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Finite Third-order Gradient Elasticity and Thermoelasticity

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Abstract A constitutive format for the third-order gradient elasticity is suggested. It includes both isotropic and anisotropic non-linear behavior under finite deformations. Appropriate invariant stress and strain variables are introduced, which allow for reduced forms of the elastic energy law that identically fulfill the objectivity requirement. After working out the transformation behavior under a change of the reference placement, the symmetry transformations for third-order materials can be introduced. After the mechanical third-order theory, an extension to thermoelasticity is given, and necessary and sufficient conditions are derived from the Clausius-Duhem inequality.

Keywords Third-order elasticity · Second strain gradient elasticity · Finite gradient elasticity

Mathematics Subject Classification (2000) 74A99 · 74B20

1 Introduction

After the seminal papers of Toupin [37] and Mindlin [28] on gradient materials, such extensions of the classical simple materials have become the subject of steadily growing research interest. Mostly surface effects are addressed as in [12, 13, 28, 34, 39], dislocation phenomena as in [16, 25, 26], or length scale effects as in shear banding [1, 40] or torsion [20] or indenter tests [27]. The majority of these publications consider only the inclusion of the second deformation gradient. However, Mindlin has already shown half a century ago, that third gradient material models, i.e., frameworks that incorporate the first, second, and the third deformation gradient, can also play an interesting role in continuum mechanics. In fact, they can model surface effects as discussed in [13, 28, 34], allow a body with edges

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and corners to sustain line and point forces as explained in [24, 28, 33, 35], and have strong regularization properties as indicated in [25, 26].

The majority of these publications consider only the case of small deformations and either model elasticity or plasticity. Especially the fundamental work [28] introduces a an elaborate concept for third-order models but is restrained to small deformations in elasticity. Therein not only the boundary conditions but applications such as the aforementioned surface effects are presented.

Therefore a unifying thermodynamically consistent framework for elasticity and afterward also for elastoplasticity of large deformations that can accommodate all these models would be desirable. In the case of second-order models this aim has been pursued in [4, 5, 21, 36], see also [6] on developments in this field. A corresponding third-order framework does not yet exist. Therefore the aim of the present work lies in setting up such a framework for elasticity as a first step. This is done by generalizing the concepts in [5].

With the introduction of a third gradient in the elastic energy, however, many questions arise and have to be answered in a constitutive framework. First of all, one has to introduce appropriate material (i.e., invariant) strain and stress measures, which is by no means straightforward in the context of finite deformations. Naturally, there are many equivalent choices, which are altogether mathematically equivalent. So one can choose those variables under a practical point of view, namely to render the constitutive framework as simple as possible. After having made such a choice, one can introduce reduced forms for the elastic energy function that identically fulfill the objectivity requirement. This step has not been done in [24].

Since such reduced forms live in the reference placement, a change of this placement has to be considered. After having established the transformations of all variables under change of reference placement, we are able to introduce the concept of material symmetry for such models. The symmetry transformations become much more complicated than in the case of simple or first-order materials, but still show the algebraic group property, which allows us to classify such models after their symmetry group. In particular, we can define centro-symmetric and isotropic behavior.

If we linearize such elastic laws, we obtain extensions of the classical St. Venant-Kirchhoff law for third-order materials, i.e., physically (objective) linear laws, but geometrically still non-linear. In the general (anisotropic) case this leads to a total number of 1,485 independent elastic constants. Again, by restricting our concern to the centro-symmetric isotropic case, this number can be drastically reduced to only 17 constants, including the two Lamé constants from classical linear elasticity as has already been found by Mindlin [28].

If one wants to assure thermodynamic consistency of such models, one has to imbed the mechanical theory into a complete thermodynamical setting. In doing so we follow the lines of [8]. One would be tempted to not only include higher deformation gradients into the list of independent variables in the constitutive equations, but also higher temperature gradients. However, it has already by shown by [32] that such dependencies are ruled out by the second law of thermodynamics. In the present setting, only the mechanical variables have been extended to non-classical ones, while the thermodynamics remain classical. We exploit the Clausius-Duhem inequality and find necessary and sufficient conditions for it to hold. In particular, we find the potential relations of the free energy for the stress tensors and the entropy, as one would expect. Finally, the concept of symmetry transformation is extended to the whole set of the thermoelastic constitutive laws. An introduction of internal constraints to gradient materials has been done in [9], see also [6]. Although such a concept if extended to the kinematics of gradients opens interesting applications, we will here limit our concern exclusively to unconstrained materials.

2 Notation

It is well-known that a compact direct notation is extremely helpful when dealing with complicated tensor equations. When working with gradient materials, we have to handle tensors of different orders and their algebraic compositions. Here, the standard notations from continuum mechanics are no longer sufficient. We will therefore extend it in the sequel. Vectors are denoted by small bold print letters, e.g., $\mathbf{v}$. Tensors are denoted by bold capital letters with an indicator of the order of the tensor as a superscript, e.g., $\overset{\langle 3\rangle}{\mathbf{A}}$. Exceptions will be made only in a few cases where the superscript is suppressed such as for the right Cauchy-Green tensor $\mathbf{C}$ where a standard notation has been established and the definition and order of the tensor are well known. Tensor contractions are denoted by $\cdot$, $:$, $\vdots$ or $::$ where the number of dots indicates the multiplicity of the contraction. For tensors $\mathbf{A} = A_{i_1\dots i_n}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n}$, $\mathbf{B} = B_{j_1\dots j_m}\mathbf{e}_{j_1}\otimes\cdots\otimes\mathbf{e}_{j_m}$, each of sufficiently high order n and m, respectively, this means

$$\mathbf{A}\cdot\mathbf{B} := A_{i_1\dots i_{n-1}a}B_{aj_2\dots j_m}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-1}}\otimes\mathbf{e}_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_m}, \tag{1}$$

$$\mathbf{A}:\mathbf{B} := A_{i_1\dots i_{n-2}ab}B_{abj_3\dots j_m}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-2}}\otimes\mathbf{e}_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_m}, \tag{2}$$

$$\mathbf{A}\vdots\mathbf{B} := A_{i_1\dots i_{n-3}abc}B_{abcj_4\dots j_m}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-3}}\otimes\mathbf{e}_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_m}, \tag{3}$$

$$\mathbf{A}::\mathbf{B} := A_{i_1\dots i_{n-4}abcd}B_{abcdj_5\dots j_m}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-4}}\otimes\mathbf{e}_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_m}, \tag{4}$$

$$\mathbf{A}\underbrace{\cdot\dots\cdot}_{\text{p times}}\mathbf{B} := A_{i_1\dots i_{n-p}k_1\dots k_p}B_{k_1\dots k_p j_{p+1}\dots j_m}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-p}}\otimes\mathbf{e}_{j_{p+1}}\otimes\cdots\otimes\mathbf{e}_{j_m}, \tag{5}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ refers to an ONB of $\mathbb{R}^3$.

For a tensor $\overset{\langle n\rangle}{\mathbf{A}}$ of order n with $i < j \leq n$ one defines $\overset{\langle n\rangle}{\mathbf{A}}{}^{[i,j]}$ as the transposed of $\overset{\langle n\rangle}{\mathbf{A}}$ with respect to the i^{th} and j^{th} index, i.e.,

$$\overset{\langle n\rangle}{A}{}^{[i,j]}_{k_1\dots k_i\dots k_j\dots k_n} = \overset{\langle n\rangle}{A}_{k_1\dots k_j\dots k_i\dots k_n} \tag{6}$$

with respect to an orthonormal basis (ONB). Symmetrization of a tensor of order $n > 2$ will be abbreviated as follows.

$$2\overset{[i,j]}{sym}\left[\overset{\langle n\rangle}{\mathbf{A}}\right] := \overset{\langle n\rangle}{\mathbf{A}} + \overset{\langle n\rangle}{\mathbf{A}}{}^{[i,j]}, \tag{7}$$

$$3\overset{[i,j][k,l]}{sym}\left[\overset{\langle n\rangle}{\mathbf{A}}\right] := \overset{\langle n\rangle}{\mathbf{A}} + \overset{\langle n\rangle}{\mathbf{A}}{}^{[i,j]} + \overset{\langle n\rangle}{\mathbf{A}}{}^{[k,l]}. \tag{8}$$

If $\overset{\langle n\rangle}{\mathbf{A}} = \overset{\langle 2\rangle}{\mathbf{A}}$ is a second-order tensor one obtains the classic definition of the symmetric part of a tensor.

The second-order and fourth-order identity tensors are denoted by $\overset{\langle 2\rangle}{\mathbf{I}}$ and $\overset{\langle 4\rangle}{\mathbf{I}}$, respectively. The zero tensor of any order is denoted by $\overset{\langle n\rangle}{\mathbf{0}}$. The determinant of a second-order tensor $\overset{\langle 2\rangle}{\mathbf{A}}$ is denoted by $J_{\overset{\langle 2\rangle}{\mathbf{A}}}$.

Gradients are denoted as follows.

$$Grad(\cdot) \quad \text{denotes the first material gradient,} \tag{9}$$

$$Grad^{\mathrm{II}}(\cdot) \quad \text{denotes the second material gradient,} \tag{10}$$

$$Grad^{\mathrm{III}}(\cdot) \quad \text{denotes the third material gradient.} \tag{11}$$

A higher-order gradient of undetermined order n is denoted by $Grad^n$. Similarly, repeated application of the spatial divergence operator to a tensor field of order n (with n being sufficiently high) is denoted by

$$div^{\mathrm{II}}\big(\overset{\langle n\rangle}{\mathbf{T}}\big) := div\big(div\big(\overset{\langle n\rangle}{\mathbf{T}}\big)\big) \quad \text{or} \quad div^{\mathrm{III}}\big(\overset{\langle n\rangle}{\mathbf{T}}\big) := div\big(div\big(div\big(\overset{\langle n\rangle}{\mathbf{T}}\big)\big)\big), \tag{12}$$

where the Roman numbers prevent confusion with an exponent. One defines the following useful abbreviations, which will be frequently used in what follows.

$$\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} := \overset{\langle 2\rangle}{\mathbf{A}}^{-1} \cdot Grad\big(\overset{\langle 2\rangle}{\mathbf{A}}\big), \tag{13}$$

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} := \overset{\langle 2\rangle}{\mathbf{A}}^{-1} \cdot Grad^{\mathrm{II}}\big(\overset{\langle 2\rangle}{\mathbf{A}}\big), \tag{14}$$

for an invertible second-order tensor field $\overset{\langle 2\rangle}{\mathbf{A}}$. If $\overset{\langle 2\rangle}{\mathbf{A}}$ is the Jacobian matrix of a sufficiently smooth mapping, then Schwartz' theorem implies that $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}$ is symmetric with respect to its last two indices and $\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}$ with respect to the last three indices. In the context of the present work these quantities are understood to have these symmetries. The following sets will be used throughout the present work.

$\mathscr{Inv}$ denotes the group of all invertible second-order tensors.

$\mathscr{Sym}$ denotes the space of all symmetric, second-order tensors.

$\mathscr{Sym}_+$ denotes the set of all symmetric, positive definite second-order tensors.

$\mathscr{Orth}^+$ denotes the group of all orthogonal second-order tensors with positive determinant.

