Entrainment and boundary-layer separation: a modeling history

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Abstract For many decades since Prandtl's presentation of the boundary-layer equations, all attempts to calculate separated boundary layers ended up in a break-down of the numerical algorithm. It was not until the late 1970s that the first successful calculations were reported. The paper describes the history and philosophy of the steps that led to understanding where the break-down originates and how to circumvent it. Especially, the role of Head's entrainment will be highlighted.

Keywords Boundary-layer flow · Entrainment · Flow separation · Goldstein singularity · Quasi-simultaneous coupling · Turbulent flow · Viscous–inviscid interaction

1 Prologue

An issue celebrating the 50th birthday of this journal gives an opportunity to present an essential episode from a scientific development that has been discussed earlier here, e.g., in [1,2]. In fact, the paper concerns a history that goes back to the middle of the 18th century, to the days when Leonhard Euler and Daniel Bernoulli were shaping the mathematical description of fluid mechanics. In those days, the French mathematician Jean le Rond d'Alembert wondered how aerodynamic drag was created, later known as the *Paradox of d'Alembert* [3]. It was not until the beginning of the 20th century that the German scientist Ludwig Prandtl could shed some light on this matter [4]. Our history will start there.

2 Prandtl's "Grenzschicht"

With his specially designed *Wasserversuchskanal* (English: water channel) at the Technische Hochschule in Hannover (see Fig. 1), Prandtl discovered in 1904 that very close to a solid wall the flow behaves quite differently from

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Fig. 1 Ludwig Prandtl at his *Wasserversuchskanal* in Hannover (Photo DLR-Archiv, Göttingen [5])

Fig. 2 The original Figs. 1 and 2 as they appeared, next to each other, in Prandtl's landmark paper [4]. It is remarkable that from the very beginning he was interested in separated flow. With permission of Springer Nature

that of an inviscid fluid as was hitherto assumed by Euler and Bernoulli. The extremely thin region, where the flow felt the retarding influence of the viscous stiction to the wall, Prandtl termed *Grenzschicht*. This term was later literally translated into the English language as *boundary layer*.

In his landmark presentation [4] at the 3rd International Congress of Mathematicians in Heidelberg (1904), Prandtl sketched in his first figure, reproduced in Fig. 2, how the velocity profile close to the wall shows the retarding influence of a solid wall. This event was the actual birth of boundary-layer theory, as we know it nowadays. For a modern account, the reader is referred to another 'classic': the monograph [6] by Herman Schlichting (one of Prandtl's students). Overview papers of boundary-layer theory can be found at several places in the literature, e.g., in the Annual Review of Fluid Mechanics [7–10].

In the thin boundary layer, a simplified version of the Navier–Stokes equations holds. In two dimensions, it can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \quad \mathbf{u}|_{\text{wall}} = \mathbf{0} \text{ and } u|_{\text{edge}} = U(x), \quad (1)$$

where $U(x)$ is the 'driving' velocity, prescribed from the outer inviscid flow. Further $\mathbf{u} \equiv (u, v)$ denotes the velocity vector in the (x, y) -coordinate system, and ν denotes the kinematic viscosity. At the solid wall, a no-slip boundary condition applies, which is responsible for the formation of the boundary layer.

Fig. 3 The influence of the thin boundary layer is shown by comparing an inviscid solution (based on the Euler equations) and a viscous solution (based on the Navier–Stokes equations). The *red color* shows the supersonic flow region; *green* corresponds with low velocities. The boundary layer along the airfoil surface in the right-hand picture is mostly invisible **a** inviscid computation (Euler) **b** viscous computation (Navier–Stokes). (Color figure online)

Prandtl’s second figure, also reproduced in Fig. 2, shows a sketch of streamlines in a boundary layer that separates from a solid wall. Little did Prandtl know that it would take three quarters of a century before such a separated boundary layer could be calculated as the solution of ‘his’ boundary-layer equation (1). The current paper describes the essential steps towards this result.

