Topological bifurcations of spatial central configurations in the N -body problem

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Abstract We study topological bifurcations of classes of spatial central configurations (s.c.c.) from the following highly symmetrical families: two nested regular tetrahedra, octahedra and cubes, two nested rotated regular tetrahedra and two dual regular polyhedra for 14 bodies. We prove the existence of local and global topological bifurcations of s.c.c. from these families. We seek new classes of s.c.c. by using equivariant bifurcation theory. It is worth pointing out that the shapes of the bifurcating families are less symmetrical than the shapes of the considered families of s.c.c.

Keywords N -body problem · Spatial central configurations · Bifurcations of critical orbits · Equivariant potentials · Nested polyhedra

1 Introduction

One of the most important problems of Celestial Mechanics is the problem of classification of all central configurations (c.c.) in the N -body problem. It has a long history and has been studied by various mathematicians. Collinear configurations for 3 bodies (see [Euler 1767](#)) discovered by Euler and an equilateral triangle with arbitrary positive masses at the vertices (see [Lagrange 1772](#)) found by Lagrange are the first known c.c. Examples of c.c. provide us with highly symmetrical configurations with restrictions on the masses. By using the symmetries of configurations and imposing restrictions on the masses the system of equations for c.c. can be reduced to the simpler one. We cannot find in the literature many of c.c. with an irregular shape, there are only a few non-symmetric ones known in the history of this problem. For example, [Meyer and Schmidt \(1988a, b\)](#) proved numerically the existence of bifurcations of less symmetrical families from highly symmetrical ones and then gave us the examples of c.c. with poorer symmetries.

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In our paper we consider the following highly symmetrical families:

- a spatial central configuration of two nested regular tetrahedra, octahedra and cubes studied by [Corbera and Llibre \(2008\)](#),
- a spatial central configuration of two nested rotated regular tetrahedra studied by [Corbera and Llibre \(2009\)](#),
- a spatial central configuration of two dual regular polyhedra for 14 bodies studied by [Corbera et al. \(2014\)](#).

Note that the studied families consist of non-collinear s.c.c. It is known that the isotropy group of any non-collinear spatial central configuration is trivial. Moreover, the collinear c.c. are non-degenerate as critical orbits of an appropriate potential (see [Pacella 1987](#)) and thus there are no topological bifurcations from these c.c. [see Theorem 3.1].

The existence of symmetric central configurations in general was considered in the recent paper of [Montaldi \(2015\)](#) where the uniform approach was given by using some variational methods in symmetric case. The principal conclusion of it is the existence of a central configuration of any symmetry type and for any symmetric distribution of masses. In particular, Montaldi proved the existence of families of s.c.c. mentioned above for any symmetric choice of masses. We study topological bifurcations from those families and use the precise formulas for positions and masses provided in the above listed papers of Corbera, Llibre and Pérez-Chavela.

There are numerous reasons for studying c.c., the most important ones are the following: c.c. generate the unique explicit solutions in the N -body problem; planar central configurations (p.c.c.) also give rise to families of periodic solutions; c.c. are important in analysing total collision orbits (see [Saari 1980](#); [Wintner 1941](#)); the hypersurfaces of constant energy in a level set of the angular momentum change their topology exactly at the energy level sets which contain c.c. (see [Smale 1970](#)).

C.c. are invariant under homotheties and rotations because of the homogeneity of the potential and the fact that the potential depends only on the mutual distances between the particles and not on their positions. Besides, Newton's equations are of gradient nature which allows us to apply methods which benefit from this.

In [Kowalczyk \(2015\)](#) we used equivariant versions of classical topological invariants such as the degree for equivariant gradient maps (see [Balanov et al. 2006](#); [Balanov et al. 2010](#); [Gęba 1997](#); [Rybicki 2005](#)) and the equivariant Conley index (see [Bartsch 1993](#); [Floer 1987](#); [Gęba 1997](#)) to formulate abstract theorems giving necessary and sufficient conditions for the existence of topological bifurcations (local and global ones) in the vicinity of a given family (a so-called trivial family) of solutions of equivariant gradient equations [see Theorems 3.1, 3.2 and 3.3 of [Kowalczyk \(2015\)](#) in the case of global bifurcation and Theorems 3.1, 3.5 and 3.6 of [Kowalczyk \(2015\)](#) for local one]. So we proved some abstract results in equivariant bifurcation theory and applied them to the study of bifurcations of new families of p.c.c. [see Theorems 4.1 and 4.2 of [Kowalczyk \(2015\)](#)]. This approach using equivariant theory has been used in [Maciejewski and Rybicki \(2004\)](#) and [Pérez-Chavela and Rybicki \(2013\)](#). Moreover, we showed that the new families are less symmetrical than those trivial ones.

Now, applying the above-mentioned theorems, we shall study classes of s.c.c. which are treated as $SO(3)$ -orbits of critical points of $SO(3)$ -invariant potentials. Similarly as in [Kowalczyk \(2015\)](#), we shall also address the shapes of the found families of s.c.c. and thus we prove that the bifurcating families have poorer symmetries or less regular shapes. Namely, we exclude that the bifurcating families of s.c.c. have the same type of symmetry as the trivial ones. We emphasise that the information about the shapes of new families of s.c.c. is only local.

It is worth pointing out that in the planar case the action of the symmetry group $SO(2)$ is free, so a natural thing to consider is the quotient space $\Omega/SO(2)$ of the action of the group $SO(2)$ on the configuration space Ω (see [Lei and Santoprete 2006](#); [Meyer 1987](#); [Meyer and Schmidt 1988a, b](#)). In the spatial case, the action of the symmetry group $SO(3)$ is not free, i.e. there is a non-trivial isotropy group appearing in the configuration space. On the other hand, the only s.c.c. which provide non-trivial isotropy groups are collinear and, as we mentioned above, there are no topological bifurcations from the families of collinear s.c.c. Thus we get that the quotient space $\Omega/SO(3)$ of the action of the group $SO(3)$ on Ω is a manifold if we exclude collinear s.c.c. from the configuration space. But Theorem 3.2 of [Kowalczyk \(2015\)](#) which we apply to seek bifurcations (global ones) was proved for orthogonal representations. We do not have the corresponding global bifurcation theorem on manifolds and so we study Ω instead of the quotient space $\Omega/SO(3)$ to study the existence of global bifurcations.

This paper is organised as follows: in Sect. 2 we introduce the spatial N -body problem and investigate s.c.c. as the $SO(3)$ -orbits of solutions of the equation

$$\nabla_q \varphi(q, \rho) = 0. \quad (1.1)$$

Next, we apply our abstract results of [Kowalczyk \(2015\)](#) to the spatial N -body problem and we prove the existence of topological bifurcations from the following families of s.c.c.: (2.4), (2.5), (2.6), (2.7) and (2.8). Tedious computations of this section are aided by the algebraic processor MAPLE™. Section 3 contains some facts of equivariant topology and theorems giving necessary and sufficient conditions for the existence of local and global bifurcations from a given family of s.c.c.