$\mathscr{Unim}$ denotes the unimodular group, i.e., the group of all second-order tensors with determinant of absolute value one.

$$\mathscr{SubSym}_3 := \big\{\overset{\langle 3\rangle}{\mathbf{P}}\text{, triadic} \mid \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \big(\overset{\langle 3\rangle}{\mathbf{P}} \cdot \mathbf{u}\big) \cdot \mathbf{v} = \big(\overset{\langle 3\rangle}{\mathbf{P}} \cdot \mathbf{v}\big) \cdot \mathbf{u}\big\}.$$

$$\mathscr{SubSym}_4 := \big\{\overset{\langle 4\rangle}{\mathbf{P}}\text{, tetradic} \mid \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \big(\big(\overset{\langle 4\rangle}{\mathbf{P}} \cdot \mathbf{u}\big) \cdot \mathbf{v}\big) \cdot \mathbf{w} = \big(\big(\overset{\langle 4\rangle}{\mathbf{P}} \cdot \mathbf{v}\big) \cdot \mathbf{u}\big) \cdot \mathbf{w}$$
$$= \big(\big(\overset{\langle 4\rangle}{\mathbf{P}} \cdot \mathbf{w}\big) \cdot \mathbf{v}\big) \cdot \mathbf{u} = \big(\big(\overset{\langle 4\rangle}{\mathbf{P}} \cdot \mathbf{u}\big) \cdot \mathbf{w}\big) \cdot \mathbf{v}\big\}.$$

$$\mathscr{SubSym} := \mathscr{Sym} \times \mathscr{SubSym}_3 \times \mathscr{SubSym}_4.$$

$\mathscr{Config} := \mathscr{Sym}_+ \times \mathscr{SubSym}_3 \times \mathscr{SubSym}_4$ denotes the space of generalized configurations.

The **Rayleigh product** between a second-order tensor and a tensor of arbitrary order is denoted by "$*$". For a second-order tensor $\overset{\langle 2\rangle}{\mathbf{F}}$ its action on a tensor basis element $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$ with respect to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined as

$$\overset{\langle 2\rangle}{\mathbf{F}} *(\mathbf{e}_{i_1} \otimes\ \mathbf{e}_{i_2} \otimes \cdots \otimes\ \mathbf{e}_{i_n}) := \big(\overset{\langle 2\rangle}{\mathbf{F}} \cdot \mathbf{e}_{i_1}\big) \otimes \big(\overset{\langle 2\rangle}{\mathbf{F}} \cdot \mathbf{e}_{i_2}\big) \otimes \cdots \otimes \big(\overset{\langle 2\rangle}{\mathbf{F}} \cdot \mathbf{e}_{i_n}\big). \tag{15}$$

If $\overset{\langle 2\rangle}{\mathbf{F}}$ is the differential of a diffeomorphism, the Rayleigh product can be interpreted as the pushforward of a contravariant n-th-order tensor.

By "$\circ$" another product will be denoted, which is very similar to the Rayleigh product. Its action on $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$ with respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined as

$$\overset{\langle 2\rangle}{\mathbf{F}} \circ(\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}) := \left(\overset{\langle 2\rangle}{\mathbf{F}}^{-T} \cdot \mathbf{e}_{i_1}\right) \otimes \big(\overset{\langle 2\rangle}{\mathbf{F}} \cdot \mathbf{e}_{i_2}\big) \otimes \cdots \otimes \big(\overset{\langle 2\rangle}{\mathbf{F}} \cdot \mathbf{e}_{i_n}\big). \tag{16}$$

Let $\overset{\langle 2\rangle}{\mathbf{P}}$ be a smooth tensor field of order two that is invertible everywhere. One interprets $\overset{\langle 2\rangle}{\mathbf{P}}$ as the Jacobian of a change of the reference placement that maps one reference placement B onto another one denoted by $\underline{B}$. Then the following relations hold for all differentiable tensor fields $\overset{\langle 2\rangle}{\mathbf{P}}$ that are invertible at each point.

$$Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right) = -\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \left(\left(Grad\big(\overset{\langle 2\rangle}{\mathbf{P}}\big)\right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)^{[2,3]}, \tag{17}$$

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ\left(-\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} + \overset{[2,3][2,4]}{3sym}\left[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\right]\right). \tag{18}$$

Relation (17) is obtained by applying the product rule to the identity.

$$\overset{\langle 2\rangle}{\mathbf{I}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{P}} \tag{19}$$

$$\Rightarrow \overset{\langle 2\rangle}{\mathbf{0}} = Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{P}}\right) \tag{20}$$

$$\Rightarrow \overset{\langle 2\rangle}{\mathbf{0}} = \left[Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}\right]^{[2,3]} + \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot Grad\big(\overset{\langle 2\rangle}{\mathbf{P}}\big) \tag{21}$$

$$\Leftrightarrow \left[Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}\right]^{[2,3]} = -\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot Grad\big(\overset{\langle 2\rangle}{\mathbf{P}}\big) \tag{22}$$

$$\Leftrightarrow Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right) = -\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \left[Grad\big(\overset{\langle 2\rangle}{\mathbf{P}}\big)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right]^{[2,3]}. \tag{23}$$

The proof of (18) will be given by computing components with respect to an ONB $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$ for $\underline{B}$ and an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for B. In index notation coordinates for $\underline{B}$ are denoted by underlined capital letters while coordinates for B are denoted by lower case letters. Furthermore one will make use of the fact that (17) can be rewritten as

$$Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right) = -\left[\left(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right]^{[2,3]}. \tag{24}$$

Using (24) one can write

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{\mathbf{P}} \cdot Grad\left(Grad\left(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)\right) \tag{25}$$

$$= -\overset{\langle 2\rangle}{\mathbf{P}} \cdot Grad\left(\left[\left(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}\right]^{[2,3]}\right). \tag{26}$$

In index notation the components of $\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}$ can thus be written as

$$\left(\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}\right)_{abcd} = -\overset{\langle 2\rangle}{P}{}_a{}^{\underline{A}} \left[\overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{A}}{}^{\underline{BC}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{C}}{}^{\mathrm{c}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{B}}{}^{\mathrm{b}}\right]^{d}, \tag{27}$$

$$= -\overset{\langle 2\rangle}{P}{}_a{}^{\underline{A}} \left\{\left[\overset{\langle 4\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{A}}{}^{\underline{BCD}} - \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{A}}{}^{\underline{DE}} \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{E}}{}^{\underline{CB}}\right] \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{D}}{}^{\mathrm{d}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{B}}{}^{\mathrm{b}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{C}}{}^{\mathrm{c}} \right. \tag{28}$$

$$- \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{A}}{}^{\underline{BC}} \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{B}}{}^{\underline{DE}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{D}}{}^{\mathrm{b}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{E}}{}^{\mathrm{d}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{C}}{}^{\mathrm{c}}$$

$$\left. - \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{A}}{}^{\underline{BC}} \overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}}{}_{\underline{B}}{}^{\underline{DE}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{C}}{}^{\mathrm{c}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{D}}{}^{\mathrm{b}} \overset{\langle 2\rangle}{P}{}^{-1}{}_{\underline{E}}{}^{\mathrm{d}} \right\},$$

where the components of $\overset{\langle 2\rangle}{\mathbf{P}}^{-1}$ are denoted by $\overset{\langle 2\rangle}{P}{}^{-1}{}_{i}{}^{j}$. This yields in direct notation

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \left(-\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} + \overset{[2,3][2,4]}{3sym}\left[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\right]\right). \tag{29}$$

3 Stress Power and Material Variables

The region occupied by a body in the reference placement is denoted by B_0 and its boundary by ∂B_0. As usual the motion of a body is denoted by χ and the deformation gradient by

$$\mathbf{F} := Grad(\chi). \tag{30}$$

In this context $Grad()$ denotes the gradient with respect to material coordinates while $grad()$ denotes the spatial gradient. The right Cauchy-Green tensor is then defined as $\mathbf{C} := \mathbf{F}^T \cdot \mathbf{F}$.

Following [7], the stress power of a body at time t within a third-order theory is

$$P = \int_{B_t} p\, dv, \tag{31}$$

with the stress power density

$$p := \overset{\langle 2\rangle}{\mathbf{T}} : grad(\mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} : grad^{\mathrm{II}}(\mathbf{v}) + \overset{\langle 4\rangle}{\mathbf{T}} : grad^{\mathrm{III}}(\mathbf{v}). \tag{32}$$

Since the first three gradients of the velocity occur in (32), one refers the presented model as a third-order material. The tensors $\overset{\langle 2\rangle}{\mathbf{T}}$, $\overset{\langle 3\rangle}{\mathbf{T}}$, $\overset{\langle 4\rangle}{\mathbf{T}}$ are generalized spatial stress tensors. The first one is the Cauchy stress tensor from the classical theory. Following the approach in [7], based on the principle of virtual power, one deduces that for a third-order material Cauchy's equations have the form

$$\big(div(\overset{\langle 2\rangle}{\mathbf{T}}) - div^{\mathrm{II}}(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{\mathrm{III}}(\overset{\langle 4\rangle}{\mathbf{T}})\big) + \rho\mathbf{b} = \rho\ddot{\mathbf{a}}, \tag{33}$$

$$\overset{\langle 2\rangle}{\mathbf{T}} = \overset{\langle 2\rangle}{\mathbf{T}}{}^{T}, \tag{34}$$

with the specific body force $\mathbf{b}$, the acceleration $\mathbf{a}$, and the mass density ρ. The boundary conditions are shown in [24, 28] or [6]. The following equations are the foundation of the present work. Equation (35) is well-known and can easily be verified. Equation (36) is more difficult. It has been shown in detail in [4] and [6]. The derivation of Eq. (37) is rather lengthy and is therefore omitted here. It is given in detail in [6].

$$grad(\mathbf{v}) = \mathbf{F}^{-T} * \left(\frac{1}{2}\mathbf{C}^{\cdot}\right), \tag{35}$$

$$grad^{\mathrm{II}}(\mathbf{v}) = \mathbf{F}^{-T} \circ \overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}}, \tag{36}$$

$$grad^{\mathrm{III}}(\mathbf{v}) = \mathbf{F}^{-T} \circ \left(\overset{\langle 4\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} - \overset{[2,3][2,4]}{3sym}\left[\overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right]\right), \tag{37}$$

where the dot denotes the derivative with respect to time. The tensor triple $\{\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\} \in \mathscr{Config}$ describes the configuration of a third-order material point and is considered to be an element of the generalized configuration space for a third-order material.

It is invariant under both changes of observer (Euclidean transformations) and superimposed rigid body modifications of the motion. With this we can express the stress power in different forms.

$$P = \int_{B_t} \frac{1}{\rho}\big(\overset{\langle 2\rangle}{\mathbf{T}} : grad(\mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} \vdots grad^{\mathrm{II}}(\mathbf{v}) + \overset{\langle 4\rangle}{\mathbf{T}} :: grad^{\mathrm{III}}(\mathbf{v})\big) dm \tag{38}$$

$$= \int_{B_t} \frac{1}{\rho}\Big(\frac{1}{2}\big[\mathbf{F}^{-1} * \overset{\langle 2\rangle}{\mathbf{T}}\big] : \mathbf{C}^{\cdot} + \big[\mathbf{F}^{-1} \circ \overset{\langle 3\rangle}{\mathbf{T}}\big] \vdots \overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} \tag{39}$$

$$+ \big[\mathbf{F}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{T}}\big] :: \big[\mathbf{F}^{T} \circ grad^{\mathrm{III}}(\mathbf{v})\big]\Big) dm$$

$$= \int_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\cdot} + \overset{\langle 3\rangle}{\mathbf{S}} \vdots \overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \left\{\overset{\langle 4\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} - \overset{[2,3][2,4]}{3sym}\left[\overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right]\right\}\right) dm \tag{40}$$

$$= \int_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\cdot} + \left(\overset{\langle 3\rangle}{\mathbf{S}} - 3\,\overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}{}^{[1,3]}\right) \vdots \overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}}\right) dm \tag{41}$$

$$= \int_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\cdot} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} \vdots \overset{\langle 3\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle^{\cdot}}{\mathbf{K}}_{\mathbf{F}}\right) dm, \tag{42}$$

with

stress measures **strain measures**

$$\overset{\langle 2\rangle}{\mathbf{S}} := \mathbf{F}^{-1} * \big(J_{\mathbf{F}}\, \overset{\langle 2\rangle}{\mathbf{T}}\big), \qquad \mathbf{C} := \mathbf{F}^{T} \cdot \mathbf{F}, \tag{43}$$

$$\overset{\langle 3\rangle}{\mathbf{S}} := \mathbf{F}^{-1} \circ \big(J_{\mathbf{F}}\, \overset{\langle 3\rangle}{\mathbf{T}}\big), \qquad \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} := \mathbf{F}^{-1} \cdot Grad(\mathbf{F}), \tag{44}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} := \mathbf{F}^{-1} \circ \big(J_{\mathbf{F}}\, \overset{\langle 4\rangle}{\mathbf{T}}\big), \qquad \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} := \mathbf{F}^{-1} \cdot Grad^{\mathrm{II}}(\mathbf{F}), \tag{45}$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} := \left(\overset{\langle 3\rangle}{\mathbf{S}} - 3\, \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}\right). \tag{46}$$

While $\overset{\langle 3\rangle}{\mathbf{S}}$ is a stress tensor in the second-order framework, in the third-order theory it plays only the role of a partial stress. $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}$ is the material third-order stress tensor. An interesting aspect of the third-order theory in the present work is the fact that both, $\overset{\langle 3\rangle}{\mathbf{S}}$ and $\overset{\langle 4\rangle}{\mathbf{S}}$, contribute to $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}$. This is due to the fact that $grad^{\mathrm{III}}(\mathbf{v})$ disperses into $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\boldsymbol{\cdot}}$ and $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{\boldsymbol{\cdot}}$ when pulled back to the reference placement as can be seen in (37). It has been proven in [6] that this dispersion cannot be avoided by choosing other variables.