2.1 The ‘invisible’ boundary layer

Let me first give an impression of the thickness (or should I say ‘thinness’) of a boundary layer. With modern CFD codes, it is easy to switch the effect of viscosity off (Euler equations) and on (Navier–Stokes equations). As an example, Fig. 3 shows the transonic 2D flow past a supercritical airfoil: the inviscid solution of the Euler equations is compared to the viscous solution of the Navier–Stokes equations. These calculations have been carried out with the CFD code Enflow [11, 12] developed at the Netherlands Aerospace Center NLR.

The red area in Fig. 3 represents the supersonic (i.e., low pressure) part of the flow, which is terminated by a shock wave. The green color, visible, e.g., near the leading and trailing edges, corresponds with low flow velocities. There is also slow flow between the supersonic region and the airfoil surface, but the boundary layer is thinner than one pixel of the original picture. Yet, this ‘invisible’ boundary layer is responsible for a large decrease of the supersonic region, creating a significant reduction of the aerodynamic lift.

In spite of being very thin, a turbulent boundary layer can display much flow details. Van Dyke’s beautiful *Album of Fluid Motion* [13] shows many snapshots that clearly reveal the intermittent character of a turbulent shear layer, with ‘clouds’ of turbulent flow penetrating in the laminar outer flow. From these pictures, one can infer the tremendous challenge to model this type of flow.

3 Early calculation methods

Theodore Von Kármán (another student of Prandtl) realized that in such a thin boundary layer there is not always a need to model the details of the flow. Thus, he integrated (averaged) the boundary-layer equations over its thickness δ , to end up with

$$\frac{d\theta}{dx} + \frac{\theta}{U}(2 + H)\frac{dU}{dx} = \frac{1}{2}c_f. \quad (2)$$

This *momentum integral equation* was introduced in 1921 [14]. It contains the main boundary-layer ‘actors’ that will feature further on in the paper (Fig. 5):

Fig. 4 A sketch of a laminar velocity profile with separated flow showing the main boundary-layer ‘actors’

Fig. 5 A collage of graphs collected by Head [19] in his quest for universal closure relations

- $\delta^* \equiv \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ *displacement thickness* effective shape of body with equal inviscid mass transport past $y = \delta^*$
- $\theta \equiv \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$ *momentum thickness* equal to inviscid momentum transport past $y = \theta + \delta^*$; directly related to drag
- $H \equiv \frac{\delta^*}{\theta} (\geq 1)$ *shape factor* turbulent separated flow corresponds roughly with $H \approx 3$
- $c_f \equiv \frac{2\nu}{U^2} \frac{\partial u}{\partial y} \Big|_{\text{wall}}$ *skin friction coefficient*.

Equation (2) was solved by Ernst Pohlhausen [15] (yet another of Prandtl's students) by assuming a polynomial velocity profile with unknown coefficients plus some additional conditions on the polynomial. From their definition, the above quantities can then be expressed in the unknown polynomial coefficients and substituted in Von Kármán's equation. Schlichting's monograph [6] (until the 7th edition) gives a detailed description of this approach.

Fig. 6 An illustration of the entrainment mechanism

It appeared impossible, however, to compute separated boundary layers in this way. All calculations ended up in diverging or singular results. In 1948, a thorough analysis of this break-down was presented by Sydney Goldstein in another landmark paper [16]. He proposed a number of possible causes for this failure, but the computational power to deeper investigate these options was not available in those days. It would take three decades before further research into this issue was possible along computational lines, eventually revealing that Goldstein had been close. Prandtl, who passed away in 1953, never has seen the outcome as we understand it nowadays.

4 M.R. Head and his ideas on entrainment

In the mid-1950s, at the University of Cambridge (UK), M.R. Head, a New Zealand-born former World War II flight lieutenant [17], was studying boundary layers both from the experimental side as well as the theoretical (calculational) side [18, 19]. In the latter vain, he was interested in developing flow models suitable for engineering purposes. Hereto, he collected a large number of data from various experimental measurements of boundary layers—nowadays we would call these ‘big data.’

