Erratum to: On the Number of Invariants in the Strain Energy Density of an Anisotropic Nonlinear Elastic Material with Two Material Symmetry Directions

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This erratum concerns a correction in the formulae (19) and (23) of the original article. It also concerns the addition of Remark 5, which contains two relations obtained from (19) and their correspondence to two syzygys found by Shariff and Bustamante [1], which also appear in [2]. These works were not known to the authors at the time of publication of the article.

The correct formulae in (19) are

$$\begin{aligned}b_1^2 &= J_{10}, & C_{11} &= I_5, & b_1 b_2 C_{12} &= J_{11} - I_5 J_{10}, \\C_{22} &= (I_8 + I_5 J_{10} - 2J_{11})/(1 - J_{10}), & C_{33} &= J_1 - (I_8 + I_5 - 2J_{11})/(1 - J_{10}), \\C_{13}^2 &= J_{15}/(1 - J_{10}), & C_{12}^2 &= I_6 - I_5^2 - J_{15}/(1 - J_{10}), \\C_{23}^2 &= -(4J_{11}^2 + J_{10}^2 J_2 + J_{10}^2 I_6 - 2J_{10} J_2 + J_{10} I_5^2 - 2J_{10} I_6 + J_2 + I_6 + I_5(J_{10} + 1)I_8 \\&\quad + I_8^2 + J_1(J_{10} - 1)(-2J_{11} + I_5 + I_8) - 2J_{11}(J_{10} I_5 + I_5 + 2I_8))/(J_{10} - 1)^2,\end{aligned}$$

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$$\begin{aligned}
C_{12}C_{13}C_{23} &= (J_{10}^2 J_3 + J_{10} J_{15} I_5 - J_{10}^2 J_2 I_5 - J_{10}^2 I_5 I_6 - 2J_{10} J_3 + J_{15} I_5) \\
&\quad + 2J_{10} J_2 I_5 - J_{10} I_5^3 + 3J_{10} I_5 I_6 + J_3 - J_2 I_5 + I_5^3 - 2I_5 I_6 \\
&\quad + J_1(J_{10} - 1)(J_{15} + I_6(J_{10} - 1)) - 2J_{11}(2J_{15} + (I_5^2 - I_6)(-J_{10} + 1)) \\
&\quad + (2J_{15} + (I_5^2 - I_6)(-J_{10} + 1))I_8)/(2(J_{10} - 1)^2), \\
b_1 b_2 C_{13} C_{23} &= (J_{10}^2 (J_2 + I_6) - J_{10} (J_{15} + 2J_{11} I_5 + 2J_2 - I_5^2 + I_6 - I_5 I_8 \\
&\quad - J_1(-2J_{11} + I_5 + I_8) + I_9) + (4J_{11}^2 + 2J_{11}(J_1 - 2I_5 - I_8) \\
&\quad + (J_{15} + J_2 - (J_1 - I_5)(I_5 + I_8) + I_9))) / (2(1 - J_{10})).
\end{aligned} \tag{19}$$

Substituting the expressions of $b_1 b_2 C_{12}$, $b_1 b_2 C_{13} C_{23}$, and $C_{12} C_{13} C_{23}$, given in (19), into $b_1 b_2 C_{12} b_1 b_2 C_{13} C_{23} = (b_1 b_2)^2 C_{12} C_{13} C_{23}$, we obtain the correct expression for J_3 , which is given by

$$\begin{aligned}
J_3 &= (J_{11}^3 + J_1 J_{10} (-J_{11}^2 - J_{10} I_5 I_8 + J_{11} (I_5 + I_8)) \\
&\quad + J_{10} J_{11} ((-1 + J_{10}) J_2 - I_6 - I_5 I_8 - I_9) \\
&\quad + J_{10}^2 (I_6 I_8 + I_5 I_9)) / (J_{10}^2 (-1 + J_{10})).
\end{aligned} \tag{23}$$

Remark 5 Below we use both (22) and (23) in all the calculations and consider that J_{10} does not vanish. Substituting the expressions of C_{13}^2 , C_{23}^2 , and $b_1 b_2 C_{13} C_{23}$, given in (19), into $A_1 \triangleq (b_1 b_2)^2 C_{13}^2 C_{23}^2 - (b_1 b_2 C_{13} C_{23})^2$, we obtain

$$\begin{aligned}
A_1 &\triangleq (4(J_{11}^2 - 2J_{10} J_{11} I_5 + J_{10} (I_5^2 + (-1 + J_{10}) I_6)) (4J_{11}^2 + J_2 + I_6 \\
&\quad + J_{10} ((-2 + J_{10}) J_2 + I_5^2 + (-2 + J_{10}) I_6) - J_1 (-1 + J_{10}) (2J_{11} - I_5 - I_8) \\
&\quad + (1 + J_{10}) I_5 I_8 + I_8^2 - 2J_{11} (I_5 + J_{10} I_5 + 2I_8)) - (((-1 + 5J_{10}) J_{11}^2 \\
&\quad - J_1 (-1 + J_{10}) J_{10} (2J_{11} - I_5 - I_8) - 2J_{10} J_{11} (I_5 + 2J_{10} I_5 + I_8) \\
&\quad + J_{10} ((-1 + J_{10})^2 J_2 + I_6 + 2J_{10}^2 I_6 + I_5 I_8 \\
&\quad + J_{10} (-3I_6 + I_5 (2I_5 + I_8) - I_9) + I_9))^2) / (J_{10}^2) / (4(-1 + J_{10})^2).
\end{aligned} \tag{24}$$

