

Non-integrability of the three-body problem

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Abstract We consider the planar problem of three bodies which attract mutually with the force proportional to a certain negative integer power of the distance between the bodies. We show that such generalisation of the gravitational three-body problem is not integrable in the Liouville sense.

Keywords Integrability · Non-integrability criteria · Monodromy group · Differential Galois group · Hypergeometric equation · Hamiltonian equations · Three-body problem

1 Introduction and results

We consider three point masses m_1, m_2 and m_3 in a plane. We assume that they interact mutually according to a generalized law of gravitation. Namely, the force of the attraction between two points is proportional to the product of their masses and inversely proportional to a certain power of the distance between them.

Let $\mathbf{r}_1 := (x_1, x_2)$, $\mathbf{r}_2 := (x_3, x_4)$ and $\mathbf{r}_3 := (x_5, x_6)$ denote the inertial Cartesian coordinates of the masses and (y_1, y_2) , (y_3, y_4) , (y_5, y_6) their respective linear momenta in this frame. Then the Hamiltonian of the problem has the form

$$K = \frac{1}{2m_1} (y_1^2 + y_2^2) + \frac{1}{2m_2} (y_3^2 + y_4^2) + \frac{1}{2m_3} (y_5^2 + y_6^2) + U(\mathbf{x}), \quad (1.1)$$

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where

$$U(\mathbf{x}) = -\frac{m_1 m_2}{r_{12}^{2n}} - \frac{m_2 m_3}{r_{23}^{2n}} - \frac{m_3 m_1}{r_{31}^{2n}} \tag{1.2}$$

and

$$\begin{aligned} r_{12} &:= \sqrt{(x_1 - x_3)^2 + (x_2 - x_4)^2}, \\ r_{23} &:= \sqrt{(x_3 - x_5)^2 + (x_4 - x_6)^2}, \\ r_{31} &:= \sqrt{(x_5 - x_1)^2 + (x_6 - x_2)^2}. \end{aligned}$$

We assume that $2n \in \mathbb{N}$. The system admits three additional classical first integrals: two components of the total momentum, and the total angular momentum

$$\begin{aligned} Y_1 &:= y_1 + y_3 + y_5, & Y_2 &:= y_2 + y_4 + y_6, \\ C &:= x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5. \end{aligned} \tag{1.3}$$

However, those first integrals do not pairwise commute, as we have

$$\{C, Y_1\} = Y_2, \quad \text{and} \quad \{Y_2, C\} = Y_1. \tag{1.4}$$

We perform a canonical reduction and we eliminate two degrees of freedom. To this end we make a linear canonical transformation

$$\mathbf{x} = S\tilde{\mathbf{q}}, \quad \mathbf{y} = S^{-T}\tilde{\mathbf{p}}, \quad S^{-T} := (S^{-1})^T, \tag{1.5}$$

where

$$\begin{aligned} \tilde{\mathbf{q}} &:= [q_1, \dots, q_6]^T, & \tilde{\mathbf{p}} &:= [p_1, \dots, p_6]^T, \\ S &:= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ \sigma_1 & 0 & \sigma_2 & 0 & \sigma_3 & 0 \\ 0 & \sigma_1 & 0 & \sigma_2 & 0 & \sigma_3 \end{bmatrix}, \end{aligned} \tag{1.6}$$

and

$$\sigma_i = \frac{m_i}{m} \quad \text{for } i = 1, 2, 3; \quad m = m_1 + m_2 + m_3.$$

In other words (q_1, q_2) is the vector between m_3 and m_1 , i.e., $(q_1, q_2) := \mathbf{r}_1 - \mathbf{r}_3$; similarly, $(q_3, q_4) := \mathbf{r}_2 - \mathbf{r}_3$ is the vector between m_3 and m_2 , but (q_5, q_6) are coordinates of the mass centre

$$(q_5, q_6) := \frac{1}{m} \sum_{i=1}^3 m_i \mathbf{r}_i.$$

The transformed Hamiltonian (1.1) reads

$$H(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) := K(S\tilde{\mathbf{q}}, S^T\tilde{\mathbf{p}}) = H_r(\mathbf{q}, \mathbf{p}) + \frac{1}{2m}(p_5^2 + p_6^2), \tag{1.7}$$

where

$$\begin{aligned} \mathbf{q} &:= [q_1, \dots, q_4]^T, & \mathbf{p} &:= [p_1, \dots, p_4]^T, \\ H_r &= H_r(\mathbf{q}, \mathbf{p}) = T_r(\mathbf{p}) + U_r(\mathbf{q}), \end{aligned} \tag{1.8}$$

$$T_r(\mathbf{p}) := \frac{1}{2\mu_1} (p_1^2 + p_2^2) + \frac{1}{2\mu_2} (p_3^2 + p_4^2) + \frac{1}{m_3} (p_1 p_3 + p_2 p_4), \tag{1.9}$$

$$U_r(\mathbf{q}) := -\frac{m_1 m_2}{[(q_1 - q_3)^2 + (q_2 - q_4)^2]^n} - \frac{m_2 m_3}{[q_3^2 + q_4^2]^n} - \frac{m_3 m_1}{[q_1^2 + q_2^2]^n}, \tag{1.10}$$

and

$$\mu_1 := \frac{m_1 m_3}{m_1 + m_3}, \quad \mu_2 := \frac{m_2 m_3}{m_2 + m_3}. \tag{1.11}$$

Clearly q_5 and q_6 are cyclic coordinates. Moreover, Hamilton’s equations with Hamiltonian (1.7) split into a direct product of Hamilton’s equations with Hamiltonian H_r and Hamiltonian $H_c := (p_5^2 + p_6^2)/(2m)$.