In his report from September 1958 for the Aeronautical Research Council [19], Head presents these data in a large number of graphs featuring several variables (see Fig. 6). His hope was to find some ‘magic’ pattern between the variables which could have universal value. Apparently disappointed by what he found, and guided by his own physical intuition, he decided to come up with a new variable. He proposed that the *entrainment* of the boundary layer would play an important and, above all, universal role.

Head’s entrainment idea is schematized in Fig. 7. He started with defining a new shape function $H_{\delta-\delta^*}$, based on $\delta - \delta^*$:

$$\text{Head's shape factor} \quad H_{\delta-\delta^*} \equiv \frac{\delta - \delta^*}{\theta}. \quad (3)$$

This shape function is related to the mass transport $Q(x)$ per cross section:

$$\text{Mass transport} \quad Q \equiv \int_0^{\delta} u \, dy = U(\delta - \delta^*) = U\theta H_{\delta-\delta^*}.$$

By mass conservation, the variation of Q over a distance Δx has to be compensated by an influx of mass from outside the boundary layer. This is called the entrainment E (per unit of length):

$$\text{Entrainment} \quad E \equiv \frac{dQ}{dx} = \frac{d}{dx}[U(\delta - \delta^*)] = \frac{d}{dx}(U\theta H_{\delta-\delta^*}). \quad (4)$$

Connecting these quantities, Head made two assumptions:

$$\text{— Assumption 1} \quad E/U \text{ is a function of only } H_{\delta-\delta^*} : E/U = F(H_{\delta-\delta^*}); \quad (5)$$

$$\text{— Assumption 2} \quad H_{\delta-\delta^*} \text{ is a function of only } H = \delta^*/\theta : H_{\delta-\delta^*} = H_{\delta-\delta^*}(H). \quad (6)$$

FIG. 1. $\frac{1}{U} \frac{d}{dx} [U(\delta - \delta^*)]$ as a function of $H_{\delta-\delta^*}$.

FIG. 2. Variation of $H_{\delta-\delta^*}$ with conventional form parameter H .

Fig. 7 The two essential figures in Head's report on entrainment [19] (including the original captions) illustrating his main assumptions

Fig. 8 Experimental data (symbols) and theoretical proposals (lines) for the relation between the conventional shape factor H and Head's shape factor $H_{\delta-\delta^*}$

It is good to realize that the thickness δ of the boundary layer is only vaguely defined, not only in turbulent flow, but also in laminar flow (Fig. 5). Yet, as we will see, Head's approach turned out extremely useful.

Head fitted the assumed functional relations to experimental data [20,21]. Figure 8, copied from [19], shows these relations together with the used measurement data. Observe that the $H-H_{\delta-\delta^*}$ curve in Fig. 8(right) is only drawn up to $H = 3$, where (roughly) flow separation sets in. Also, observe that the curve runs more or less horizontal

there. Yet it would take another two decades before the consequences thereof were realized. Head was still not able to compute separated boundary layers!

5 A minimum enters the scene

In the second half of the 1960s, further light was shed on the numerical break-down at separation. Catherall and Mangler [22], in 1966, were the first to continue the calculations (slightly) past the point of flow separation. However, inside the region of reversed flow, once more they ran into a break-down of the calculations. In 1969, Stewartson and Williams [23] presented an asymptotic view of the flow field near separation, introducing a so-called triple-deck structure around the singularity. Their theory gave insight into the physical flow of information near a separation point; see also [1, 9, 24]. One decade later, the last pieces of the puzzle would fall into place.