It is worth pointing out that in this paper we study topological bifurcations [local and global ones, see Definition 3.1]. Namely, in the case of local ones, we consider known family of solutions of the Eq. (1.1) (the so-called trivial family of solutions) and seek non-trivial solutions nearby. In the case of global ones, we seek connected sets of non-trivial solutions nearby the trivial ones which satisfy some additional condition [see Definition 3.1(2)]. The notion of a topological bifurcation is different from the notion of a bifurcation in the sense of a change in the number of solutions of the Eq. (1.1). These two definitions, the topological bifurcation and the bifurcation in the sense of a change in the number of solutions, are independent. So, in other words, the first one can occur while the second one does not occur and inversely. To shorten the notation, throughout this paper, we will write bifurcations instead of topological bifurcations.

2 Bifurcations of spatial central configurations

In this section we present the existence of new families of s.c.c. which bifurcate from certain known families of s.c.c. We consider N bodies of positive masses $m_1, \dots, m_N$ in the 3-dimensional Euclidean vector space, whose positions are denoted by $q_1, \dots, q_N \in \mathbb{R}^3$. We define the configuration space $\Omega \subset \mathbb{R}^{3N}$ as follows

$$\Omega = \{q = (q_1, \dots, q_N) \in \mathbb{R}^{3N} : q_i \neq q_j \text{ for } i \neq j\}.$$

By $SO(3)$ we will understand the group of special orthogonal matrices of order 3. From now on we will treat the space $\mathbb{R}^{3N}$ as an $SO(3)$ -representation $\mathbb{V}$, i.e. a representation of the Lie group $SO(3)$ which is a direct sum of N copies of the natural orthogonal $SO(3)$ -representation (i.e. the usual multiplication of a vector by a matrix). Moreover, $\mathbb{R}$ denotes the one-dimensional trivial representation of $SO(3)$, i.e. $\mathbb{R} = (\mathbb{R}, \mathbf{1})$, where $\mathbf{1}(g) = 1$ for any $g \in SO(3)$. The action of $SO(3)$ on $\mathbb{V} \times \mathbb{R}$ is given by

$$\begin{aligned} SO(3) \times (\mathbb{V} \times \mathbb{R}) \ni (g, (q, \rho)) &= (g, (q_1, \dots, q_N, \rho)) \\ \mapsto g \cdot (q, \rho) &= (g \cdot q_1, \dots, g \cdot q_N, \rho) \in \mathbb{V} \times \mathbb{R}. \end{aligned}$$

By $q \in \mathbb{V}$ we mean $q \in \mathbb{R}^{3N}$ and, for simplicity of notation, we write gq instead of $g \cdot q$. If $q \in \mathbb{V}$ then $SO(3)_q = \{g \in SO(3) : gq = q\}$ is the isotropy group of q and the set $SO(3)(q) = \{gq : g \in SO(3)\} \subset \mathbb{V}$ is called the $SO(3)$ -orbit through q .

The equations of motion of the spatial N -body problem are given by

$$m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}(q, m), \quad j = 1, \dots, N \quad (2.1)$$

where the Newtonian potential $U : \Omega \times (0, +\infty)^N \rightarrow \mathbb{R}$ is defined by

$$U(q, m) = U(q_1, \dots, q_N, m_1, \dots, m_N) = \sum_{1 \leq i < j \leq N} \mathbf{G} \frac{m_i m_j}{|q_i - q_j|}$$

and without loss of generality the gravitational constant $\mathbf{G}$ can be taken equal to one. We will use the symbol $\nabla_q U$ to denote the gradient of U with respect to the first coordinate.

Definition 2.1 A configuration $q = (q_1, \dots, q_N) \in \Omega$ is a central configuration of the system (2.1) if there exists a positive constant λ such that $\ddot{q} = -\lambda q$.

Equivalently, the following condition is fulfilled:

$$-\lambda \nabla_q I(q, m) = \nabla_q U(q, m) \quad (2.2)$$

where $I : \Omega \times (0, +\infty)^N \rightarrow \mathbb{R}$ given by the formula $I(q, m) = 1/2 \sum_{j=1}^N m_j |q_j|^2$ is the moment of inertia. Note that the centre of mass of the bodies is at the origin of the coordinate chart for any solution of the Eq. (2.2). Moreover, one can show that $\lambda = U(q, m)/(2I(q, m))$.

First, observe that the set $\Omega \subset \mathbb{V}$ is open and $SO(3)$ -invariant, i.e. for any $q \in \Omega$ we have $SO(3)(q) \subset \Omega$. Moreover, the potential U is $SO(3)$ -invariant map of class C^∞ , i.e. $U(gq, m) = U(q, m)$ for any $g \in SO(3)$ and $(q, m) \in \Omega \times (0, +\infty)^N$. In this approach our problem of studying s.c.c. becomes a problem of studying critical $SO(3)$ -orbits of a smooth $SO(3)$ -invariant potential $\hat{\varphi} : \Omega \times (0, +\infty)^N \rightarrow \mathbb{R}$ given by $\hat{\varphi}(q, m) = U(q, m) + \lambda I(q, m)$. Additionally, assume there exist continuous maps $w : \mathbb{R} \rightarrow \Omega$ and $m : \mathbb{R} \rightarrow (0, +\infty)^N$ such that $\nabla_q \hat{\varphi}(w(\rho), m(\rho)) = 0$. Now, define a potential $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by $\varphi(q, \rho) = \hat{\varphi}(q, m(\rho))$. Thus we will consider the following equation:

$$\nabla_q \varphi(q, \rho) = 0, \quad (2.3)$$

where $\lambda = \lambda(\rho) = U(w(\rho), m(\rho))/(2I(w(\rho), m(\rho)))$. Because of $SO(3)$ -equivariance of $\nabla_q \varphi$, i.e. $\nabla_q \varphi(gq, \rho) = g \nabla_q \varphi(q, \rho)$ for any $g \in SO(3)$ and $(q, \rho) \in \Omega \times \mathbb{R}$, if there exists a point $(q_0, \rho_0) \in (\nabla_q \varphi)^{-1}(0)$ then $SO(3)(q_0) \times \{\rho_0\} \subset (\nabla_q \varphi)^{-1}(0)$, so

$$\mathcal{F} = \bigcup_{\rho \in \mathbb{R}} SO(3)(w(\rho)) \times \{\rho\} \subset (\nabla_q \varphi)^{-1}(0).$$

The family $\mathcal{F}$ is called the trivial family of solutions of the Eq. (2.3) and for any subset $X \subset \mathbb{R}$ let $\mathcal{F}_X$ denote the set of $SO(3)$ -orbits $\bigcup_{\rho \in X} SO(3)(w(\rho)) \times \{\rho\} \subset \mathcal{F}$.