Similarly, substituting the expressions of C_{12}^2 , C_{13}^2 , C_{23}^2 , and $C_{12} C_{13} C_{23}$, given in (19), into $A_2 \triangleq (C_{12} C_{13} C_{23})^2 - C_{12}^2 C_{13}^2 C_{23}^2$, we obtain

$$\begin{aligned}
A_2 &= (J_{11} - J_{10} I_5)^2 ((-1 + 9J_{10}) J_{11}^4 - 4J_{10} J_{11}^3 (I_5 + I_8) \\
&\quad + J_1^2 (-1 + J_{10}) J_{10}^2 (-2J_{11} + I_5 + I_8)^2 + 2J_{10} J_{11}^2 ((-1 + J_{10}) (-1 + 3J_{10}) J_2 \\
&\quad - (-1 + 5J_{10}) (I_6 + I_5 I_8 + I_9)) + 4J_{10}^2 J_{11} (I_5^2 I_8 - (-1 + J_{10}) J_2 (I_5 + I_8) \\
&\quad + I_8 (I_6 + 2J_{10} I_6 + I_9) + I_5 (I_6 + I_8^2 + I_9 + 2J_{10} I_9)) + J_{10}^2 ((-1 + J_{10})^3 J_2^2 \\
&\quad + (-1 + J_{10}) I_6^2 + (-1 + J_{10}) I_5^2 I_8^2 - 2I_5 (I_8 + J_{10} (2I_5 + I_8)) I_9 + (-1 + J_{10}) I_9^2 \\
&\quad - 2(-1 + J_{10}) J_2 ((-1 + J_{10}) I_6 - (1 + J_{10}) I_5 I_8 + (-1 + J_{10}) I_9))
\end{aligned}$$

$$\begin{aligned}
& -2I_6(I_8(I_5 + J_{10}I_5 + 2J_{10}I_8) + I_9 + J_{10}(-3 + 2J_{10})I_9)) \\
& -2J_1J_{10}(2J_{11} - I_5 - I_8)((-1 + 3J_{10})J_{11}^2 - 2J_{10}J_{11}(I_5 + I_8) + J_{10}((-1 + J_{10})^2J_2 \\
& + I_6 + I_5I_8 + I_9 - J_{10}(I_6 - I_5I_8 + I_9))) / (4(-1 + J_{10})^3J_{10}^4). \quad (25)
\end{aligned}$$

By definition, however, both A_1 and A_2 are related to each other through the expression $(J_{11} - J_{10}I_5)^2A_1 = -J_{10}^2(-1 + J_{10})^2A_2$, where we have used the expressions of b_1^2 from (19) and C_{12} in (21). It is then clear that only one relation between the invariants can be obtained from both (24) and (25).

On the other hand, Shariff and Bustamante [1] have obtained three relations, given by (30), (34), and (39), between the invariants in the set (3) of their work. These invariants correspond to the invariants listed in (7) of our article. The authors claim that these three relations can be used to reduce the number of invariants in (3) of their work from ten to seven, or, from nine to six if their ninth invariant, I_9 , is known. We show below that these relations can only give two independent invariants. The third invariant is precisely J_3 , given by (23) above.

- (a) Their relation (30) corresponds to the polynomial relation (8) in our article, yielding, for instance, $(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{C}^2\mathbf{b})$ in terms of the other invariants.
- (b) Their relation (34) is not correct. Its right-hand side should be multiplied by $(1 - I_9)^2I_9^2$ in their notation. Passing the correct right-hand side to the left-hand side of the relation (34) and denoting by B_1 the resulting left-hand side, we find that $B_1 = A_1R_1$, where A_1 is given by the expression (24) above and R_1 is a polynomial of the invariants, which, in general, does not vanish. This means that both A_1 and B_1 yield the same syzygy.
- (c) Passing the right-hand side of their relation (39) to the left-hand side and denoting by B_2 the resulting left-hand side, we find that $B_2 = 4A_2$, where A_2 is given by the expression (25) above. We also find that B_1 and B_2 are related by the expression

$$-4(J_{11} - J_{10}I_5)^2B_1 / (J_{10}^2(-1 + J_{10})^2) = B_2R_1, \quad (26)$$

where R_1 is the polynomial that appears in Item b) above. By definition, both sides of (26) must be zero. Since, in general, both $J_{11} - J_{10}I_5$ and R_1 do not vanish, we must have that an invariant that satisfies $B_1 = 0$ must also satisfy $B_2 = 0$ and conversely. Therefore, B_1 and B_2 yield one independent invariant, provided that J_3 is known from (23) above.

To conclude this remark, we still need a set of *seven* invariants, given by $\{J_1, J_2, I_5, I_6, I_8, I_9, J_{11}\}$, to characterize the strain energy function of an anisotropic elastic material with two non-orthogonal symmetry directions. Similarly to the case of orthogonal directions, these invariants satisfy only one syzygy, given by either $A_1 = 0$ or $A_2 = 0$. Recall from above that Shariff and Bustamante [1] claim the existence of two syzygies, given by both $B_1 = 0$ and $B_2 = 0$.

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