From (38) it is clear that $\overset{\langle 2\rangle}{\mathbf{T}}$ and $\overset{\langle 3\rangle}{\mathbf{T}}$ can be submitted to the same symmetries as $grad^{\mathrm{II}}(\mathbf{v})$ and $grad^{\mathrm{III}}(\mathbf{v})$, respectively. In addition to that, $\overset{\langle 2\rangle}{\mathbf{T}}$ is symmetric due to the same arguments as in classical continuum mechanics namely, the balance of moment of momentum, which does not impose any restrictions on $\overset{\langle 3\rangle}{\mathbf{T}}$ and $\overset{\langle 4\rangle}{\mathbf{T}}$. Each of the three material stress tensors can be assumed to have the same symmetries as the corresponding Cauchy like stress tensor. In the case of $\overset{\langle 2\rangle}{\mathbf{S}}$ and $\overset{\langle 2\rangle}{\mathbf{T}}$ the pullback preserves the well known symmetry of $\overset{\langle 2\rangle}{\mathbf{T}}$. The same argument leads to $\overset{\langle 4\rangle}{\mathbf{S}}$ having the same symmetries as $\overset{\langle 4\rangle}{\mathbf{T}}$. $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}$ can be assumed to have the same symmetries as $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ because only the symmetric parts enter in (42).

4 Third-order Elasticity

Definition 1 (Third-order elastic material) A material is called a third-order elastic material if the stress tensors are functions of the motion χ, of $Grad(\chi)$, $Grad^{\mathrm{II}}(\chi)$ and $Grad^{\mathrm{III}}(\chi)$, thus

$$\overset{\langle 2\rangle}{\mathbf{T}} = \overset{\langle 2\rangle}{f}\big(\chi, Grad(\chi), Grad^{\mathrm{II}}(\chi), Grad^{\mathrm{III}}(\chi)\big), \tag{47}$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{T}}} = \overset{\langle 3\rangle}{\widehat{f}}\big(\chi, Grad(\chi), Grad^{\mathrm{II}}(\chi), Grad^{\mathrm{III}}(\chi)\big), \tag{48}$$

$$\overset{\langle 4\rangle}{\mathbf{T}} = \overset{\langle 4\rangle}{f}\big(\chi, Grad(\chi), Grad^{\mathrm{II}}(\chi), Grad^{\mathrm{III}}(\chi)\big), \tag{49}$$

where all variables are evaluated at the same material point at the same instant of time. These constitutive equations can be reduced by taking into account the principle of objectivity or invariance under rigid body motions (see [11, 38]). This yields reduced forms, which are chosen in the present case as

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big), \qquad \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big), \qquad \overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big). \tag{50}$$

All involved strain and stress variables in these constitutive laws are invariant under both changes of observer and superimposed rigid body motions. In the context of the present work one does not introduce a constitutive equation for $\overset{\langle 3\rangle}{\mathbf{S}}$ because it is not a stress measure. It is a partial stress that also helps making a comparison to the second-order theory from [5] and facilitates the understanding of transformation rules. The reduced forms that have been chosen in the present form are not the only option. One could as well choose $\mathbf{E} := \frac{1}{2}(\mathbf{C} - \overset{\langle 2\rangle}{\mathbf{I}})$ instead of $\mathbf{C}$, and $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ instead of $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$. There are, as it is already in the classical theory the case, infinitely many choices, which are mathematically altogether equivalent. Our choice here is only motivated by practical considerations, namely to render the resulting equations as simple as possible.

Definition 2 (Third-order hyperelasticity) A material is called hyperelastic if there exists a specific elastic energy

$$w : \mathscr{Config} \mapsto \mathbb{R},$$

such that the time-derivative of this energy equals the stress power

$$p = w\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big)^{\cdot}. \tag{51}$$

The chain rule gives then

$$p = \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \mathbf{C}} : \mathbf{C}^{\cdot} + \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\cdot} + \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{\cdot}. \tag{52}$$

A comparison with the components in (42) then reveals for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \in \mathscr{Config}$ the potentials

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = 2\rho_0 \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \mathbf{C}}, \tag{53}$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \rho_0 \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}, \tag{54}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \rho_0 \frac{\partial w(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})}{\partial \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}}. \tag{55}$$

In the rest of the paper we will not anymore distinguish between elasticity and hyperelasticity.

5 Changes of the Reference Placement

While the spatial stress tensors $\overset{\langle 2\rangle}{\mathbf{T}}, \overset{\langle 3\rangle}{\mathbf{T}}, \overset{\langle 4\rangle}{\mathbf{T}}$ do not depend on the reference placement, material variables do depend on the choice of the reference placement. Therefore it is important to understand the transformation behavior of the stress and strain measure under changes of the reference placement.

Theorem 1 (Transformation of stress and strain measures under changes of the reference placement) *Let κ and $\underline{\kappa}$ be two reference placements. The composition $\kappa(\underline{\kappa}^{-1})$ is referred to as the* ***change of the reference placement****. Its gradient is denoted by $\boldsymbol{A} := \underline{Grad}(\kappa(\underline{\kappa}^{-1}))$, the pullback of its gradient is denoted by $\overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}} := \boldsymbol{A}^{-1} \cdot \underline{Grad}(\boldsymbol{A})$, the pullback of its second gradient is denoted by $\overset{\langle 4\rangle}{\boldsymbol{K}_{\boldsymbol{A}}} := \boldsymbol{A}^{-1} \cdot \underline{Grad}^{\mathrm{II}}(\boldsymbol{A})$, and one defines the determinant $J_{\boldsymbol{A}} := det(\boldsymbol{A})$. Furthermore one defines*

$$\widehat{\gamma} : \mathscr{SubSym}_3 \times \mathscr{SubSym}_4 \times \mathscr{Inv} \times \mathscr{SubSym}_3 \times \mathscr{SubSym}_3 \mapsto \mathscr{SubSym}_3, \tag{56}$$

$$\begin{aligned} &\widehat{\gamma}\left(\overset{\langle 3\rangle}{\widehat{\boldsymbol{S}}}, \overset{\langle 4\rangle}{\boldsymbol{S}}, \boldsymbol{A}, \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}, \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}}\right) \\ &= \boldsymbol{A}^{-1} \circ \left[J_{\boldsymbol{A}} \left(\overset{\langle 3\rangle}{\widehat{\boldsymbol{S}}} + 3\, \overset{\langle 4\rangle}{\boldsymbol{S}} : \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}}^{[1,3]} \right) \right] - \left(\boldsymbol{A}^{-1} \circ \left(J_{\boldsymbol{A}} \overset{\langle 4\rangle}{\boldsymbol{S}}\right)\right) : \left[\boldsymbol{A}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}} + \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}\right]^{[1,3]}. \end{aligned} \tag{57}$$

Then the stress and strain variables transform under changes of the reference placement as

$$\underline{\boldsymbol{C}} = \boldsymbol{A}^T * \boldsymbol{C}, \qquad \overset{\langle 2\rangle}{\underline{\boldsymbol{S}}} = \boldsymbol{A}^{-1} * J_A \overset{\langle 2\rangle}{\boldsymbol{S}}, \tag{58}$$

$$\overset{\langle 3\rangle}{\boldsymbol{K}_{\underline{\boldsymbol{F}}}} = \boldsymbol{A}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}} + \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}, \qquad \overset{\langle 3\rangle}{\widehat{\underline{\boldsymbol{S}}}} = \widehat{\gamma}\left(\overset{\langle 3\rangle}{\widehat{\boldsymbol{S}}}, \overset{\langle 4\rangle}{\boldsymbol{S}}, \boldsymbol{A}, \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}, \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}}\right), \tag{59}$$

$$\begin{aligned} \overset{\langle 4\rangle}{\boldsymbol{K}_{\underline{\boldsymbol{F}}}} &= \boldsymbol{A}^T \circ \overset{\langle 4\rangle}{\boldsymbol{K}_{\boldsymbol{F}}} + \overset{\langle 4\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}, \qquad \overset{\langle 4\rangle}{\underline{\boldsymbol{S}}} = \boldsymbol{A}^{-1} \circ J_A \overset{\langle 4\rangle}{\boldsymbol{S}}\,. \\ &+ \overset{[2,4][2,3]}{3sym} \left[\left(\boldsymbol{A}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{F}}}\right) \cdot \overset{\langle 3\rangle}{\boldsymbol{K}_{\boldsymbol{A}}}\right], \end{aligned} \tag{60}$$

Proof The transformation behavior of $\overset{\langle 2\rangle}{\mathbf{S}}$, $\mathbf{C}$ and $\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}$ is derived in [5] and [6]. Therefore the proof starts with the transformation of the fourth-order stress tensor.

$$\overset{\langle 4\rangle}{\underline{\mathbf{S}}} = \underline{\mathbf{F}}^{-1} \circ \left(J_{\underline{\mathbf{F}}} \overset{\langle 4\rangle}{\mathbf{T}}\right) = \left(\mathbf{A}^{-1} \cdot \mathbf{F}^{-1}\right) \circ \left(J_{\mathbf{F}} J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{T}}\right) = \mathbf{A}^{-1} \circ \left(J_A \overset{\langle 4\rangle}{\mathbf{S}}\right). \tag{61}$$

Next the transformation behavior of $\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}$ is derived.

$$\overset{\langle 4\rangle}{\mathbf{K}_{\underline{\mathbf{F}}}} = \left(\underline{\mathbf{F}}^{-1} \cdot \underline{Grad}^{\mathrm{II}}(\underline{\mathbf{F}})\right) \tag{62}$$

$$= \left(\mathbf{A}^{-1} \cdot \mathbf{F}^{-1} \cdot Grad\big(Grad(\mathbf{F} \cdot \mathbf{A}) \cdot \mathbf{A}\big) \cdot \mathbf{A}\right). \tag{63}$$