Now, we will apply Theorems 3.1 and 3.2 to prove the existence of bifurcations from the following families of s.c.c. $w : (2.4)$, (2.5), (2.6), (2.7) and (2.8). First, observe that, according to Lemma 3.1, for any parameter ρ we have $\dim \ker \nabla_q^2 \varphi(w(\rho), \rho) \geq \dim SO(3)(w(\rho))$. Additionally, parameters which satisfied the necessary condition for the existence of local bifurcation, given by Theorem 3.1, are the ones for which the strict inequality holds. Next,

Fig. 1 Central configurations of two nested regular polyhedra. **a** Eight bodies in nested regular tetrahedra configuration. **b** Twelve bodies in nested regular octahedra configuration. **c** Sixteen bodies in nested regular cubes configuration

we will use Theorem 3.2 giving the sufficient conditions for the existence of local and global bifurcations. Note that for families w considered in this paper, for any parameter ρ , we have $SO(3)_{w(\rho)} = \{I\}$, where by the symbol I we denote the identity matrix. Thus the matrix $\mathcal{C}(w(\rho))$ [see a special form of $\nabla_q^2 \varphi(w(\rho), \rho)$ given by the formula (3.1)] is zero-dimensional and in particular, its Morse index is equal to zero, i.e. the number of negative eigenvalues (counting multiplicities) of the matrix $\mathcal{C}(w(\rho))$. Therefore for any $\rho \in \mathbb{R}$ we have $m^-(\mathcal{C}(w(\rho))) = 0$ and $m^-(\nabla_q^2 \varphi(w(\rho), \rho)) = m^-(B(w(\rho)))$ where by the symbol $m^-(\mathcal{A})$ we denote the Morse index of a symmetric matrix $\mathcal{A}$.

First, we consider the case of 8 bodies and address bifurcations from the families of s.c.c. which were studied by Corbera and Llibre (2009). This type of families consists of 8 bodies with the masses located at the vertices of two nested regular tetrahedra, where the masses on each tetrahedron are equal but the masses on different tetrahedra could be different. They have proved that there are only two different classes of s.c.c. of this type, either when one of the tetrahedra is homothetic to the other one (Type I) or when one of the tetrahedra is rotated with a rotation matrix of Euler angles $\alpha = \gamma = 0$ and $\beta = \pi$ (Type II).

Firstly, we will show bifurcations from the family of Type I (see Fig. 1a), which was also studied by these authors earlier (see Corbera and Llibre 2008). There is worth noting that Nezhinskii also studied this kind of s.c.c., i.e. he constructed the central configuration of $1 + 4l$ bodies consisting of l nested regular tetrahedra and an additional particle at the centre of mass (see Nezhinskii 1975). Now, we consider 8 bodies with the following positions:

$$\begin{aligned} \hat{q}_1 &= \left(0, 0, \sqrt{\frac{3}{2}}\right), \quad \hat{q}_2 = \left(0, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), \quad \hat{q}_3 = \left(1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), \quad \hat{q}_4 = \left(-1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), \\ \hat{q}_5 &= \rho \hat{q}_1, \quad \hat{q}_6 = \rho \hat{q}_2, \quad \hat{q}_7 = \rho \hat{q}_3, \quad \hat{q}_8 = \rho \hat{q}_4, \end{aligned}$$

where ρ is a scale factor. Let $m_1 = m_2 = m_3 = m_4 = 1$ and $m_5 = m_6 = m_7 = m_8 = m_I(\rho) = n(\rho)/d(\rho)$ [the formulas for $n(\rho)$ and $d(\rho)$ are given in Theorem 2.(b) of Corbera and Llibre (2009)]. We define $w : (\alpha, +\infty) \rightarrow \Omega$ by the formula

$$w(\rho) = (\hat{q}_1, \dots, \hat{q}_8), \quad (2.4)$$

where scale factor $\rho > \alpha = 1.8899915758\dots$ is treated as a parameter. Next, a map $m : (\alpha, +\infty) \rightarrow (0, +\infty)^8$ is given as follows

$$m(\rho) = (1, 1, 1, 1, m_I(\rho), m_I(\rho), m_I(\rho), m_I(\rho)).$$

Lemma 2.1 Put $\rho_1 = 6$, $\rho_2 = 7$, $\rho_3 = 12$ and $\rho_4 = 13$. Then $\dim \ker \nabla_q^2 \varphi(w(\rho_i), \rho_i) = \dim SO(3)(w(\rho_i)) = 3$ for $i = 1, 2, 3, 4$ and the Morse index of $\nabla_q^2 \varphi(w(\cdot), \cdot)$ evaluated at ρ_i is

$$m^-(\nabla_q^2 \varphi(w(\rho_i), \rho_i)) = \begin{cases} 8, & \text{for } i = 1 \\ 5, & \text{for } i = 2, 3 \\ 3, & \text{for } i = 4 \end{cases}.$$

Proof Let us outline the main ideas of the proof. We first show that $\dim \ker \nabla_q^2 \varphi(w(\rho_i), \rho_i) = \dim SO(3)(w(\rho_i)) = 3$ for $i = 1, 2, 3, 4$. To do this, we calculate the Hessian $\nabla_q^2 \varphi$ of the potential φ and its characteristic polynomial W_ρ along the curve w . Denote the symbolic form of the latter by $W_\rho(x) = x^{24} - a_1(\rho)x^{23} + \cdots - a_{23}(\rho)x + a_{24}(\rho)$. That $\dim \ker \nabla_q^2 \varphi(w(\rho), \rho) \geq \dim SO(3)(w(\rho)) = 3$ follows from Lemma 3.1. Since $\dim \ker \nabla_q^2 \varphi(w(\rho), \rho) = k$ if and only if $a_{24}(\rho) = \cdots = a_{24-k+1}(\rho) = 0$ and $a_{24-k}(\rho) \neq 0$, it is sufficient to show that $a_{21}(\rho_i) \neq 0$ for $i = 1, 2, 3, 4$. We do not provide the coefficients $a_{21}(\rho_i)$ because they are too large. It remains to compute the Morse indices $m^-(\nabla_q^2 \varphi(w(\rho_i), \rho_i))$ for $i = 1, 2, 3, 4$, i.e. the number of negative eigenvalues (counting multiplicities), which completes the proof. $\square$

Theorem 2.1

- (1) *There exists a connected set of s.c.c. bifurcating from the family (2.4) from the segment (ρ_1, ρ_2) , i.e. there exists a global bifurcation parameter in the segment $(\rho_1, \rho_2) \cap \mathcal{GLOB} \neq \emptyset$. Moreover, the bifurcating families are locally less symmetrical.*
- (2) *There exists a sequence of s.c.c. bifurcating from the family (2.4) from the segment (ρ_3, ρ_4) , i.e. there exists a local bifurcation parameter in the segment $(\rho_3, \rho_4) \cap \mathcal{BLF} \neq \emptyset$. Moreover, the bifurcating families are locally less symmetrical.*