At ONERA in Paris, Le Balleur realized that the horizontal slope in the $H-H_{\delta-\delta^*}$ curve in Fig. 8(right) was a sign that a minimum existed. Herewith the reason for the problems near separation was found [25, 26]. To understand this, let us go into detail through the steps in a boundary-layer calculation as carried out in those days:

- Step 1* The first step is to solve the inviscid flow equations (e.g., Euler) for a given airfoil shape including an estimate for the displacement effects δ^* . This results in the streamwise flow velocity U , which then drives the boundary layer.
- Step 2* The second step is to solve, for this given U , the two differential equations that describe the boundary layer: von Kármán's equation (2) and Head's entrainment definition (4). This provides new values for the momentum thickness θ and Head's shape factor $H_{\delta-\delta^*}$.
- Step 3* In the third step, algebraic closure relations are evaluated for the remaining variables:
 - (a) The entrainment E follows from $H_{\delta-\delta^*}$ using Head's first assumption (5).
 - (b) The conventional shape factor H follows from $H_{\delta-\delta^*}$ using Head's second assumption (6).
 - (c) The displacement thickness δ^* follows from $\delta^* = H \theta$.
 - (d) The shear stress coefficient c_f follows from, e.g., a Ludwig–Tillmann relation [27] or Green's modification [28] (details turn out not to be essential, therefore we refrain from showing these relations explicitly).

When the $H-H_{\delta-\delta^*}$ curve has a minimum, it is clear that Step 3b) cannot always be made. This step breaks down as soon as $H_{\delta-\delta^*}$ drops below its minimum, which occurs (roughly) at a point of flow separation. Note also that as soon as Step 1 is completed, i.e., U has been computed, there is no way to circumvent this problem (e.g., through a reordering of the remaining steps). The prescription of U to the boundary-layer equations is the culprit!!

In 1948, Goldstein formulated one of several possible causes for the break-down at separation as “*Or does a singularity always occur except for certain pressure distributions near separation...?*” [16, p. 51]. In 1977, some 30 years later, this possibility was confirmed via the existence of a minimum!

Experimental evidence for this minimum came at the end of the 1970s [29, 30] and in the 1980s [31, 32]; see Fig. 9 (which has been inspired by a graph in [33]). The figure also shows the theoretical models that were proposed for the $H-H_{\delta-\delta^*}$ relation. The first proposals by Green [34], East [35], and Cebeci and Bradshaw [36] were hesitating to include a minimum in the $H-H_{\delta-\delta^*}$ relation. Le Balleur at ONERA [26] was the first to explicitly consider such a minimum. He was followed by Houwink and the author at NLR [28] and Lock at RAE [37].

Fig. 9 Flow diagram of modern VII methods: semi-inverse (left) and quasi-simultaneous (right)

Fig. 10 The δ^* – U relation in a boundary layer with a minimum (roughly) at separation. An intersection with the (schematically indicated) inviscid flow relation provides existence of a steady solution of the flow equations. When the (approximate) interaction law does not possess such an intersection (e.g., when it runs horizontally as in prescribing U), the calculations break down, similar to Goldstein’s singularity

6 Separated flow, at last ...

Knowing about the existence of a minimum, ways to circumvent it can be designed. In 1977, Le Balleur [25,26] switched to prescribing δ^* to the boundary-layer equations. In this way, the latter are solved in an inverse order: δ^* goes in, U comes out. The new guess for δ^* comes from a relaxation formula comparing the U^B from the boundary-layer solution with the U^E from a (conventional) inviscid flow calculation:

$$\delta^{*\text{new}} = \delta^{*\text{old}} + \text{function}(U^E - U^B)$$

with a suitably chosen function of the difference between U^E and U^B . Because the boundary layer is solved ‘the other way around’ with respect to the conventional approach, this method was coined *semi-inverse*. A block sketch is presented in Fig. 10(left).