Thus, with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ one can write the components of $\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$ as

$$
\begin{aligned}
& A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[\left(F_b{}^{C} A_{C}{}^{\underline{B}} \right)^{,D} A_{D}{}^{\underline{C}} \right]^{,E} A_{E}{}^{\underline{D}} \\
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[F_b{}^{C,D} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} + F_b{}^{C} A_{C}{}^{\underline{B},D} A_{D}{}^{\underline{C}} \right]^{,E} A_{E}{}^{\underline{D}} && (64) \\
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[F_b{}^{C,DE} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} + F_b{}^{C,D} A_{C}{}^{\underline{B},E} A_{D}{}^{\underline{C}} + F_b{}^{C,D} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C},E} \right. \\
&\quad \left. + F_b{}^{C,E} A_{C}{}^{\underline{B},D} A_{D}{}^{\underline{C}} + F_b{}^{C} A_{C}{}^{\underline{B},DE} A_{D}{}^{\underline{C}} + F_b{}^{C} A_{C}{}^{\underline{B},D} A_{D}{}^{\underline{C},E} \right] A_{E}{}^{\underline{D}} && (65) \\
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[F_b{}^{C,DE} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} \right. \\
&\quad + F_b{}^{C,D} A_{C}{}^{\underline{\mathrm{B},\mathrm{F}}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} \\
&\quad + F_b{}^{C,D} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{\mathrm{C},\mathrm{F}}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} A_{E}{}^{\underline{D}} \\
&\quad + F_b{}^{C,E} A_{C}{}^{\underline{\mathrm{B},\mathrm{F}}} A^{-1}{}_{\underline{F}}{}^{\mathrm{D}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} + F_b{}^{C} A_{C}{}^{\underline{\mathrm{B},\mathrm{DE}}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} \\
&\quad \left. + F_b{}^{C} A_{C}{}^{\underline{\mathrm{B},\mathrm{G}}} A^{-1}{}_{\underline{G}}{}^{\mathrm{D}} A_{D}{}^{\underline{\mathrm{C},\mathrm{F}}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} A_{E}{}^{\underline{D}} \right] && (66) \\
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[F_b{}^{C,DE} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} + F_b{}^{C,D} A_{C}{}^{\underline{\mathrm{B},\mathrm{D}}} A_{D}{}^{\underline{C}} \right. \\
&\quad + F_b{}^{D,C} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{\mathrm{C},\mathrm{D}}} + F_b{}^{C,E} A_{C}{}^{\underline{B,C}} A_{E}{}^{\underline{D}} \\
&\quad \left. + F_b{}^{C} A_{C}{}^{\underline{\mathrm{B},\mathrm{DE}}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} + F_b{}^{C} A_{C}{}^{\underline{\mathrm{B},\mathrm{G}}} A^{-1}{}_{\underline{G}}{}^{\mathrm{D}} A_{D}{}^{\underline{\mathrm{C},\mathrm{D}}} \right], && (67)
\end{aligned}
$$

where a comma as a sub- or superscript denotes the derivative with respect to the coordinates represented by the index to follow. In the next step one makes use of the transformation

$$
A_{C}{}^{\underline{\mathrm{B},\mathrm{DE}}} = A_{C}{}^{\underline{B,GF}} A^{-1}{}_{\underline{G}}{}^{\mathrm{D}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} - A_{C}{}^{\underline{\mathrm{B},\mathrm{G}}} A^{-1}{}_{\underline{G}}{}^{\mathrm{X}} A_{X}{}^{\underline{F,Y}} A^{-1}{}_{\underline{Y}}{}^{\mathrm{D}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}}, \tag{68}
$$

which can be easily verified by applying the same approach as used for the proof of (17). One therefore continues equation (67)

$$
\begin{aligned}
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} \left[F_b{}^{C,DE} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} + F_b{}^{C,D} A_{C}{}^{\underline{\mathrm{B},\mathrm{D}}} A_{D}{}^{\underline{C}} \right. \\
&\quad + F_b{}^{D,C} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{\mathrm{C},\mathrm{D}}} + F_b{}^{C,E} A_{C}{}^{\underline{B,C}} A_{E}{}^{\underline{D}} \\
&\quad + F_b{}^{C} \left[A_{C}{}^{\underline{B,GF}} A^{-1}{}_{\underline{G}}{}^{\mathrm{D}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} - A_{C}{}^{\underline{\mathrm{B},\mathrm{G}}} A^{-1}{}_{\underline{G}}{}^{\mathrm{X}} A_{X}{}^{\underline{F,Y}} \right] A^{-1}{}_{\underline{Y}}{}^{\mathrm{D}} A^{-1}{}_{\underline{F}}{}^{\mathrm{E}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} \\
&\quad \left. + F_b{}^{C} A_{C}{}^{\underline{\mathrm{B},\mathrm{G}}} A^{-1}{}_{\underline{G}}{}^{\mathrm{D}} A_{D}{}^{\underline{\mathrm{C},\mathrm{D}}} \right] && (69) \\
&= A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} F_b{}^{C,DE} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{C}} A_{E}{}^{\underline{D}} + A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} F_b{}^{C,D} A_{C}{}^{\underline{B,D}} A_{D}{}^{\underline{C}} \\
&\quad + A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} F_b{}^{D,C} A_{C}{}^{\underline{B}} A_{D}{}^{\underline{\mathrm{C},\mathrm{D}}} + A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} F^{-1}{}_{A}{}^{\mathrm{b}} F_b{}^{C,E} A_{C}{}^{\underline{B,C}} A_{E}{}^{\underline{D}} \\
&\quad + A^{-1}{}_{\underline{A}}{}^{\mathrm{A}} A_{A}{}^{\underline{B,CD}}. && (70)
\end{aligned}
$$

In direct notation this gives

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} + \overset{[2,4][2,3]}{3sym} \left[\mathbf{A}^{-1} \cdot \left(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \mathbf{A}\right)^{[2,3]} \cdot \underline{Grad}(\mathbf{A})\right] \tag{71}$$

$$= \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} + \overset{[2,4][2,3]}{3sym} \left[\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]. \tag{72}$$

The transformation of the third-order stress tensor is proved in a similar manner.

$$\overset{\langle 3\rangle}{\widehat{\underline{\mathbf{S}}}} = \overset{\langle 3\rangle}{\underline{\mathbf{S}}} - \overset{\langle 4\rangle}{\underline{\mathbf{S}}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}^{[1,3]} \tag{73}$$

$$= \mathbf{A}^{-1} \circ \left(J_{\mathbf{A}} \overset{\langle 3\rangle}{\mathbf{S}}\right) - \left(\mathbf{A}^{-1} \circ \left(J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}\right)\right) : \left[\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]^{[1,3]} \tag{74}$$

$$= \mathbf{A}^{-1} \circ \left[J_{\mathbf{A}} \left(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} + 3 \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}\right)\right] - \left(\mathbf{A}^{-1} \circ \left(J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}\right)\right) : \left[\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]^{[1,3]}. \tag{75}$$

□

Equation (59) as well as (70) and (72) show that in comparison the direct notation is more compact. In direct notation (59) takes the form

$$\overset{\langle 3\rangle}{\widehat{\underline{\mathbf{S}}}} = \widehat{\gamma}\left(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)$$

$$= \mathbf{A}^{-1} \circ \left[J_{\mathbf{A}} \left(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} + 3 \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}\right)\right] - \left(\mathbf{A}^{-1} \circ \left(J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}\right)\right) : \left[\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]^{[1,3]}, \tag{76}$$

while in Ricci notation it has the form

$$\overset{\langle 3\rangle}{\widehat{\underline{S}}}{}^{\underline{A}}{}_{\underline{BC}} = A^{T\underline{A}}{}_X \left[J_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 3\rangle}{\widehat{S}}{}^X{}_{YZ} + 3 \overset{\langle 4\rangle}{S}{}^X{}_{YMN} \overset{\langle 3\rangle}{K}_{\mathbf{F}_Z}{}^{NM} \right] A^{-T^Y}{}_{\underline{B}} A^{-T^Z}{}_{\underline{C}}$$

$$- A^{T\underline{A}}{}_M J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}^M{}_{NOP} A^{-T^N}{}_{\underline{B}} A^{-T^O}{}_{\underline{X}} A^{-T^P}{}_{\underline{Y}} \left(A^{-1}{}_{\underline{C}}{}^D \overset{\langle 3\rangle}{K}_{\mathbf{F}_D}{}^{EF} A_E{}^{\underline{Y}} A_F{}^{\underline{X}} + \overset{\langle 3\rangle}{K}_{\mathbf{A}_{\underline{C}}}{}^{\underline{YX}} \right). \tag{77}$$

One also needs to state how the elastic energy changes under changes of the reference placement. The value of the energy should be independent of reference placement up to a constant. This leads to the following theorem.

Theorem 2 (Transformation of elastic energies and elastic laws under a change of reference placement) *For two reference placements κ and $\underline{\kappa}$ there exists a constant $\underline{w}_0 \in \mathbb{R}$ such that the elastic energies transform as*

$$w\left(\overset{\langle 2\rangle}{\boldsymbol{C}}, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}}\right) = \underline{w}\left(\boldsymbol{A}^T * \boldsymbol{C}, \boldsymbol{A}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}} + \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{A}}, \boldsymbol{A}^T \circ \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}} + \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{A}}\right.$$

$$\left. + \overset{[2,4][2,3]}{3sym} \left[\left(\boldsymbol{A}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}}\right) \cdot \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{A}}\right]\right) + \underline{w}_0. \tag{78}$$

Theorem 2 is a direct consequence of Theorem 1 and does therefore not require a proof. The elastic laws are obtained from the elastic energy by taking its derivative with respect to the strain variables. Therefore Theorem 2 implies for the elastic laws how they transform under a change of the reference placement, which is prescribed by $\overset{\langle 2\rangle}{\mathbf{A}}$, $\overset{\langle 3\rangle}{\mathbf{A}}$ and $\overset{\langle 4\rangle}{\mathbf{A}}$. One obtains equations that allow to calculate the elastic laws in a new reference placement (where all quantities are underlined) from the elastic laws in the old reference placement.

$$\begin{aligned}\overset{\langle 2\rangle}{f}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) = \mathbf{A} * J_{\mathbf{A}}^{-1} \underline{\overset{\langle 2\rangle}{f}}\big(&\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \\ &+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]\big),\end{aligned} \tag{79}$$

$$\begin{aligned}\widehat{\overset{\langle 3\rangle}{f}}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) = \widehat{\gamma}\Bigg(\underline{\widehat{\overset{\langle 3\rangle}{f}}}\big(&\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \\ &+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]\big), \underline{\overset{\langle 4\rangle}{f}}\big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \\ &+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]\big), \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Bigg),\end{aligned} \tag{80}$$

$$\begin{aligned}\overset{\langle 4\rangle}{f}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) = \mathbf{A} \circ J_{\mathbf{A}}^{-1} \underline{\overset{\langle 4\rangle}{\mathbf{f}}}\Bigg(&\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \\ &+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]\Bigg),\end{aligned} \tag{81}$$

with $\widehat{\gamma}$ as defined in (57) allowing for the transformations from (58)–(60).

6 Elastic Isomorphy

The concept of elastic isomorphy has been introduced in [38] in Sect. 27. It plays an important role in elasticity and elastoplasticity (see [2, 3] and [5]). It defines what it means that two points have the *same elastic behavior*. It has been generalized for second-order materials in [5]. In the same spirit it is generalized for third-order materials in the following Definition 3 and makes use of the transformation rules for the strain measures in Theorem 1. For the elastic laws it implies that two material points are elastically isomorphic if one can find reference placements for the two material points such that their elastic laws for both placements are identical.

Definition 3 (Elastic isomorphy) Two elastic material points X and Y are called elastically isomorphic if one can find reference placements κ_X for X and κ_Y for Y such that

1. in κ_X and κ_Y the mass densities are identical

$$\rho_{0X} = \rho_{0Y}, \tag{82}$$

2. with respect to κ_X and κ_Y the elastic energies w_X and w_Y are identical up to a constant w_0

$$w_x(\kappa_X, \cdot) = w_x(\kappa_X, \cdot) + w_0. \tag{83}$$

In practice, this definition is not easy to handle. In the following theorem, the change of the reference placement is substituted by some tensorial variables and, thus, facilitated.