Proof We first show that $(\rho_1, \rho_2) \cap \mathcal{BLF} \neq \emptyset$ and $(\rho_3, \rho_4) \cap \mathcal{BLF} \neq \emptyset$. In consequence of Lemma 2.1 and Theorem 3.2, since the numbers $m^-(\nabla_q^2 \varphi(w(\rho_1), \rho_1))$ and $m^-(\nabla_q^2 \varphi(w(\rho_2), \rho_2))$ are different, there exists a local bifurcation parameter in the segment (ρ_1, ρ_2) . Similarly, $m^-(\nabla_q^2 \varphi(w(\rho_3), \rho_3)) \neq m^-(\nabla_q^2 \varphi(w(\rho_4), \rho_4))$ once again implies the existence of a local bifurcation parameter in (ρ_3, ρ_4) . Furthermore, since the numbers $m^-(\nabla_q^2 \varphi(w(\rho_1), \rho_1))$ and $m^-(\nabla_q^2 \varphi(w(\rho_2), \rho_2))$ are of different parity, $(\rho_1, \rho_2) \cap \mathcal{GLOB} \neq \emptyset$ by Theorem 3.2.

What is left is to show that the bifurcating families are less symmetrical. To see this, notice that we can consider a subset of the full configuration space Ω which is invariant for the gradient flow, i.e. the set of s.c.c. of two nested tetrahedra type $(\hat{q}_1, \dots, \hat{q}_8, M, M, M, M, m, m, m, m)$ (for short), then studying c.c. in this set becomes a problem of studying zeros of a function $F : (\alpha, +\infty) \times (0, +\infty)^2 \rightarrow \mathbb{R}$ given by the formula $F(\rho, M, m) = Mn(\rho) - md(\rho)$ (see Corbera and Llibre 2008). For the trivial family of solutions (2.4), for any $\rho \in (\alpha, +\infty)$, we have $F(\rho, 1, n(\rho)/d(\rho)) = 0$ and since $n'(\rho)$ is negative and $d'(\rho)$ is positive for $\rho > 1$, we get $F'_\rho(\rho, 1, n(\rho)/d(\rho)) = n'(\rho) - (n(\rho)/d(\rho))d'(\rho) < 0$, so by the implicit function theorem there is no bifurcation of s.c.c. of two nested tetrahedra type from the trivial family. By the above, we obtain that the families which bifurcate from the family (2.4) are not of two nested tetrahedra type, which completes the proof. $\square$

Next, we will show bifurcations from the family of Type II (see Fig. 2), in which bodies are considered with the following positions:

$$\begin{aligned} \hat{q}_1 &= \left(0, 0, \sqrt{\frac{3}{2}}\right), & \hat{q}_2 &= \left(0, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), & \hat{q}_3 &= \left(1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), & \hat{q}_4 &= \left(-1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right), \\ \hat{q}_5 &= a \left(\frac{1}{3}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right), & \hat{q}_6 &= a \left(-\frac{1}{3}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right), & \hat{q}_7 &= a \left(0, -\frac{2}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right), & \hat{q}_8 &= a \left(0, 0, -\frac{1}{\sqrt{6}}\right), \end{aligned}$$

where a is a scale factor. This configuration was also studied by Corbera et al. (2014). So, let $m_1 = m_2 = m_3 = m_4 = 1$ and $m_5 = m_6 = m_7 = m_8 = m_{1I}(a) = -g(a)/f(a)$ [the

Fig. 2 Central configurations of two nested rotated regular tetrahedra. **a** $a \in (0, 1)$, **b** $a \in [1, 9]$, **c** $a \in (9, +\infty)$

formulas for $g(a)$ and $f(a)$ can be found in Theorem 11 of Corbera et al. (2014)]. We define $w : (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty) \rightarrow \Omega$ by the formula

$$w(a) = (\hat{q}_1, \dots, \hat{q}_8), \quad (2.5)$$

where $\alpha_1 = 2.145669 \dots$, $\alpha_2 = 19.60823 \dots$, $\beta_1 = 0.4589907 \dots$ and $\beta_2 = 4.194495 \dots$. Next, a map $m : (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty) \rightarrow (0, +\infty)^8$ is given as follows

$$m(a) = (1, 1, 1, 1, m_{II}(a), m_{II}(a), m_{II}(a), m_{II}(a)).$$

Lemma 2.2 Put $a_1 = 25/100$, $a_2 = 26/100$, $a_3 = 34/100$, $a_4 = 35/100$, $a_5 = 236/100$, $a_6 = 237/100$, $a_7 = 380/100$, $a_8 = 381/100$, $a_9 = 26$, $a_{10} = 27$, $a_{11} = 35$ and $a_{12} = 36$. Then $\dim \ker \nabla_q^2 \varphi(w(a_i), a_i) = \dim SO(3)(w(a_i)) = 3$ for $i = 1, \dots, 12$ and the Morse index of $\nabla_q^2 \varphi(w(\cdot), \cdot)$ evaluated at a_i is

$$m^-(\nabla_q^2 \varphi(w(a_i), a_i)) = \begin{cases} 4, & \text{for } i = 4, 9 \\ 3, & \text{for } i = 2, 3, 10, 11 \\ 2, & \text{for } i = 6, 7 \\ 0, & \text{for } i = 1, 5, 8, 12 \end{cases}.$$

To prove the above lemma we apply an analysis similar to that used in the proof of Lemma 2.1.

Theorem 2.2

- (1) There exists a sequence of s.c.c. bifurcating from the family (2.5) from the segments (a_1, a_2) , (a_3, a_4) , (a_5, a_6) , (a_7, a_8) , (a_9, a_{10}) and (a_{11}, a_{12}) , i.e. $(a_i, a_{i+1}) \cap \mathcal{BLF} \neq \emptyset$ for $i = 1, 3, 5, 7, 9, 11$. Moreover, the families which bifurcate from the segments (a_1, a_2) , (a_5, a_6) , (a_7, a_8) and (a_{11}, a_{12}) are locally less symmetrical.
- (2) There exists a connected set of s.c.c. bifurcating from the family (2.5) from the segments (a_1, a_2) , (a_3, a_4) , (a_9, a_{10}) and (a_{11}, a_{12}) , i.e. $(a_i, a_{i+1}) \cap \mathcal{GLOB} \neq \emptyset$ for $i = 1, 3, 9, 11$. Moreover, the families which bifurcate from the segments (a_1, a_2) and (a_{11}, a_{12}) are locally less symmetrical.

