Corrigendum to A Krein space approach to PT symmetry

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We comment on some inaccuracies in the above paper ([I]); they do not affect the applications to PT -symmetric Hamiltonians given therein.

It is necessary to modify two assumptions. The first is required in Corollary 3.2 if the spectrum of the family A_ε , $0 \leq \varepsilon \leq 1$, does not only consist of eigenvalues. The second concerns Corollary 3.4 and Theorem 3.5, where it has to be assumed that A_0 is also self-adjoint (or, more generally, normal) with respect to some Hilbert space inner product $(\cdot, \cdot)$ on $\mathcal{K}$ (compare [II]).

The application in Section 4 of [I] to the operator $-d^2/dx^2 + V$ with PT -symmetric potential V is not affected by these alterations; in this case all the above assumptions are satisfied.

For the convenience of the reader, we do not only point out the modifications, but we repeat the full correct statements of the respective theorems. We begin with Corollary 3.2 in [I] where the last sentence should be modified as follows:

Corollary 3.2 *Let A_ε , $0 \leq \varepsilon \leq 1$, be a family of self-adjoint operators in a Krein space $\mathcal{K}$ such that for one (and hence for all) $z \in \rho(A_\varepsilon)$ the resolvents $(A_\varepsilon - z)^{-1}$ depend continuously on ε in the uniform operator topology for $0 \leq \varepsilon \leq 1$. Let λ_0 be an isolated real eigenvalue of A_0 with a one-dimensional algebraic eigenspace. Denote by $\lambda(\varepsilon)$, $0 \leq \varepsilon \leq \varepsilon_0$, the corresponding eigenvalue of A_ε which depends continuously on ε and is such that $\lambda(0) = \lambda_0$. If for $0 \leq \varepsilon \leq \varepsilon_0$ the eigenvalue $\lambda(\varepsilon)$ does not meet any other point of $\sigma(A_\varepsilon)$, then $\lambda(\varepsilon)$ is real and algebraically simple and of the same (positive or negative) type as λ_0 for all $0 \leq \varepsilon \leq \varepsilon_0$.*

The assumption in [I] that $\lambda(\varepsilon)$ does not meet any other eigenvalue of A_ε is only sufficient if the whole spectrum of A_ε consists of eigenvalues (which was not assumed there).

In Corollary 3.4 and in Theorem 3.5, the assumption that A_0 is self-adjoint (or, more generally, normal) with respect to some Hilbert space inner product $(\cdot, \cdot)$ on $\mathcal{K}$ has to be added; this is necessary to guarantee the estimate

$$\|(A_0 - z)^{-1}\| \leq \frac{1}{\text{dist}(z, \sigma(A_0))},$$

which is used in the proof. As a consequence, the sentence following Corollary 3.4 should read: “The above corollary has also been proved in [II, Theorem 1.2].” The correct formulations of Corollary 3.4 and Theorem 3.5 are as follows:

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Corollary 3.4 *Let A_0 be a self-adjoint operator in the Krein space $\mathcal{K}$, which is also self-adjoint with respect to some Hilbert space inner product $(\cdot, \cdot)$ on $\mathcal{K}$, with discrete real spectrum consisting of eigenvalues $\lambda_1^0 < \lambda_2^0 < \dots$ of definite type (the latter holds, e.g., if the eigenvalues are algebraically simple). If V is a bounded self-adjoint operator in $\mathcal{K}$ such that*

$$\|V\| < \frac{1}{2} \inf_{j=1,2,\dots} (\lambda_{j+1}^0 - \lambda_j^0),$$

then the spectrum of $A_0 + V$ is real and consists of simple eigenvalues of definite type.

Theorem 3.5 *Let A_0 be a self-adjoint operator in a Krein space $\mathcal{K}$, which is also self-adjoint with respect to some Hilbert space inner product $(\cdot, \cdot)$ on $\mathcal{K}$. Suppose that the spectrum of A_0 consists of a sequence of real eigenvalues $\lambda_1^0 < \lambda_2^0 < \dots$ which are all algebraically simple and of alternating type. Suppose further that the distances $\delta_j := \lambda_{j+1}^0 - \lambda_j^0$, $j = 1, 2, \dots$, form an increasing sequence: $\delta_{j+1} > \delta_j$. If for some $k \in \mathbb{N}$*

$$\|V\| < \frac{1}{2} \delta_k,$$

then for the operator $A_1 = A_0 + V$ the number of non-real eigenvalues, counted with multiplicities, plus the number of critical eigenvalues, also counted with multiplicities, is at most k . Moreover, with the exception of non-real eigenvalues or real eigenvalues $< \lambda_{k+1}^0$ of total multiplicity k , the spectrum of A_1 consists of a sequence of real eigenvalues λ_j , $j = k+1, k+2, \dots$, such that $|\lambda_j - \lambda_j^0| \leq \|V\|$, and λ_j is of the same type as λ_j^0 for $j = k+1, k+2, \dots$.

Note that in the assumptions about the perturbation V , the operator norm is the one which is generated by the Hilbert space norm $\|x\| = (x, x)^{1/2}$ on $\mathcal{K}$.

Moreover, the inequality in the relation $|\lambda_j - \lambda_j^0| \leq \|V\|$ is not strict (as in [I, Theorem 3.5]). Accordingly, in Theorem 4.1 it has to read $|\lambda_j - j\pi^2/4| \leq \|V\|_\infty$.

References

- [I] H. Langer and C. Tretter: Czech. J. Phys. **54** (2004) 1113.
- [II] E. Caliceti, S. Graffi, and J. Sjöstrand: J. Phys. A **38** (2005) 185. ¹⁾

¹⁾ Updated reference [2] in [I].