Theorem 3 (Criterion for elastic isomorphy) *Let X and Y be two elastic material points with arbitrary reference placements $\underline{\kappa}_X$ and $\underline{\kappa}_Y$ and elastic energies w_X and w_Y. Then these two points are elastically isomorphic if and only if there exist tensors*

$$\overset{\langle 2\rangle}{\boldsymbol{P}} \in \mathscr{Inv}, \qquad \overset{\langle 3\rangle}{\boldsymbol{P}} \in \mathscr{SubSym}_3, \qquad \overset{\langle 4\rangle}{\boldsymbol{P}} \in \mathscr{SubSym}_4, \tag{84}$$

such that the following equations hold for all $(\underline{\boldsymbol{C}}_X, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\underline{\boldsymbol{F}}_X}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\underline{\boldsymbol{F}}_X}) \in \mathscr{Config}$.

$$\rho_{0Y} = det\big(\overset{\langle 2\rangle}{\boldsymbol{P}}\big)\rho_{0X} \quad \textit{and} \tag{85}$$

$$\begin{aligned}\underline{w}_X\big(\underline{\boldsymbol{C}}_X, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\underline{\boldsymbol{F}}_X}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\underline{\boldsymbol{F}}_X}\big) = \underline{w}_0 + \underline{w}_Y\Bigg(&\overset{\langle 2\rangle}{\boldsymbol{P}}^T * \underline{\boldsymbol{C}}_X, \overset{\langle 2\rangle}{\boldsymbol{P}}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\underline{\boldsymbol{F}}_X} + \overset{\langle 3\rangle}{\boldsymbol{P}}, \boldsymbol{P}^T \circ \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}} + \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{P}} \\ &+ \overset{[2,4][2,3]}{3sym}\big[\big(\boldsymbol{P}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}}\big) \cdot \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{P}}\big]\Bigg).\end{aligned} \tag{86}$$

Proof The proof follows directly from the transformation laws in Theorem 1. The isomorphy criteria for the elastic laws are then obtained directly by taking partial derivatives of the elastic energies with respect to the corresponding strain variables in each reference placement. □

The tensor $\overset{\langle 2\rangle}{\mathbf{A}}$ in Theorem 3 can be interpreted as the gradient of a change of the reference placement, the tensor $\overset{\langle 3\rangle}{\mathbf{A}}$ as $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}$ and the tensor $\overset{\langle 4\rangle}{\mathbf{A}}$ as $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}$. As long as two isolated material points are considered, these tensors can be regarded as mutually independent, which means that they do not have to fulfill any integrability condition.

Let

$$\Big\{\overset{\langle 2\rangle}{\underline{f}}_X, \overset{\langle 3\rangle}{\widehat{\underline{f}}}_X, \overset{\langle 4\rangle}{\underline{f}}_X\Big\} \quad \text{and} \quad \Big\{\overset{\langle 2\rangle}{\underline{f}}_Y, \overset{\langle 3\rangle}{\widehat{\underline{f}}}_Y, \overset{\langle 4\rangle}{\underline{f}}_Y\Big\} \tag{87}$$

be the respective sets of elastic laws corresponding to w_X and w_Y from Theorem 3. Just as in the case of (79)–(81) this theorem implies the equations, which the elastic laws of two points X and Y have to fulfill for these points to be elastically isomorphic.

$$\begin{aligned}\overset{\langle 2\rangle}{\underline{f}}_X\big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\big) = \overset{\langle 2\rangle}{\mathbf{P}} * det^{-1}\big(\overset{\langle 2\rangle}{\mathbf{P}}\big)\, \overset{\langle 2\rangle}{f}_Y\Bigg(&\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\underline{\mathbf{F}}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \mathbf{P}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}} \\ &+ \overset{[2,4][2,3]}{3sym}\big[\big(\mathbf{P}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\big) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\big]\Bigg),\end{aligned} \tag{88}$$

$$\begin{aligned}\overset{\langle 3\rangle}{\widehat{\underline{f}}}_X\big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\big) = \widehat{\gamma}\Bigg(\overset{\langle 3\rangle}{\widehat{\underline{f}}}_Y\Bigg(&\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \mathbf{P}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}} \\ &+ \overset{[2,4][2,3]}{3sym}\big[\big(\mathbf{P}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\big) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\big]\Bigg), \\ \overset{\langle 4\rangle}{\underline{f}}_Y\Bigg(&\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \mathbf{P}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}}\end{aligned}$$

$$+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{P}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\right)\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\right]\Bigg),$$

$$\overset{\langle 2\rangle}{\mathbf{P}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 2\rangle}{\mathbf{P}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}\Bigg), \tag{89}$$

$$\overset{\langle 4\rangle}{\underline{f}}_X\left(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\right) = \overset{\langle 2\rangle}{\mathbf{P}} \circ det^{-1}\left(\overset{\langle 2\rangle}{\mathbf{P}}\right) \overset{\langle 4\rangle}{f}_Y \Bigg(\overset{\langle 2\rangle}{\mathbf{P}}^{T} * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \mathbf{P}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}}$$

$$+ \overset{[2,4][2,3]}{3sym}\left[\left(\mathbf{P}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\right)\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\right]\Bigg), \tag{90}$$

with $\widehat{\gamma}$ as defined in (57).

7 Material Symmetry

A priori knowledge on the symmetry properties of a specific material is highly useful to further specify the constitutive equations. For this purpose we will next introduce the concept of a symmetry group of a gradient material. Such introductions have already been made for other non-classical cases, like in [14, 15, 17–19, 29–31] and others.

Applying the concept of elastic isomorphy to only one point, i.e., by setting $X \equiv Y$ in Definition 3, defines elastic symmetry. In this case one can drop the notation for the reference point. As explained in Theorem 1, a change of reference placement defines three tensors $\mathbf{A} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \in \mathscr{SubSym}_3$ and $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \in \mathscr{SubSym}_4$. So in this case the isomorphism $\mathbf{A}$ becomes an automorphism since it maps the tangent space at a point onto itself. One can set

$$\overset{\langle 2\rangle}{\mathbf{A}} = \mathbf{A}, \qquad \overset{\langle 3\rangle}{\mathbf{A}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \qquad \overset{\langle 4\rangle}{\mathbf{A}} = \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\,. \tag{91}$$

The tensors $\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}$ can then be regarded as independent from each other because they are only considered at one point. The behavior around this point (necessary for derivatives) is not of interest here.

In the following definition of symmetry the idea is to express the fact that certain changes of the reference placement at a point do not change the elastic law at this point.

Definition 4 (Symmetry Transformation) For a third-order elastic material a symmetry transformation is a triple

$$\overset{\langle 2\rangle}{\mathbf{A}} \in \mathscr{Unim}, \qquad \overset{\langle 3\rangle}{\mathbf{A}} \in \mathscr{SubSym}_3, \qquad \overset{\langle 4\rangle}{\mathbf{A}} \in \mathscr{SubSym}_4, \tag{92}$$

such that

$$\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}} = 0, \qquad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = \overset{\langle 2\rangle}{\mathbf{I}} : \left[\overset{\langle 3\rangle}{\mathbf{A}} \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right]^{[2,4]} \tag{93}$$

and

$$w\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = w\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \mathbf{P}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}} + \overset{[2,4][2,3]}{3sym}\Big[\big(\mathbf{P}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\Big]\Bigg) \tag{94}$$

for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \in \mathscr{Config}$ with $\widehat{\gamma}$ as defined in (57).

The set of all symmetry transformations forms the symmetry group of the material.

Definition 4 could also be set up without condition (93) as in [5]. Condition (93) comes from the following reasoning. If one assumes that

$$\overset{\langle 2\rangle}{\mathbf{A}} = \underline{Grad}\big(\kappa\big(\underline{\kappa}^{-1}\big)\big) \tag{95}$$

is the Jacobian of a change of the reference placement with

$$det\big(\overset{\langle 2\rangle}{\mathbf{A}}\big) = 1 \tag{96}$$

everywhere in the body, then

$$Grad\big(det\big(\overset{\langle 2\rangle}{\mathbf{A}}\big)\big) = 0 \tag{97}$$

must hold everywhere. Since *det* is a differentiable matrix function one can write

$$0 = Grad\big(det\big(\overset{\langle 2\rangle}{\mathbf{A}}\big)\big) = \frac{d(det)}{d\overset{\langle 2\rangle}{\mathbf{A}}} : Grad\big(\overset{\langle 2\rangle}{\mathbf{A}}\big). \tag{98}$$

Applying Jacobi's formula to the term $\frac{d(det)}{d\overset{\langle 2\rangle}{\mathbf{A}}}$ with the adjungate of $\overset{\langle 2\rangle}{\mathbf{A}}$ being equal to $J_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 2\rangle}{\mathbf{A}}^{-1}$ yields

$$0 = \overset{\langle 2\rangle}{\mathbf{I}} : \Big[J_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 2\rangle}{\mathbf{A}}^{-T} \cdot Grad\big(\overset{\langle 2\rangle}{\mathbf{A}}\big)\Big] = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}}. \tag{99}$$

Furthermore, since (97) holds everywhere in the body one obtains

$$0 = Grad\big(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\big) = \overset{\langle 2\rangle}{\mathbf{I}} : Grad\big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\big) = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} \tag{100}$$

$$= \overset{\langle 2\rangle}{\mathbf{I}} : \Big(\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} - \big[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}\big]^{[2,4]}\Big) \tag{101}$$

$$\Rightarrow \quad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} = \overset{\langle 2\rangle}{\mathbf{I}} : \big[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}\big]^{[2,4]} \quad \Leftrightarrow \quad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = \overset{\langle 2\rangle}{\mathbf{I}} : \big[\overset{\langle 3\rangle}{\mathbf{A}} \cdot \overset{\langle 3\rangle}{\mathbf{A}}\big]^{[2,4]}. \tag{102}$$

The set of all symmetry transformations is the **symmetry group** of a material. In fact the symmetry group is an algebraic group under composition.

The **composition** is defined as

$$\big(\overset{\langle 2\rangle}{\mathbf{B}}, \overset{\langle 3\rangle}{\mathbf{B}}, \overset{\langle 4\rangle}{\mathbf{B}}\big)\big(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big) := \Bigg(\overset{\langle 2\rangle}{\mathbf{B}} \cdot \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{B}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{B}} + \overset{\langle 4\rangle}{\mathbf{A}} + \overset{[2,3][2,4]}{3sym}\Big[\big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{B}}\big) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\Big]\Bigg). \tag{103}$$

The **neutral element** is

$$\big(\overset{\langle 2\rangle}{\mathbf{I}},\overset{\langle 3\rangle}{\mathbf{0}},\overset{\langle 4\rangle}{\mathbf{0}}\big). \tag{104}$$

The **inverse element** is

$$\big(\overset{\langle 2\rangle}{\mathbf{A}},\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 4\rangle}{\mathbf{A}}\big)^{-1} := \Big(\overset{\langle 2\rangle}{\mathbf{A}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{A}}^{-T}\circ\overset{\langle 3\rangle}{\mathbf{A}}, \big(\overset{\langle 2\rangle}{\mathbf{A}}^{-T}\circ\big(-\overset{\langle 4\rangle}{\mathbf{A}}+\overset{[2,4][2,3]}{3sym}\,\big[\overset{\langle 3\rangle}{\mathbf{A}}\cdot\overset{\langle 3\rangle}{\mathbf{A}}\big]\big)\big)\Big). \tag{105}$$

The group definitions are routed in the idea that the group elements stem from the gradients of symmetry transformations, e.g., the entries of the inverse element are calculated by assuming that

$$\big(\overset{\langle 2\rangle}{\mathbf{A}},\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 4\rangle}{\mathbf{A}}\big)^{-1} = \big(\mathbf{A},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)^{-1} = \big(\mathbf{A}^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}\big) \tag{106}$$

and then using (17)–(18).

The group operation as well as the definition of the neutral and inverse elements follow from the definition of the strain variables.