Proof To prove this theorem we can proceed analogously to the proof of Theorem 2.1. Let us first show that $(a_i, a_{i+1}) \cap \mathcal{BLF} \neq \emptyset$ for $i = 1, 3, 5, 7, 9, 11$. By Lemma 2.2 and Theorem 3.2, since the numbers $m^-(\nabla_q^2 \varphi(w(a_1), a_1))$ and $m^-(\nabla_q^2 \varphi(w(a_2), a_2))$ are different, we provided a local bifurcation from the segment (a_1, a_2) . By a similar argument, there exists a local bifurcation parameter in (a_3, a_4) , (a_5, a_6) , (a_7, a_8) , (a_9, a_{10}) and (a_{11}, a_{12}) . Moreover, since some differences of Morse indices are odd numbers, a global bifurcation from the segments (a_1, a_2) , (a_3, a_4) , (a_9, a_{10}) and (a_{11}, a_{12}) occurs.

It remains to prove that the bifurcating families are less symmetrical. To do this, notice that we can consider a subset of the full configuration space Ω which is invariant for the gradient flow, i.e. the set of s.c.c. of two nested rotated tetrahedra type

$(\hat{q}_1, \dots, \hat{q}_8, M, M, M, M, m, m, m, m)$ ((a, M, m) for short), then studying c.c. in this set becomes a problem of studying zeros of a function $F : ((0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty)) \times (0, +\infty)^2 \rightarrow \mathbb{R}$ given by the formula $F(a, M, m) = Mg(a) + mf(a)$ (see Corbera et al. 2014). For the trivial family of solutions (2.5), for any $a \in (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty)$, we have $F(a, 1, -g(a)/f(a)) = 0$. Additionally, for any $a \in (a_1, a_2) \cup (a_5, a_6) \cup (a_7, a_8) \cup (a_{11}, a_{12})$, we obtain numerically that $F'_a(a, 1, -g(a)/f(a)) = g'(a) - (g(a)/f(a))f'(a) < 0$, so by the implicit function theorem there is no bifurcation of s.c.c. of two nested rotated tetrahedra type from these segments. From what has already been proved, we conclude that the families which bifurcate from the family (2.5) from those segments are not of two nested rotated tetrahedra type and the proof is complete. $\square$

Further, we study bifurcations from the family of spatial configurations of the 12-body problem where the masses lie at the vertices of two nested regular octahedra (see Fig. 1b). The masses on each octahedron are equal but the masses on different octahedra could be different. This configuration was analysed by Corbera and Llibre (2008). Note that similar family of s.c.c. was also described by Nezhinskii in 1975. He constructed the central configuration of $1 + 6l$ bodies consisting of l nested regular octahedra and an additional particle at the centre of mass (see Nezhinskii 1975). Now, we consider 12 bodies with the following positions:

$$\begin{aligned} \hat{q}_1 &= (1, 0, 0), & \hat{q}_2 &= (-1, 0, 0), & \hat{q}_3 &= (0, 1, 0), & \hat{q}_4 &= (0, -1, 0), & \hat{q}_5 &= (0, 0, 1), & \hat{q}_6 &= (0, 0, -1), \\ \hat{q}_7 &= \rho \hat{q}_1, & \hat{q}_8 &= \rho \hat{q}_2, & \hat{q}_9 &= \rho \hat{q}_3, & \hat{q}_{10} &= \rho \hat{q}_4, & \hat{q}_{11} &= \rho \hat{q}_5, & \hat{q}_{12} &= \rho \hat{q}_6, \end{aligned}$$

where ρ is a scale factor. Let $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$ and $m_7 = m_8 = m_9 = m_{10} = m_{11} = m_{12} = f_{12}(\rho) = b(\rho)/f(\rho)$ [the formulas for $b(\rho)$ and $f(\rho)$ can be found in Proposition 3.(a) of Corbera and Llibre (2008)]. We define $w : (\alpha, +\infty) \rightarrow \Omega$ by the formula

$$w(\rho) = (\hat{q}_1, \dots, \hat{q}_{12}), \quad (2.6)$$

where $\rho > \alpha = 1.7298565115043054 \dots$. Next, a map $m : (\alpha, +\infty) \rightarrow (0, +\infty)^{12}$ is given as follows

$$m(\rho) = (1, 1, 1, 1, 1, 1, f_{12}(\rho), f_{12}(\rho), f_{12}(\rho), f_{12}(\rho), f_{12}(\rho), f_{12}(\rho)).$$

Lemma 2.3 Put $\rho_1 = 3$, $\rho_2 = 31/10$, $\rho_3 = 36/10$, $\rho_4 = 37/10$, $\rho_5 = 7$ and $\rho_6 = 71/10$. Then $\dim \ker \nabla_q^2 \varphi(w(\rho_i), \rho_i) = \dim SO(3)(w(\rho_i)) = 3$ for $i = 1, \dots, 6$ and the Morse index of $\nabla_q^2 \varphi(w(\cdot), \cdot)$ evaluated at ρ_i is

$$m^-(\nabla_q^2 \varphi(w(\rho_i), \rho_i)) = \begin{cases} 12, & \text{for } i = 1 \\ 9, & \text{for } i = 2, 3 \\ 6, & \text{for } i = 4, 5 \\ 3, & \text{for } i = 6 \end{cases}.$$

Theorem 2.3 There exists a connected set of s.c.c. bifurcating from the family (2.6) from the segments (ρ_1, ρ_2) , (ρ_3, ρ_4) and (ρ_5, ρ_6) , i.e. there exists a global bifurcation parameter in the segments (ρ_1, ρ_2) , (ρ_3, ρ_4) and (ρ_5, ρ_6) $((\rho_i, \rho_{i+1}) \cap \mathcal{GLOB} \neq \emptyset$ for $i = 1, 3, 5)$. Moreover, the bifurcating families are locally less symmetrical.

Proof We begin by proving that $(\rho_i, \rho_{i+1}) \cap \mathcal{GLOB} \neq \emptyset$ for $i = 1, 3, 5$ by the same method as in the proof of Theorem 2.1. Using Lemma 2.3 and Theorem 3.2, since the Morse indices $m^-(\nabla_q^2 \varphi(w(\rho_1), \rho_1))$ and $m^-(\nabla_q^2 \varphi(w(\rho_2), \rho_2))$ differ by an odd number, we prove

the existence of a global bifurcation from the segment (ρ_1, ρ_2) . By a similar argument, we get a global bifurcation from the segments (ρ_3, ρ_4) and (ρ_5, ρ_6) .