Around the same time at the Netherlands Aerospace Center NLR in Amsterdam, the author, inspired by the paper by Catherall and Mangler [22] but not yet aware of Le Balleur’s achievements, was studying the relation between U and δ^* of solutions of the full boundary-layer equation (1) [38,39]. It turned out that this relation shows a similar minimum (Fig. 11) at or close to the point of flow separation. However, this relation is position dependent and far from being universal, unlike Head’s expectation for the H – H_{δ} – δ^* relation.

As a way out, the author replaced the conventional prescription of U by a (simple) linear relation between U and δ^* , called *interaction law*. Effectively, it acts as a boundary condition to the boundary-layer equation (1). From the physical side, the interaction law should be a good approximation of the ‘exact’ external inviscid flow to allow for quick convergence. From a mathematical point of view, it should have a sufficiently positive slope in Fig. 11 to intersect with the boundary-layer relation between U and δ^* , and herewith prevent a Goldstein-type singularity. The method was coined *quasi-simultaneous* because it attempts to solve the two subdomains simultaneously as good as possible, Fig. 10(right).

Thus, since the late 1970s two methods were available to simulate separated boundary layers. Their common essential idea is that the edge velocity U is *not prescribed* to the boundary layer, but results from the solution of the *coupled* viscous–inviscid interaction (VII) problem. A comparison between both approaches has been made in the survey paper by Lock and Williams [33].

It is interesting to see what the effect is of the various H – H_{δ} – δ^* closure relations shown in Fig. 9. Hereto calculations have been performed of flow around a NACA 0012 airfoil at a Reynolds number of 9 million at varying angle of attack. The flow was tripped at 0.014 chord at the upper side and at 0.7 chord at the lower side. The turbulent parts of the flow were modeled with the above entrainment model for various H – H_{δ} – δ^* closures. The laminar part was modeled using Falkner–Skan closure. Further details can be found in the PhD theses of Edith Coenen [40] and

Fig. 11 Lift polar for incompressible flow past a NACA 0012 airfoil at $Re = 9$ million, as computed with Head's entrainment model and various $H-H_{\delta-\delta^*}$ closures from Fig. 9

Fig. 12 Comparison of displacement thickness δ^* (left) and wall shear stress c_f (right) for incompressible flow past a NACA 0012 airfoil at $Re = 9$ million at an angle of attack of 10° (top) and 16° (bottom), as computed with various $H-H_{\delta-\delta^*}$ closures. **a** Angle of attack 10° . **b** Angle of attack 16°

Henny Bijleveld [41]. The latter thesis also describes the case of unsteady flow and the link between the $H-H_{\delta-\delta^*}$ closure and the Van Dommelen–Shen [42] singularity, which occurs when $dH_{\delta-\delta^*}/dH = H_{\delta-\delta^*}/H$.

Looking at the slope of the $H-H_{\delta-\delta^*}$ relation for larger values of H , it is to be expected that Cebeci–Bradshaw's closure is the most robust, whereas Le Balleur's closure is the least robust. The most accurate results are to be expected from the closures by Lock and Houwink. This is confirmed by the simulation results. Figure 12 shows the lift polar compared with experimental data [43, App. IV]. The four curves have been continued until the calculations break down. For larger angles of attack eventually all calculations break down because the physics of the flow does

Fig. 13 Axial slices of the instantaneous vorticity magnitude for flow past a delta wing at $Re_c = 150,000$, computed with DNS on a grid with 133 million cells [55]

not allow a steady solution anymore. Observe that the comparison with experimental data can be pretty good, even beyond maximum lift. Not bad for such simple models...!

The difference between the four variants is shown in more detail in Fig. 13 where the displacement thickness δ^* and the wall shear stress c_f are plotted along the upper side of the profile for an angle of attack of 10° (where all methods generate results) and 16° (where only Houwink's and Cebeci's closures 'survive'). It is clear, as forecasted from the $H-H_{\delta-\delta^*}$ relation, that Le Balleur's closure predicts the most separation and Cebeci's closure the least.