Definition 5 (Isotropic material) An elastic material is called isotropic if there exists a reference placement such that the symmetry group contains the proper orthogonal group in the first entry:

$$\big\{\big(\overset{\langle 2\rangle}{\mathbf{Q}},\overset{\langle 3\rangle}{\mathbf{0}},\overset{\langle 4\rangle}{\mathbf{0}}\big)\in\mathscr{Config}\ \mid\ \overset{\langle 2\rangle}{\mathbf{Q}}\in\mathscr{Orth}^{+}\big\}. \tag{107}$$

It is clear from Theorem 3 and Definition 4 that for an isotropic material the elastic laws

$$\overset{\langle 2\rangle}{f}\quad\text{and}\quad\overset{\langle 4\rangle}{f} \tag{108}$$

are isotropic tensor functions: One applies the fact that for isotropic materials $\overset{\langle 2\rangle}{\mathbf{A}}$ is orthogonal. This yields $J_{\overset{\langle 2\rangle}{\mathbf{A}}}=1$ and in this case the product "$\circ$" can be replaced by the product "$*$":

$$\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 2\rangle}{f}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \overset{\langle 2\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C},\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big), \tag{109}$$

$$\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 4\rangle}{f}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \overset{\langle 4\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C},\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big). \tag{110}$$

The elastic law $\overset{\langle 3\rangle}{\widehat{f}}$ is not an isotropic tensor function. The reason for this is the fact that $\overset{\langle 3\rangle}{\widehat{f}}$ transforms with the function $\widehat{\gamma}$ as defined in (57) and not as a pullback like the other elastic laws. In the case of an isotropic third-order material Theorem 3 implies

$$\begin{aligned}\overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \mathbf{A}^{-1} * \Big[\Big(\overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) + 3\,\overset{\langle 4\rangle}{f}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}\Big)\Big] \\ -\big(\mathbf{A}^{-1} * \big[\overset{\langle 4\rangle}{f}\big(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big)\big]\big) : \big[\mathbf{A}^{-1} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big]^{[1,3]}.\end{aligned} \tag{111}$$

Definition 6 (Centro-symmetric materials) A gradient material is centro-symmetric if $(-\overset{\langle 2\rangle}{\mathbf{I}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}})$ is an element of the symmetry group. A simple material is always centro-symmetric since

$$\left(-\overset{\langle 2\rangle}{\mathbf{A}}^T\right) * \overset{\langle 2\rangle}{\mathbf{C}} = \left(\overset{\langle 2\rangle}{\mathbf{A}}^T\right) * \overset{\langle 2\rangle}{\mathbf{C}} \tag{112}$$

implies for a symmetry transformation $\overset{\langle 2\rangle}{\mathbf{A}} \in \mathscr{Inv}$

$$w\left(\left(-\overset{\langle 2\rangle}{\mathbf{A}}^T\right) * \mathbf{C}\right) = w\left(\overset{\langle 2\rangle}{\mathbf{A}}^T * \mathbf{C}\right) = w(\mathbf{C}). \tag{113}$$

For even-order tensors the minus sign in the symmetry transformation cancels out. Therefore the even-order stress variables cannot depend on odd-order configuration variables in the case of centro-symmetry and in a second gradient of the strain framework one has to distinguish between symmetric and centro-symmetric materials. In a similar way one could also define solid materials as those for which a reference placement exists such that the orthogonal group from Eq. (107) contains the symmetry group. However, such a definition does not include the other two higher-order tensors in the symmetry transformation. Their role needs further investigations.

8 Linear Third-order Theory

In many cases the elastic deformations are small enough to justify the linearization of the hyperelastic laws. Naturally, an elastic law is mainly of interest for solid materials, although such a restriction is not compulsory in the present context. In the physically linear third-order elasticity theory the elastic energy is assumed to be a symmetric square form of the configuration in analogy to the St. Venant-Kirchhoff model. The reference placement is chosen to be stress free. In this linear case it is common to use Green's strain tensor instead of $\mathbf{C}$:

$$\mathbf{E}^G := \frac{1}{2}\left(\mathbf{C} - \overset{\langle 2\rangle}{\mathbf{I}}\right) \in \mathscr{Sym}. \tag{114}$$

One assumes that the higher-order terms $\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}$ and $\overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}}$ are small which means that

$$\left\|\mathbf{E}^G\right\| \ll 1, \qquad L\left\|\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}\right\| \ll 1, \qquad L^2\left\|\overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}}\right\| \ll 1, \tag{115}$$

with a scaling parameter L that has the dimension of a length. In tensor notation the quadratic elastic energy can be written as

$$\begin{aligned}\rho_0 w\left(\mathbf{E}^G, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}}\right) = {} & \frac{1}{2}\mathbf{E}^G : \overset{\langle 4\rangle}{\mathbb{E}} : \mathbf{E}^G + \frac{1}{2}\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} \vdots \overset{\langle 6\rangle}{\mathbb{E}} \vdots \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} + \frac{1}{2}\overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}} :: \overset{\langle 8\rangle}{\mathbb{E}} :: \overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}} \\ & \mathbf{E}^G : \overset{\langle 5\rangle}{\mathbb{E}} \vdots \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} + \mathbf{E}^G : \overset{\langle 6\rangle}{\mathbb{E}} :: \overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} \vdots \overset{\langle 7\rangle}{\mathbb{E}} :: \overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}},\end{aligned} \tag{116}$$

with higher-order elasticity tensors

$$\overset{\langle 4\rangle}{\mathbf{E}}, \overset{\langle 5\rangle}{\mathbf{E}}, \overset{\langle 6\rangle}{\mathbf{E}}, \overset{\langle \underline{6}\rangle}{\mathbf{E}}, \overset{\langle 7\rangle}{\mathbf{E}}, \overset{\langle 8\rangle}{\mathbf{E}}, \tag{117}$$

which inherit the corresponding subsymmetries from $\mathbf{E}^G$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ and $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ according to (116). Furthermore they have the following symmetries because of the integrability conditions

$$\overset{\langle 4\rangle}{\mathbf{E}}_{ab\ cd} = \overset{\langle 4\rangle}{\mathbf{E}}_{cd\ ab}, \qquad \overset{\langle 6\rangle}{\mathbf{E}}_{abc\ def} = \overset{\langle 6\rangle}{\mathbf{E}}_{def\ abc}, \qquad \overset{\langle 8\rangle}{\mathbf{E}}_{abcd\ efgh} = \overset{\langle 8\rangle}{\mathbf{E}}_{efgh\ abcd}\,. \tag{118}$$

Thus $\overset{\langle 4\rangle}{\mathbf{E}}$ has 21 independent constants, $\overset{\langle 5\rangle}{\mathbf{E}}$ has 180, $\overset{\langle 6\rangle}{\mathbf{E}}$ has 171, $\overset{\langle \underline{6}\rangle}{\mathbf{E}}$ has 180, $\overset{\langle 7\rangle}{\mathbf{E}}$ has 540, and $\overset{\langle 8\rangle}{\mathbf{E}}$ has 465. In total these are 1485 independent constants. In the general anisotropic case the elastic energy is a potential for the stresses and one obtains

$$\overset{\langle 2\rangle}{f}\left(\mathbf{E}^G, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) = \overset{\langle 4\rangle}{\mathbf{E}} : \mathbf{E}^G + \overset{\langle 5\rangle}{\mathbf{E}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle \underline{6}\rangle}{\mathbf{E}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \tag{119}$$

$$\begin{aligned}\overset{\langle 3\rangle}{f}\left(\mathbf{E}^G, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) &= \overset{\langle 6\rangle}{\mathbf{E}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \mathbf{E}^G : \overset{\langle 5\rangle}{\mathbf{E}} + \overset{\langle 7\rangle}{\mathbf{E}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} \\ &\quad + \left(\mathbf{E}^G : \overset{\langle \underline{6}\rangle}{\mathbf{E}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \,\vdots\, \overset{\langle 7\rangle}{\mathbf{E}} + \overset{\langle 8\rangle}{\mathbf{E}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}\end{aligned} \tag{120}$$

$$\approx \overset{\langle 6\rangle}{\mathbf{E}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \mathbf{E}^G : \overset{\langle 5\rangle}{\mathbf{E}} + \overset{\langle 7\rangle}{\mathbf{E}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \tag{121}$$

$$\overset{\langle 4\rangle}{f}\left(\mathbf{E}^G, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) = \overset{\langle 8\rangle}{\mathbf{E}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \mathbf{E}^G : \overset{\langle \underline{6}\rangle}{\mathbf{E}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \,\vdots\, \overset{\langle 7\rangle}{\mathbf{E}}, \tag{122}$$

where (121) is a linearization. These elastic laws are a generalization of the St.-Venant-Kirchhoff law to gradient elasticity. They are geometrically nonlinear but physically linear.

In [28] the case of linear elasticity for a centrosymmetric and isotropic third-order material is presented. The corresponding elastic energy is given therein and requires 17 material constants including the usual Lamé constants. The elastic energy is characterized by 18 constants and takes the form

$$\begin{aligned}\rho_0 w &= a_1\left(\mathbf{E}^G : \overset{\langle 2\rangle}{\mathbf{I}}\right)^2 + a_2 \mathbf{E}^G : \mathbf{E}^G + b_1\left(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right)\cdot\left(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) \\ &\quad + b_2\, \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}} + b_3\left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)\cdot\left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) + b_4\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \\ &\quad + b_5\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} + c_1\left(\mathbf{I} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \mathbf{I}\right)^2 + c_2\left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) : \left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \\ &\quad + c_3\left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\right) : \left(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) + c_4\left(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) : \left(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) \\ &\quad + c_5\left(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) : \left(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} : \overset{\langle 2\rangle}{\mathbf{I}}\right) + c_6\, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + c_7\, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} \\ &\quad + d_1\left(\mathbf{E}^G : \overset{\langle 2\rangle}{\mathbf{I}}\right)\left(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}}\right) + d_2 \mathbf{E}^G : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \overset{\langle 2\rangle}{\mathbf{I}} + d_3\, \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} : \mathbf{E}^G.\end{aligned} \tag{123}$$

The linear relationship between ($\mathbf{C}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$) and ($\overset{\langle 2\rangle}{\mathbf{S}}$, $\overset{\underset{\sim}{\langle 3\rangle}}{\mathbf{S}}$, $\overset{\langle 4\rangle}{\mathbf{S}}$) yields for a centrosymmetric gradient material that $\overset{\langle 2\rangle}{\mathbf{S}}$ can only depend on $\mathbf{C}$ as well as that $\overset{\underset{\sim}{\langle 3\rangle}}{\mathbf{S}}$, $\overset{\langle 4\rangle}{\mathbf{S}}$ can only depend on $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ and $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$.

The form (123) of the energy with its 17 constants is obtained by using the following facts. The only isotropic third-order tensors are multiples of the Levi-Civita permutation tensor, which is ruled out by the centrosymmetry. Bilinear terms of odd and even-order tensors cancel out due to centrosymmetry.