The proof is completed by showing that the bifurcating families are less symmetrical. To do this, notice that we can consider a subset of the full configuration space Ω which is invariant for the gradient flow, i.e. the set of s.c.c. of two nested octahedra type $(\hat{q}_1, \dots, \hat{q}_{12}, M, M, M, M, M, M, m, m, m, m, m, m)$ $((\rho, M, m)$ for short), then studying c.c. in this set becomes a problem of studying zeros of a function $F : (\alpha, +\infty) \times (0, +\infty)^2 \rightarrow \mathbb{R}$ given by the formula $F(\rho, M, m) = mf(\rho) - Mb(\rho)$ (see Corbera and Llibre 2008). For the trivial family of solutions (2.6), for any $\rho \in (\alpha, +\infty)$, we have $F(\rho, 1, b(\rho)/f(\rho)) = 0$ and since $b'(\rho)$ is negative and $f'(\rho)$ is positive for $\rho > 1$, we get $F'_\rho(\rho, 1, b(\rho)/f(\rho)) = (b(\rho)/f(\rho))f'(\rho) - b'(\rho) > 0$, so by the implicit function theorem there is no bifurcation of s.c.c. of two nested octahedra type from the trivial family. By the above, we obtain that the families which bifurcate from the family (2.6) are not of two nested octahedra type, which proves the theorem. $\square$

Now, we study bifurcations from the family of spatial configurations of the 16-body problem where the masses lie at the vertices of two nested regular cubes (see Fig. 1c). Similarly, the masses on each cube are equal and the masses on different cubes could be different. This configuration was studied by Corbera and Llibre (2008), where bodies are considered with the following positions:

$$\begin{aligned} \hat{q}_1 &= (1, 1, 1), & \hat{q}_2 &= (1, 1, -1), & \hat{q}_3 &= (1, -1, 1), & \hat{q}_4 &= (-1, 1, 1), \\ \hat{q}_5 &= (1, -1, -1), & \hat{q}_6 &= (-1, 1, -1), & \hat{q}_7 &= (-1, -1, 1), & \hat{q}_8 &= (-1, -1, -1), \\ \hat{q}_9 &= \rho \hat{q}_1, & \hat{q}_{10} &= \rho \hat{q}_2, & \hat{q}_{11} &= \rho \hat{q}_3, & \hat{q}_{12} &= \rho \hat{q}_4, \\ \hat{q}_{13} &= \rho \hat{q}_5, & \hat{q}_{14} &= \rho \hat{q}_6, & \hat{q}_{15} &= \rho \hat{q}_7, & \hat{q}_{16} &= \rho \hat{q}_8, \end{aligned}$$

where ρ is a scale factor. Let $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 = 1$ and $m_9 = m_{10} = m_{11} = m_{12} = m_{13} = m_{14} = m_{15} = m_{16} = \tilde{m}(\rho) = b(\rho)/f(\rho)$ [the formulas for $b(\rho)$ and $f(\rho)$ can be found in Proposition 4.(a) of Corbera and Llibre (2008)]. We define $w : (\alpha, +\infty) \rightarrow \Omega$ by the formula

$$w(\rho) = (\hat{q}_1, \dots, \hat{q}_{16}), \quad (2.7)$$

where $\rho > \alpha = 1.643646762940176 \dots$. Next, a map $m : (\alpha, +\infty) \rightarrow (0, +\infty)^{16}$ is given as follows

$$m(\rho) = (1, 1, 1, 1, 1, 1, 1, 1, \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho), \tilde{m}(\rho)).$$

Lemma 2.4 Put $\rho_1 = 2$, $\rho_2 = 3$ and $\rho_3 = 4$. Then $\dim \ker \nabla_q^2 \varphi(w(\rho_i), \rho_i) = \dim SO(3)(w(\rho_i)) = 3$ for $i = 1, 2, 3$ and the Morse index of $\nabla_q^2 \varphi(w(\cdot), \cdot)$ evaluated at ρ_i is

$$m^-(\nabla_q^2 \varphi(w(\rho_i), \rho_i)) = \begin{cases} 18, & \text{for } i = 1 \\ 10, & \text{for } i = 2 \\ 7, & \text{for } i = 3 \end{cases}.$$

Theorem 2.4

- (1) There exists a sequence of s.c.c. bifurcating from the family (2.7) from the segment (ρ_1, ρ_2) , i.e. a local bifurcation from the segment (ρ_1, ρ_2) occurs. Moreover, the bifurcating families are locally less symmetrical.
- (2) There exists a connected set of s.c.c. bifurcating from the family (2.7) from the segment (ρ_2, ρ_3) , i.e. a global bifurcation from the segment (ρ_2, ρ_3) occurs. Moreover, the bifurcating families are locally less symmetrical.

Fig. 3 Fourteen bodies located at the vertices of two dual regular polyhedra. **a** $a \in (0, 1)$, **b** $a \in [1, 3]$, **c** $a \in (3, +\infty)$

Proof We first show that $(\rho_1, \rho_2) \cap \mathcal{BLF} \neq \emptyset$ and $(\rho_2, \rho_3) \cap \mathcal{GLOB} \neq \emptyset$. As in the proof of Theorem 2.1, applying Lemma 2.4 and Theorem 3.2, we conclude that there exists a local bifurcation parameter in (ρ_1, ρ_2) and a global one in (ρ_2, ρ_3) .

Now, notice that we can consider a subset of the full configuration space Ω which is invariant for the gradient flow, i.e. the set of s.c.c. of two nested cubes type $(\hat{q}_1, \dots, \hat{q}_{16}, M, M, M, M, M, M, m, m, m, m, m, m, m, m)$ $((\rho, M, m)$ for short), then studying c.c. in this set becomes a problem of studying zeros of a function $F : (\alpha, +\infty) \times (0, +\infty)^2 \rightarrow \mathbb{R}$ given by the formula $F(\rho, M, m) = mf(\rho) - Mb(\rho)$ (see Corbera and Llibre 2008). For the trivial family of solutions (2.7), for any $\rho \in (\alpha, +\infty)$, we have $F(\rho, 1, b(\rho)/f(\rho)) = 0$ and because $b'(\rho)$ is negative and $f'(\rho)$ is positive for $\rho > 1$, we get $F'_\rho(\rho, 1, b(\rho)/f(\rho)) = (b(\rho)/f(\rho))f'(\rho) - b'(\rho) > 0$, so by the implicit function theorem there is no bifurcation of s.c.c. of two nested cubes type from the trivial family. It follows that the families which bifurcate from the family (2.7) are not of two nested cubes type, which proves our assertion. $\square$