6.1 Mathematical basis

Attempts to complete Goldstein's analysis with a firm mathematical basis have failed thus far; existing mathematical theory is still restricted to accelerating flow [44, Chap. 2]. Yet, the evidence for a relation between Goldstein's singularity and non-existence of a solution of the boundary-layer equations at prescribed edge velocity U is substantial. Next to the already-mentioned conjectured singularity in the δ^*-U relation from the full boundary-layer equations, approximate calculations using (laminar) Falkner–Skan velocity profiles analytically show such a singularity [45, 46]. Also, the strong interaction between boundary layer and local inviscid flow region in the asymptotic triple-deck theory, with its reversal of hierarchy, points in this direction; see, e.g., [1, 2, 23, 47, 48]. The latter asymptotic theory can be extended to describe boundary-layer flow with marginal separation [9, 49], until the separation bubble becomes unsteady and vortex-shedding sets in [50] (in terms of Fig. 11, there is no intersection anymore of the inviscid flow relation and the boundary-layer relation).

6.2 Relation with Navier–Stokes

The singularity for prescribed U will not disappear when the full Navier–Stokes equations are used in the boundary layer. Only when a thicker computational domain is used containing part of the adjacent inviscid flow region (which gives the prescribed U the opportunity to adapt itself to the presence of the boundary layer), the singularity can be avoided (unpublished results).

During the 1980s, Navier–Stokes methods had a tough job in keeping up with these VII methods. In the viscous transonic flow workshop [51] in 1987, many methods have been compared for simulating several airfoils under transonic flow conditions. In those days, the state-of-the-art in turbulence modeling for the Navier–Stokes methods was not good enough to predict maximum lift (most Navier–Stokes methods did not even produce a maximum). In contrast, the VII integral methods, using the above approach, did a good job.

7 What will the future bring?

In the three decades since the mid-1980s, much progress has been made in simulating separated flow using a Navier–Stokes model. This progress has been achieved partly because of improvements in computational hardware. But also improvements in computational algorithms can be mentioned, of which the most important one is the advent of energy-preserving discretization methods, e.g., [52–54]. The current state-of-the-art are direct numerical simulations at Reynolds numbers around 100,000. An example is presented in Fig. 13 showing a direct numerical simulation (DNS) of a massively separated flow around a delta wing at $Re = 150,000$ featuring natural transition to turbulence. The flow has been calculated by Wybe Rozema [55] using a numerical method without artificial diffusion and implemented in the earlier mentioned NLR method Enflow.

For larger Reynolds numbers, in cruise flight in the order of 50 million, DNS will be possible in a few decades from now. Until that time, resort has to be sought to large-eddy turbulence models, again using as little eddy diffusion as possible. A modern example is the anisotropic minimum dissipation model AMD [56,57].

8 Epilogue

Describing the physics in aerodynamic boundary layers has provided a century-long modeling inspiration. In particular, the quest for a model that can describe separated boundary layers spanned three quarters of a century. Also, as shortly touched upon, it strongly stimulated the development of singular perturbation theory, e.g., [24,58,59].

With hindsight, we can say that Head’s experiment-fitted $H-H_{\delta-\delta^*}$ curve, based on a simple entrainment model, tells it all! Yet, Head may not have foreseen its impact in the mid-1950s when he proposed to make this relation central to his boundary-layer model. It was not until two decades later that a minimum in this relation showed up, explaining the poorly understood break-down of boundary-layer calculations for separated flow. This then opened the way for cheap engineering calculations of aerodynamic boundary layers, which are still competitive for airfoil and wing design with limited separated flow regions (as in cruise flight), e.g., [60–62].

This leaves us with the question: “Where did Head find the inspiration that made him focus on the $H-H_{\delta-\delta^*}$ relation?” It was not just ‘big data’ analytics *avant la lettre*. The story shows that to extract useful information from large amounts of data requires expertise in the domain area, and—to end in the same language as we began—*Fingerspitzengefühl* (literally: finger tip feeling).

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