In [28] another term $c_0(\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}})$ is added to (123). It is linear in $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ and therefore produces a constant fourth-order stress. In the present setting a stress free reference placement is assumed and thus c_0 must vanish. In [12, 28] or [13] it is explained how c_0 can be used to model surface tensions. The stresses, here presented for the case $c_0 = 0$, are shown to be

$$\overset{\langle 2\rangle}{\mathbf{S}} = 2a_1\big(\mathbf{E}^G:\overset{\langle 2\rangle}{\mathbf{I}}\big)\overset{\langle 2\rangle}{\mathbf{I}} + 2a_2\mathbf{E}^G + d_1\big(\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}}\big)\overset{\langle 2\rangle}{\mathbf{I}} + d_2\, sym\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}}\big) + d_3\,\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \tag{124}$$

$$\begin{aligned}\overset{\widehat{\langle 3\rangle}}{\mathbf{S}} &= sym^{[2,3]}\Big(2b_1\,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} + b_2\big[\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}} + \overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\otimes\overset{\langle 2\rangle}{\mathbf{I}}\big] \\ &\quad + 2b_3\,\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + 2b_5\,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]}\Big) + 2b_4\,\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}},\end{aligned} \tag{125}$$

$$\begin{aligned}\overset{\langle 4\rangle}{\mathbf{S}} &= sym^{[2,3][2,4]}\Big[2c_1\big(\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}}\big)\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} + 2c_2\,\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} \\ &\quad + c_3\,\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}} + c_3\,\overset{\langle 2\rangle}{\mathbf{I}}:\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} + 2c_4\,\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}:\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} \\ &\quad + 2c_5\,\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]}:\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} + 2c_6\,\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + 2c_7\,\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} \\ &\quad + d_1\big(\mathbf{E}^G:\overset{\langle 2\rangle}{\mathbf{I}}\big)\overset{\langle 2\rangle}{\mathbf{I}}\otimes\overset{\langle 2\rangle}{\mathbf{I}} + d_2\mathbf{E}^G\otimes\overset{\langle 2\rangle}{\mathbf{I}} + d_3\,\overset{\langle 2\rangle}{\mathbf{I}}\otimes\mathbf{E}^G\Big].\end{aligned} \tag{126}$$

In [35] a linear third-order elasticity model has been implemented in a finite element software package. It is shown how point and line forces or displacements on corners and edges of a tetrahedron and a cube can be sustained by a third-order elastic material.

The physical interpretation of the stress tensors in the third-order theory as well as of the elasticity tensors is not well understood so far and requires further research. Some explanations on how to interpret higher-order stress tensors have already been given in [28] or [33]. The first steps towards understanding the role of the components of elasticity tensors in the linear second-order theory have been taken in [22].

9 Thermodynamical Variables and Basic Concepts

In this section and the following ones it will be shown that the third-order elasticity format can be embedded in a thermodynamics setting and the thermodynamic restrictions are derived. In the thermodynamical setting we need additional variables such as

ε,	the **specific internal energy**,	(127)
Q,	the **heat supply** per unit mass and time by irradiation and conduction,	(128)
$\mathbf{q}_E$,	the **spatial heat flux** per unit area and unit time in the current placement,	(129)
$\mathbf{q} := J_\mathbf{F}\mathbf{F}^{-1} \cdot \mathbf{q}_E$,	the **material heat flux** per unit area and unit time in the reference placement,	(130)
θ,	the **absolute temperature**,	(131)
$\mathbf{g} := Grad(\theta)$,	the **material temperature gradient**,	(132)
η,	the **specific entropy**, and	(133)
$\psi := \varepsilon - \theta\eta$,	the **Helmholtz free energy**.	(134)

Again, some of these quantities depend on the choice of the reference placement. The material temperature gradient and the material heat flux are such quantities. Their transformations under a change of the reference placement are

$$\underline{\mathbf{g}} := \mathbf{A}^T \cdot \mathbf{g}, \tag{135}$$

$$\underline{\mathbf{q}} := J_\mathbf{A}\mathbf{A}^{-1} \cdot \mathbf{q}, \tag{136}$$

where quantities in a new reference placement are denoted by underlining them.

The **first law of thermodynamics** is assumed in the usual form, where we have already eliminated the kinetic energy by the mechanical balance of work:

$$Q = \varepsilon^{\cdot} - p. \tag{137}$$

The concept of the kinematical process is extended to a **thermo-kinematical process**, described by the set of variables

$$\left\{\chi(\tau), \mathbf{F}(\tau), Grad(\mathbf{F}), Grad^{\mathrm{II}}(\mathbf{F}), \theta(\tau), grad(\theta)(\tau)\right\}, \tag{138}$$

with $\tau \in [0, t]$. One assumes that this process determines the **caloro-dynamical state** at the end of the process, defined by the set

$$\left\{\overset{\langle 2\rangle}{\mathbf{T}}(t), \overset{\langle 3\rangle}{\mathbf{T}}(t), \overset{\langle 4\rangle}{\mathbf{T}}(t), \mathbf{q}_E(t), \varepsilon(t), \eta(t)\right\}. \tag{139}$$

One could include higher gradients of the temperature as independent variables. However, it has already by shown by Perzyna in 1971 [32] that such dependencies are ruled out by the second law of thermodynamics. The argument for this is rather standard. If one assumes that the free energy depends on the temperature gradient, then there is no counterpart in the Clausius-Duhem inequality, so that such a dependence is ruled out. Exactly the same happens with the dependence on higher temperature gradients.

10 Third-order Thermoelasticity

The concept of third-order elasticity has to be extended in such a way that the current calorodynamical state is determined by only the current thermo-kinematical state. This means that the past process does not directly influence the current material behavior.

Definition 7 (Third-order thermoelasticity) A material is called a **third-order thermoelastic material** if the calorodynamic state $\{\overset{\langle 2\rangle}{\mathbf{T}}(t), \overset{\langle 3\rangle}{\mathbf{T}}(t), \overset{\langle 4\rangle}{\mathbf{T}}(t), \mathbf{q}_E(t), \varepsilon(t), \eta(t)\}$ is a function of the current thermo-kinematical state $\{\chi(t), \mathbf{F}(t), Grad(\mathbf{F})(t), Grad^{\mathrm{II}}(\mathbf{F})(t), \theta(t), grad(\theta)(t)\}$.

Here one takes all variables at the same material point and instant of time. Following the line of argumentation in Sect. 4 allows the introduction of reduced forms. The constitutive equations can be reduced to the set of equations

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big), \tag{140}$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big), \tag{141}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big), \tag{142}$$

$$\mathbf{q} = q\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big), \tag{143}$$

$$\varepsilon = \varepsilon\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big), \tag{144}$$

$$\eta = \eta\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big). \tag{145}$$

After (134) the free energy is also a function of $\mathbf{C}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$, θ and $\mathbf{g}$:

$$\psi\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big) = \varepsilon\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big) - \theta\eta\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g}\big). \tag{146}$$

The **second law of thermodynamics** is assumed in the form of the **Clausius-Duhem inequality**, which is a local and momentary restriction to all admissible thermodynamical processes:

$$p - \psi^{\cdot} - \theta^{\cdot}\eta - \frac{1}{\rho_0\theta}\mathbf{g}\cdot\mathbf{q} \geq 0. \tag{147}$$

In this inequality the last term is the **thermal dissipation** and the first three terms are the mechanical dissipation. The Clausius-Duhem inequality is assumed to hold at all states and all thermo-kinematical continuations.

Theorem 4 *For a third-order thermoelastic material the Clausius-Duhem inequality* (147) *is fulfilled for every thermo-kinematical process if and only if*

1. *The free energy does not depend on the temperature gradient.*
2. *The free energy acts as a potential for the generalized stresses and for the entropy.*
3. *The heat conduction inequality holds*: $\boldsymbol{q}\cdot\boldsymbol{g} \geq 0$.

This shows that the thermoelastic behavior of a third-order material is completely determined by the two functions $\psi(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta)$ and $q(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta, \mathbf{g})$.

Proof Partial derivatives are abbreviated, e.g., $\partial_{\mathbf{C}}\psi$ denotes the partial derivative of ψ with respect to $\mathbf{C}$. Combining (146) with (147)

$$
\begin{aligned}
0 \geq & -\frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\boldsymbol{\cdot}} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} \,\vdots\, \overset{\langle 3\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}}\right) + \partial_{\mathbf{C}}\psi : \mathbf{C}^{\boldsymbol{\cdot}} + \partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi \,\vdots\, \overset{\langle 3\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}} \\
& + \partial_{\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi :: \overset{\langle 4\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}} + \partial_\theta\psi\theta^{\boldsymbol{\cdot}} + \partial_{\mathbf{g}}\psi \boldsymbol{\cdot} \mathbf{g}^{\boldsymbol{\cdot}} + \theta^{\boldsymbol{\cdot}}\eta + \frac{1}{\rho_0\theta}\mathbf{g}\boldsymbol{\cdot}\mathbf{q} \qquad (148)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(\partial_{\mathbf{C}}\psi - \frac{1}{2\rho_0}\overset{\langle 2\rangle}{\mathbf{S}}\right) : \mathbf{C}^{\boldsymbol{\cdot}} + \left(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi - \frac{1}{\rho_0}\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}\right) \vdots\, \overset{\langle 3\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}} \\
& + \left(\partial_{\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi - \frac{1}{\rho_0}\overset{\langle 4\rangle}{\mathbf{S}}\right) :: \overset{\langle 4\rangle\boldsymbol{\cdot}}{\mathbf{K}}_{\mathbf{F}} + (\partial_\theta\psi + \eta)\theta^{\boldsymbol{\cdot}} + \partial_{\mathbf{g}}\psi \boldsymbol{\cdot} \mathbf{g}^{\boldsymbol{\cdot}} + \frac{1}{\rho_0\theta}\mathbf{g}\boldsymbol{\cdot}\mathbf{q}. \qquad (149)
\end{aligned}
$$

Using (140)–(145) one obtains by standard arguments the following thermoelastic relations.

$$\partial_{\mathbf{g}}\psi = \overset{\langle 1\rangle}{\mathbf{0}}, \qquad (150)$$

$$\overset{\langle 2\rangle}{\mathbf{S}} = 2\rho_0\partial_{\mathbf{C}}\psi, \qquad (151)$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \rho_0\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi, \qquad (152)$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \rho_0\partial_{\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi, \qquad (153)$$

$$\eta = -\partial_\theta\psi, \qquad (154)$$

$$0 \geq \mathbf{g}\boldsymbol{\cdot}\mathbf{q}. \qquad (155)$$

Equation (150) means that the free energy is independent of the temperature gradient. Furthermore (151)–(153) show that the free energy is a potential for the generalized stresses, and (154) that it is a potential for the elastic part of the entropy. Finally (155) is the heat conduction inequality. □

These conditions are similar to those which we know from simple materials. The extension to the higher potentials appears to be straight forward and natural. So our gradient ansatz does not conflict with the second law of thermodynamics, in contrast to the findings of [23].

11 Material Isomorphy

The concept of elastic isomorphy can be extended for the case of thermoelasticity. One considers two thermoelastic points as isomorphic if their measurable thermoelastic behavior does not show any differences during arbitrary thermo-kinematical processes. Measurable quantities are the generalized stresses, the heat flux and the rate of the internal energy as they appear in balances. The entropy and free energy are not considered as (directly) measurable quantities.

Definition 8 (Thermoelastic isomorphy) Two thermoelastic points X and Y are **thermoelastically isomorphic** if two reference placements κ_X and κ_Y and two constants $\eta_c, \varepsilon_c \in \mathbb{R}$ exist such that

$$\rho_{0X} = \rho_{0Y}, \tag{156}$$

$$\psi_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \psi_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) - \eta_c\theta + \varepsilon_c, \tag{157}$$

$$q_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta, \mathbf{g}\big) = q_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta, \mathbf{g}\big) \tag{158}$$

hold for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}) \in \mathscr{Config}$, $\theta \in \mathbb{R}$, $\mathbf{g} \in \mathbb{R}^3$.