Next, we consider the new family of spatial bi-stacked c.c. formed by two dual regular polyhedra, which was found by Corbera et al. (2014). We prove the existence of bifurcations of s.c.c. for 8 and 14 bodies. Notice that a regular tetrahedron is itself dual and the configuration formed by the regular tetrahedron and its dual was considered as the family of Type II (see Fig. 2). In the case of 14 bodies, the configuration consists of 8 equal masses on the vertices of a regular cube and 6 additional equal masses on the vertices of its dual regular octahedron (see Fig. 3) and is considered with the following positions:

$$\begin{aligned} \hat{q}_1 &= (1, 1, 1), & \hat{q}_2 &= (-1, 1, 1), & \hat{q}_3 &= (1, -1, 1), & \hat{q}_4 &= (1, 1, -1), \\ \hat{q}_5 &= (-1, -1, 1), & \hat{q}_6 &= (-1, 1, -1), & \hat{q}_7 &= (1, -1, -1), & \hat{q}_8 &= (-1, -1, -1), \\ \hat{q}_9 &= a(1, 0, 0), & \hat{q}_{10} &= a(-1, 0, 0), & \hat{q}_{11} &= a(0, 1, 0), & \hat{q}_{12} &= a(0, -1, 0), \\ \hat{q}_{13} &= a(0, 0, 1), & \hat{q}_{14} &= a(0, 0, -1), \end{aligned}$$

where a is a scale factor. Let $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 = 1$ and $m_9 = m_{10} = m_{11} = m_{12} = m_{13} = m_{14} = \tilde{m}(a) = -g(a)/f(a)$ [the formulas for $g(a)$ and $f(a)$ can be found in Section 2 of Corbera et al. (2014)]. We define $w : (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty) \rightarrow \Omega$ by the formula

$$w(a) = (\hat{q}_1, \dots, \hat{q}_{14}), \quad (2.8)$$

where $\alpha_1 = 1.278175\dots$, $\alpha_2 = 3.628586\dots$, $\beta_1 = 0.8932884\dots$ and $\beta_2 = 2.2083166\dots$. Next, a map $m : (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty) \rightarrow (0, +\infty)^{14}$ is given as follows

$$m(a) = (1, 1, 1, 1, 1, 1, 1, 1, \tilde{m}(a), \tilde{m}(a), \tilde{m}(a), \tilde{m}(a), \tilde{m}(a), \tilde{m}(a)).$$

Lemma 2.5 Put $a_1 = 49/100$, $a_2 = 1/2$, $a_3 = 53/100$, $a_4 = 54/100$, $a_5 = 7/10$, $a_6 = 71/100$, $a_7 = 131/100$, $a_8 = 132/100$, $a_9 = 181/100$, $a_{10} = 182/100$, $a_{11} = 216/100$,

$a_{12} = 217/100$, $a_{13} = 4$, $a_{14} = 5$, $a_{15} = 6$, $a_{16} = 7$, $a_{17} = 8$, $a_{18} = 17$ and $a_{19} = 18$. Then $\dim \ker \nabla_q^2 \varphi(w(a_i), a_i) = \dim SO(3)(w(a_i)) = 3$ for $i = 1, \dots, 19$ and the Morse index of $\nabla_q^2 \varphi(w(\cdot), \cdot)$ evaluated at a_i is

$$m^-(\nabla_q^2 \varphi(w(a_i), a_i)) = \begin{cases} 8, & \text{for } i = 6, 13 \\ 7, & \text{for } i = 4, 5, 14 \\ 4 & \text{for } i = 2, 3, 15 \\ 3, & \text{for } i = 8, 9 \\ 2, & \text{for } i = 1, 12, 19 \\ 1, & \text{for } i = 16 \\ 0, & \text{for } i = 7, 10, 11, 17, 18 \end{cases}.$$

Theorem 2.5 *There exist sequences of s.c.c. bifurcating from the family (2.8) from the segments $(0, \beta_1)$, (α_1, β_2) and $(\alpha_2, +\infty)$, i.e. there exist local bifurcation parameters in the segments $(0, \beta_1)$, (α_1, β_2) and $(\alpha_2, +\infty)$. Additionally, some of them are global bifurcation parameters. Moreover, the families which bifurcate from the segments (a_1, a_2) , (a_3, a_4) , (a_7, a_8) , (a_9, a_{10}) , (a_{11}, a_{12}) , (a_{14}, a_{15}) , (a_{15}, a_{16}) , (a_{16}, a_{17}) and (a_{18}, a_{19}) are locally less symmetrical.*

Proof In consequence of Lemma 2.5 and Theorem 3.2, there exist local and global bifurcation parameters in each of the segments $(0, \beta_1)$, (α_1, β_2) and $(\alpha_2, +\infty)$.

It remains to prove that the bifurcating families are less symmetrical. For this purpose, we consider a subset of the full configuration space Ω which is invariant for the gradient flow, i.e. the set of s.c.c. formed by a cube and an octahedron $(\hat{q}_1, \dots, \hat{q}_{14}, M, M, M, M, M, M, m, m, m, m, m, m, m)$ ((a, M, m) for short), then studying c.c. in this set becomes a problem of studying zeros of a function $F : ((0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty)) \times (0, +\infty)^2 \rightarrow \mathbb{R}$ given by the formula $F(a, M, m) = Mg(a) + mf(a)$ (see Corbera et al. 2014). For the trivial family of solutions (2.8), for any $a \in (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty)$, we have $F(a, 1, -g(a)/f(a)) = 0$. Additionally, for any $a \in (a_1, a_2) \cup (a_3, a_4) \cup (a_7, a_8) \cup (a_9, a_{10}) \cup (a_{11}, a_{12}) \cup (a_{14}, a_{15}) \cup (a_{15}, a_{16}) \cup (a_{16}, a_{17}) \cup (a_{18}, a_{19})$, we obtain numerically that $F'_a(a, 1, -g(a)/f(a)) = g'(a) - (g(a)/f(a))f'(a) < 0$, so by the implicit function theorem there is no bifurcation of s.c.c. formed by a cube and an octahedron from these segments. By the above, we obtain that the families which bifurcate from the family (2.8) from those segments are not formed by a cube and an octahedron, which completes the proof. $\square$

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3 Appendix

In this section we review some facts of equivariant topology (see for instance tom Dieck (1987), Kawakubo (1991) for more details). Moreover, we formulate the theorems of Kowalczyk (2015) giving necessary and sufficient conditions for the existence of local and global bifurcations of s.c.c. in the vicinity of some known families of c.c.

Let $\Omega \subset \mathbb{V}$ be an open and $SO(3)$ -invariant subset of an $SO(3)$ -representation $\mathbb{V}$ and fix $k \in \mathbb{N} \cup \{\infty\}$, then the set of $SO(3)$ -invariant maps of class C^k is denoted by $C_{SO(3)}^k(\Omega, \mathbb{R})$. The following lemma giving a decomposition of the Hessian $\nabla^2\phi$ of the $SO(3)$ -invariant potential ϕ has been proved in Gęba (1997).

