{K} (T_k - T_{k-1})^{-\lambda/2} \leq \frac{\min(1, \lambda)}{32}.$$

Therefore, for some generic constant K not depending on k and t which may vary between lines, for every $t < k$ sufficiently large,

$$\begin{aligned} Q_{k,t} &\leq K \sum_{\substack{0 \leq i \leq J_k \\ 0 \leq j \leq J_t}} L_i^k L_j^t N_i^k N_j^t (T_k - T_{k-1})^{-\lambda/2} \exp \left(-\frac{(v_i^k)^2 + (v_j^t)^2}{2(1 + \frac{\lambda}{16})} \right) \\ &\leq K (T_k - T_{k-1})^{-\lambda/2} (L_{J_k}^k L_{J_t}^t)^2 \sum_{\substack{0 \leq i \leq J_k \\ 0 \leq j \leq J_t}} \left(a_i^k \log^{1+C_H-p} a_i^k \right)^{-\frac{1}{1+\frac{\lambda}{16}}} \left(a_j^t \log^{1+C_H-p} a_j^t \right)^{-\frac{1}{1+\frac{\lambda}{16}}} \\ &\leq K (T_k - T_{k-1})^{-\lambda/2} \log^{10} T_k (T_k)^{\frac{\frac{\lambda}{8}}{1+\frac{\lambda}{8}}} (T_t)^{\frac{\frac{\lambda}{8}}{1+\frac{\lambda}{8}}} \\ &\leq K T_k^{-\lambda/8} \leq K \exp(-\lambda k(1+\varepsilon^2)/p/8). \end{aligned}$$

Hence, we have

$$\sum_{0 \leq t < k < \infty} Q_{k,t} < \infty. \quad (27)$$

Now (24) follows from (25)–(27) and the general form of the Borel-Cantelli lemma. $\square$

Step 3. If $p \in (0, 1]$, then for every $\varepsilon \in (0, \frac{1}{4})$

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.} \quad (28)$$

and

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \leq -(1 - \varepsilon) \quad \text{a.s.} \quad (29)$$

Proof Put

$$T_k = \exp(k), \quad S_k = T_k \exp \left(-(1 + 2\varepsilon)^2 h_p(T_k) \right).$$

Proceeding the same as in the proof of (21), one can obtain that

$$\liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.}$$

On the other hand it is clear that

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} = \liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \quad \text{a.s.,}$$

since

$$\liminf_{k \rightarrow \infty} \frac{\log(T_k/T_{k+1})}{h_p(T_k)/T_k} = 0.$$

This proves (28).

Let

$$T_k = \exp(k^{1+\varepsilon^2}), \quad S_k = T_k \exp(-(1 - \varepsilon)h_p(T_k)).$$

Noting that

$$\frac{S_{k+1} - T_k}{S_{k+1}} \sim 1, \quad \text{as } k \rightarrow \infty,$$

following along the same lines as in the proof of (22), we also have

$$\liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \leq -(1 - \varepsilon) \quad \text{a.s.,}$$

which proves (29). $\square$

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