Definition 8 implies with the relations from Theorem 4 the following relations for the other thermoelastic equations

$$\overset{\langle 2\rangle}{f}_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \overset{\langle 2\rangle}{f}_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big), \tag{159}$$

$$\overset{\langle 3\rangle}{\widehat{f}}_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \overset{\langle 3\rangle}{\widehat{f}}_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big), \tag{160}$$

$$\overset{\langle 4\rangle}{f}_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \overset{\langle 4\rangle}{f}_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big), \tag{161}$$

$$\varepsilon_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \varepsilon_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) + \varepsilon_c, \tag{162}$$

$$\eta_X\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) = \eta_Y\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_\mathbf{F}, \overset{\langle 4\rangle}{\mathbf{K}}_\mathbf{F}, \theta\big) + \eta_c. \tag{163}$$

Definition 8 is derived from the following reasoning, see [10]. For third-order thermoelastic materials the mechanical dissipation is zero. By use of (134) and (137)

$$0 = p - \psi^{\boldsymbol\cdot} - \theta^{\boldsymbol\cdot}\eta \tag{164}$$

$$= p - \varepsilon^{\boldsymbol\cdot} + \theta\eta^{\boldsymbol\cdot} \tag{165}$$

$$= -Q + \theta\eta^{\boldsymbol\cdot}. \tag{166}$$

If the heat supply and the temperature are measurable, then so is the rate of the entropy. In conclusion the entropy of two isomorphic points of a thermoelastic material can only differ by a constant η_c (which cannot be measured).

$$\eta_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \theta_X\big) = \eta_Y\big(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \theta_Y\big) + \eta_c. \tag{167}$$

Integration with respect to the temperature then yields

$$\psi_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \theta_X\big) = \psi_Y\big(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \theta_Y\big) - \eta_c\theta + \varepsilon_c, \tag{168}$$

where ε_c is another constant. Finally one can use these relations together with (134) to obtain

$$\begin{aligned}
&\varepsilon_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \theta_X\big)\\
&\quad = \psi_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \theta_X\big) + \theta\eta_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \theta_X\big) && (169)\\
&\quad = \psi_Y\big(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \theta_Y\big) + \theta\eta_Y\big(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \theta_Y\big) + \varepsilon_c && (170)\\
&\quad = \varepsilon_Y\big(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, \theta_Y\big) + \varepsilon_c, && (171)
\end{aligned}$$

which motivates Definition 8. From here it is clear that the criterion for elastic isomorphy can be generalized in the same manner.

Theorem 5 (Criterion for thermoelastic isomorphy) *Let X and Y be two thermoelastic material points with arbitrary reference placements κ_X and κ_Y. Let ψ_X, q_X and ψ_Y, q_Y be the corresponding thermoelastic laws. Then these two points are* ***thermoelastically isomorphic*** *if and only if there exist three tensors $\overset{\langle 2\rangle}{\boldsymbol{P}} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\boldsymbol{P}} \in \mathscr{SubSym}_3$, $\overset{\langle 4\rangle}{\boldsymbol{P}} \in \mathscr{SubSym}_4$ and two real constants ε_c, η_c such that the following three conditions hold.*

$$\rho_{0Y} = det\big(\overset{\langle 2\rangle}{\boldsymbol{P}}\big)\rho_{0X} \tag{172}$$

and

$$\begin{aligned}
\psi_X\big(\boldsymbol{C}_X, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}, \theta\big) = \psi_Y\Bigg(&\overset{\langle 2\rangle}{\boldsymbol{P}}{}^T * \boldsymbol{C}_X, \overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X} + \overset{\langle 3\rangle}{\boldsymbol{P}}, \overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X} + \overset{\langle 4\rangle}{\boldsymbol{P}}\\
&+ \overset{[2,4][2,3]}{3sym}\Bigg[\bigg(\overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}\bigg) \cdot \overset{\langle 3\rangle}{\boldsymbol{P}}\Bigg], \theta\Bigg) - \eta_c\theta + \varepsilon_c
\end{aligned} \tag{173}$$

and

$$\begin{aligned}
det\big(\overset{\langle 2\rangle}{\boldsymbol{P}}\big)q_X\big(\boldsymbol{C}_X, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}, \theta, \boldsymbol{g}_X\big) = \overset{\langle 2\rangle}{\boldsymbol{P}} * q_Y\Bigg(&\overset{\langle 2\rangle}{\boldsymbol{P}}{}^T * \boldsymbol{C}_X, \overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X} + \overset{\langle 3\rangle}{\boldsymbol{P}}, \overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X} + \overset{\langle 4\rangle}{\boldsymbol{P}}\\
&+ \overset{[2,4][2,3]}{3sym}\Bigg[\bigg(\overset{\langle 2\rangle}{\boldsymbol{P}}{}^T \circ \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}\bigg) \cdot \overset{\langle 3\rangle}{\boldsymbol{P}}\Bigg], \theta, \overset{\langle 2\rangle}{\boldsymbol{P}}{}^T * \boldsymbol{g}_X\Bigg)
\end{aligned} \tag{174}$$

for all $(\boldsymbol{C}_X, \overset{\langle 3\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}, \overset{\langle 4\rangle}{\boldsymbol{K}}_{\boldsymbol{F}_X}) \in \mathscr{Config}$, $\theta \in \mathbb{R}^+$, $\boldsymbol{g} \in \mathbb{R}^3$.

Theorem 5 implies that the tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{SubSym}_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{SubSym}_4$ and two real constants ε_c, η_c determine the isomorphy transformation of the other constitutive laws as a consequence of (151)–(154).

For $(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}) \in \mathscr{Config}$, $\theta \in \mathbb{R}^+$ two points X and Y are thermoelastically isomorphic if the following equations hold.

$$\overset{\langle 2\rangle}{f}_X\left(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X},\theta\right)=\overset{\langle 2\rangle}{\mathbf{P}}*det^{-1}\big(\overset{\langle 2\rangle}{\mathbf{P}}\big)\,\overset{\langle 2\rangle}{f}_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\right.$$
$$\left.\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right)\quad(175)$$

and

$$\widehat{\overset{\langle 3\rangle}{f}}_X\left(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X},\theta\right)$$
$$=\widehat{\gamma}\left(\widehat{\overset{\langle 3\rangle}{f}}_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}\right.\right.$$
$$\left.+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right),\overset{\langle 4\rangle}{f}_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}\right.$$
$$\left.\left.+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right),\overset{\langle 2\rangle}{\mathbf{P}}^{-1},-\overset{\langle 2\rangle}{\mathbf{P}}^{-T}\circ\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}}\right)\qquad(176)$$

and

$$\overset{\langle 4\rangle}{f}_X\left(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X},\theta\right)$$
$$=\overset{\langle 2\rangle}{\mathbf{P}}\circ det^{-1}\big(\overset{\langle 2\rangle}{\mathbf{P}}\big)\,\overset{\langle 4\rangle}{f}_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\right.$$
$$\left.\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right)\qquad(177)$$

and

$$\varepsilon_X\left(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X},\theta\right)$$
$$=\varepsilon_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}\right.$$
$$\left.+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right)+\varepsilon_c\qquad(178)$$

and

$$\eta_X\left(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X},\theta\right)$$
$$=\eta_Y\left(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 4\rangle}{\mathbf{P}}\right.$$
$$\left.+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}\right)\cdot\overset{\langle 3\rangle}{\mathbf{P}}\right],\theta\right)+\eta_c,\qquad(179)$$

where $\widehat{\gamma}$ is defined in (57).

With these results the concept of symmetry transformations can be extended to thermoelastic materials in a straightforward manner.

12 Material Symmetry (Thermoelastic)

Definition 9 (Thermoelastic symmetry transformations) For a third-order thermoelastic material with material laws ψ and q a symmetry transformation is a triple $(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}) \in \mathscr{Unim} \times \mathscr{SubSym}_3 \times \mathscr{SubSym}_4$ that fulfills

$$\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}} = 0 \quad \text{and} \quad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = \overset{\langle 2\rangle}{\mathbf{I}} : \left[\overset{\langle 3\rangle}{\mathbf{A}} \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right]^{[2,4]} \tag{180}$$

such that for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \in \mathscr{Config}$, $\theta \in \mathbb{R}^+$, $\mathbf{g} \in \mathbb{R}^3$ the following equations hold.

$$\begin{aligned} &\psi\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta\right) \\ &\quad = \psi\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{A}} \\ &\qquad + \overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right], \theta\Bigg) \end{aligned} \tag{181}$$

and

$$\begin{aligned} &q\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta\right) \\ &\quad = \overset{\langle 2\rangle}{\mathbf{A}} * q\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{A}} \\ &\qquad + \overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right], \theta\Bigg). \end{aligned} \tag{182}$$

Definition 9 implies symmetry transformations for the thermoelastic laws.

For all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \in \mathscr{Config}$, $\theta \in \mathbb{R}$, $\mathbf{g} \in \mathbb{R}^3$

$$\begin{aligned} &\overset{\langle 2\rangle}{f}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta\right) \\ &\quad = \overset{\langle 2\rangle}{\mathbf{A}} * J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 2\rangle}{f}\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{A}} \\ &\qquad + \overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right], \theta\Bigg) \end{aligned} \tag{183}$$

and

$$\begin{aligned} &\overset{\langle 3\rangle}{\widehat{f}}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \theta\right) \\ &= \widehat{\gamma}\Bigg(\overset{\langle 3\rangle}{\widehat{f}}\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{A}} \\ &\quad + \overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right], \theta\Bigg) \overset{\langle 4\rangle}{f}\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{A}} \end{aligned}$$

$$+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)\cdot\overset{\langle 3\rangle}{\mathbf{A}}\right],\theta\Bigg),\overset{\langle 2\rangle}{\mathbf{A}}^{-1},-\overset{\langle 2\rangle}{\mathbf{A}}^{-T}\circ\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 3\rangle}{\mathbf{A}}\Bigg) \tag{184}$$

and

$$\begin{aligned}
&\overset{\langle 4\rangle}{f}\left(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}},\theta\right)\\
&\quad=\overset{\langle 2\rangle}{\mathbf{A}}\circ J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}}\,\overset{\langle 4\rangle}{f}\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T}*\mathbf{C},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 4\rangle}{\mathbf{A}}\\
&\qquad+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)\cdot\overset{\langle 3\rangle}{\mathbf{A}}\right],\theta\Bigg)
\end{aligned} \tag{185}$$

and

$$\begin{aligned}
&\varepsilon\left(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}},\theta\right)\\
&\quad=\varepsilon\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T}*\mathbf{C},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 4\rangle}{\mathbf{A}}\\
&\qquad+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)\cdot\overset{\langle 3\rangle}{\mathbf{A}}\right],\theta\Bigg)
\end{aligned} \tag{186}$$

and

$$\begin{aligned}
&\eta\left(\mathbf{C},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}},\theta\right)\\
&\quad=\eta\Bigg(\overset{\langle 2\rangle}{\mathbf{A}}^{T}*\mathbf{C},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}+\overset{\langle 4\rangle}{\mathbf{A}}\\
&\qquad+\overset{[2,4][2,3]}{3sym}\left[\left(\overset{\langle 2\rangle}{\mathbf{A}}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\right)\cdot\overset{\langle 3\rangle}{\mathbf{A}}\right],\theta\Bigg),
\end{aligned} \tag{187}$$

where $\widehat{\gamma}$ is defined in (57). The definitions of a symmetry groups, of isotropy, etc. also apply to the thermoelastic case.

13 Conclusion and Outlook

A constitutive format for third-order elasticity for finite deformations is presented. The basic assumption is the existence of an objective expression of the stress power due to the Principle of Euclidean Invariance or Objectivity. In elasticity, we assume the existence of a potential for the stress power, which is also submitted to this invariance principle. With the derived variables a third-order framework for elasticity in the spirit of [5] can be set up. For each stress and strain variable the transformation behavior under changes of the reference placements is derived. In this context it is a remarkable result that the power of the second velocity gradient disperses to both, a third- and fourth-order stress tensor. In the second-order framework from [5] such mixed dependencies could be avoided. It has been shown [6] that this dispersion in the third-order theory cannot be avoided, in principle.

The concepts of elastic isomorphy and material symmetry are extended to the third-order case. The algebraic group structure is worked out in a natural way.

These results allow to set up a thermodynamic format. The Helmholtz free energy is introduced and is shown not to depend on the gradient of the temperature. It acts as a potential for the generalized stresses and for the entropy. The heat conduction inequality is shown to hold as well. The concepts of elasticity such as isomorphy and symmetry are extended for the thermoelastic case. This proves that the ansatz with higher gradients does not conflict with thermodynamics if properly introduced.

This paper presents a rather general framework for third-order finite elasticity models. It offers the opportunity for consistent material modelling within such interesting class of materials.

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