Lemma 3.1 Assume that $\phi \in C_{SO(3)}^2(\Omega, \mathbb{R})$, $q_0 \in (\nabla\phi)^{-1}(0)$ and denote by H the isotropy group of q_0 . Then the Hessian

$$\nabla^2\phi(q_0) : \begin{array}{c} T_{q_0}SO(3)(q_0) \\ \oplus \\ \mathbb{W}^H \\ \oplus \\ (\mathbb{W}^H)^\perp \end{array} \longrightarrow \begin{array}{c} T_{q_0}SO(3)(q_0) \\ \oplus \\ \mathbb{W}^H \\ \oplus \\ (\mathbb{W}^H)^\perp \end{array}$$

is of the form

$$\nabla^2\phi(q_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{B}(q_0) & 0 \\ 0 & 0 & \mathcal{C}(q_0) \end{bmatrix}, \quad (3.1)$$

where $T_{q_0}SO(3)(q_0)$ is the tangent space to the $SO(3)$ -orbit $SO(3)(q_0)$ at q_0 and $\mathbb{W} = (T_{q_0}SO(3)(q_0))^\perp$.

According to Lemma 3.1, we always have that $\dim \ker \nabla^2\phi(q_0) \geq \dim SO(3)(q_0)$. If the strict inequality holds, a critical $SO(3)$ -orbit $SO(3)(q_0)$ is called degenerate. Thus we call the critical $SO(3)$ -orbit non-degenerate if $\dim \ker \nabla^2\phi(q_0) = \dim SO(3)(q_0)$.

We now introduce the notions of local and global bifurcations of $SO(3)$ -orbits of solutions of the Eq. (2.3), which are preceded by some additional notation. We will denote by $C(\rho_0)$ the connected component of the set $cl(\{(q, \rho) \in (\Omega \times \mathbb{R}) \setminus \mathcal{F} : \nabla_q \varphi(q, \rho) = 0\})$ containing $\mathcal{F}_{\rho_0}$ and by $C([\rho^-, \rho^+])$ the connected component of the set $cl(\{(q, \rho) \in (\Omega \times \mathbb{R}) \setminus \mathcal{F} : \nabla_q \varphi(q, \rho) = 0\}) \cup \mathcal{F}_{[\rho^-, \rho^+]}$ containing $\mathcal{F}_{[\rho^-, \rho^+]}$.

Definition 3.1 Fix parameters $\rho^\pm \in \mathbb{R}$ such that $\rho^- < \rho^+$.

- (1) A local bifurcation from the segment of $SO(3)$ -orbits $\mathcal{F}_{[\rho^-, \rho^+]} \subset \mathcal{F}$ of solutions of the Eq. (2.3) occurs if there exists an $SO(3)$ -orbit $\mathcal{F}_{\rho_0} \subset \mathcal{F}_{[\rho^-, \rho^+]}$ such that the point $(w(\rho_0), \rho_0) \in \mathcal{F}_{\rho_0}$ is an accumulation point of the set $\{(q, \rho) \in (\Omega \times \mathbb{R}) \setminus \mathcal{F} : \nabla_q \varphi(q, \rho) = 0\}$.

We call ρ_0 a parameter of local bifurcation and $\mathcal{F}_{\rho_0}$ an $SO(3)$ -orbit of local bifurcation.

- (2) A global bifurcation from the segment of $SO(3)$ -orbits $\mathcal{F}_{[\rho^-, \rho^+]} \subset \mathcal{F}$ of solutions of the Eq. (2.3) occurs if the component $C([\rho^-, \rho^+]) \subset \Omega \times \mathbb{R}$ is not compact or $(C([\rho^-, \rho^+]) \setminus \mathcal{F}_{[\rho^-, \rho^+]}) \cap \mathcal{F} \neq \emptyset$ (see Fig. 4).

We call $\rho_0 \in [\rho^-, \rho^+]$ a parameter of global bifurcation if the component $C(\rho_0) \subset \Omega \times \mathbb{R}$ is not compact or $(C(\rho_0) \setminus \mathcal{F}_{\rho_0}) \cap \mathcal{F} \neq \emptyset$. The $SO(3)$ -orbit $\mathcal{F}_{\rho_0} \subset \mathcal{F}_{[\rho^-, \rho^+]}$ is called an $SO(3)$ -orbit of global bifurcation.

We denote by BIF and GOB the sets of all parameters of local and global bifurcation, respectively.

In the spatial N -body problem we have a presence of symmetry of the Lie group $SO(3)$. Thus we formulate the necessary and sufficient conditions for the existence of local and global bifurcations from the segment of $SO(3)$ -orbits of solutions of the Eq. (2.3). Abstract results for an arbitrary compact Lie group G can be found in Kowalczyk (2015) [for details see Theorems 3.1, 3.2, 3.3 and 3.5 of Kowalczyk (2015)]. The necessary condition for the

Fig. 4 A global bifurcation from the segment of $SO(3)$ -orbits $\mathcal{F}_{[\rho^-, \rho^+]} \subset \mathcal{F}$. In particular, $\rho_0 \in [\rho^-, \rho^+]$ is a parameter of global bifurcation

existence of parameters of local bifurcation gives that only a degenerate critical $SO(3)$ -orbit of solutions of the Eq. (2.3) can be an $SO(3)$ -orbit of local bifurcation.

Theorem 3.1 Fix a parameter $\rho_0 \in \mathbb{R}$. If $\rho_0 \in \mathcal{BIF}$, then $\dim \ker \nabla_q^2 \varphi(w(\rho_0), \rho_0) > \dim SO(3)(w(\rho_0))$.

Let us introduce the notion of a conjugacy class. Two closed subgroups H and H' of $SO(3)$ are called conjugate if there exists $g \in SO(3)$ such that $H = g^{-1}H'g$. Notice that conjugacy is an equivalence relation and let $(H)_{SO(3)}$ denote the conjugacy class of a closed subgroup H of $SO(3)$. In the following theorem we give the sufficient conditions for the existence of local and global bifurcations. Theorem 3.2 states that a change of the Morse indices implies the existence of a local bifurcation and a change of parity of the Morse indices guarantees the existence of a global bifurcation.

Theorem 3.2 Fix parameters $\rho^\pm \in \mathbb{R}$, $\rho^- < \rho^+$ such that:

- (1) $\dim \ker \nabla_q^2 \varphi(w(\rho^\pm), \rho^\pm) = \dim SO(3)(w(\rho^\pm))$,
- (2) $(SO(3)_{w(\rho^-)})_{SO(3)} = (SO(3)_{w(\rho^+)})_{SO(3)}$,
- (3) $m^-(C(w(\rho^\pm))) = 0$.

If $m^-(B(w(\rho^-))) \neq m^-(B(w(\rho^+)))$, then $(\rho^-, \rho^+) \cap \mathcal{BIF} \neq \emptyset$, i.e. there exists a local bifurcation parameter $\rho_0 \in (\rho^-, \rho^+)$. Moreover, if the numbers $m^-(B(w(\rho^-)))$ and $m^-(B(w(\rho^+)))$ are of different parity, then $(\rho^-, \rho^+) \cap \mathcal{GLB} \neq \emptyset$, i.e. there exists a global bifurcation parameter $\rho_1 \in (\rho^-, \rho^+)$.

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