The reduced system with four degrees of freedom governed by Hamiltonian H_r has one additional first integral

$$F := q_1 p_2 - q_2 p_1 + q_3 p_4 - q_4 p_3, \tag{1.12}$$

which is the total angular momentum of the system. The system given by (1.8) we call the partially reduced three-body problem.

One can eliminate one more degree of freedom using first integral (1.12). This reduction is described, e.g., in Sect. 161 of Whittaker (1965). The obtained system has three degrees of freedom and we call it the fully reduced three-body problem.

We consider the Hamiltonian system generated by (1.8) in the complex phase space which is an open subset of \mathbb{C}^8 with canonical coordinates $\mathbf{q} = (q_1, \dots, q_4)$ and momenta $\mathbf{p} = (p_1, \dots, p_4)$. Our main result is formulated in the following theorem.

Theorem 1.1 *Assume that $2n \in \mathbb{N} \setminus \{2\}$ and masses m_1, m_2 and m_3 are positive. Then the Hamiltonian system given by (1.8) is not integrable in the Liouville sense.*

The non-integrability of the non-reduced classical gravitational three-body problem corresponding to $2n = 1$ was proved in the framework of the Ziglin theory in Ziglin (2000) and later by means of the so-called Morales–Ramis theory in Morales-Ruiz and Simon (2009), Simon (2007). The first proof of the non-integrability of the fully reduced classical planar three-body problem was done by Tsygvinsev in (2000; 2001; 2007), and by an application of the differential Galois approach by Boucher and Weil (2003).

In Maciejewski and Przybylska (2010), basing on ideas of Morales-Ruiz and Simon (2009), we have found a surprisingly simple proof of the non-integrability of the classical three-body problem. In this paper we generalised this result to a class of potentials of the form (1.2) with an arbitrary positive integer $2n$.

Our proof of Theorem 1.1 is based on an application of a general theorem concerning the integrability of homogeneous potentials. In the next section we reformulate this theorem according to our needs. Here it is worth to mention that this general result takes its origin from a brilliant work of Yoshida (1987). The main statement of this theorem can be derived from an analysis of the monodromy group of the variational equations, see Yoshida (2000); Ziglin (2000), as well as from an analysis of their differential Galois group, Morales-Ruiz and Ramis (2001).

Remark 1.2 In Theorem 1.1 we excluded the case of the Jacobi problem when the potential of interactions between bodies is homogeneous of degree -2 . In this case, the above mentioned Morales–Ramis theorem does not give any obstruction to the integrability. The non-integrability of the Jacobi three-body problem was investigated by Julliard Tosel in (2000). She proved, among other things, the non-integrability of the Jacobi problem when masses of two points are equal. Additionally, she also proved that the Jacobi problem (with arbitrary masses) is not integrable with rational first integrals which are meromorphic functions of the masses.

2 Theory

Let us consider a complex Hamiltonian system with n degrees of freedom given by a natural Hamiltonian function of the following form

$$H = \frac{1}{2} \mathbf{p}^T \mathbf{M} \mathbf{p} + V(\mathbf{q}), \tag{2.1}$$

where $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{C}^n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{C}^n$ are canonical coordinates and momenta, $V(\mathbf{q})$ is a homogeneous function of degree $k \in \mathbb{Z}^*$, and \mathbf{M} is a symmetric non-singular $n \times n$ matrix. The phase space of this system is \mathbb{C}^{2n} which is considered as \mathbb{C} -linear symplectic space with the canonical symplectic form

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i.$$

Hence, Hamilton’s equations have the standard canonical form

$$\frac{d}{dt} \mathbf{q} = \mathbf{M} \mathbf{p}, \quad \frac{d}{dt} \mathbf{p} = -V'(\mathbf{q}), \tag{2.2}$$

where $V'(\mathbf{q}) := \text{grad } V(\mathbf{q})$. Moreover, we assume also that the time t is a complex variable.

The basic assumption of our considerations is that there exists a non-zero vector $\mathbf{d} \in \mathbb{C}^n$ such that

$$\mathbf{M}V'(\mathbf{d}) = \mathbf{d}. \tag{2.3}$$

It is called a proper Darboux point of potential V . It defines a two dimensional plane in the phase spaces \mathbb{C}^{2n} , given by

$$\Pi(\mathbf{d}) := \{(\mathbf{q}, \mathbf{p}) \in \mathbb{C}^{2n} \mid \mathbf{q} = \varphi \mathbf{d}, \mathbf{p} = \psi \mathbf{M}^{-1} \mathbf{d}, (\varphi, \psi) \in \mathbb{C}^2\}. \tag{2.4}$$

This plane is invariant with respect to the system (2.2). Equations (2.2) restricted to $\Pi(\mathbf{d})$ have the form of one degree of freedom Hamilton’s equations

$$\frac{d}{dt} \varphi = \psi, \quad \frac{d}{dt} \psi = -\varphi^{k-1}, \tag{2.5}$$

with the following phase curves

$$\Gamma_{k,\varepsilon} := \left\{ (\varphi, \psi) \in \mathbb{C}^2 \mid \frac{1}{2} \psi^2 + \frac{1}{k} \varphi^k = \varepsilon \right\} \subset \mathbb{C}^2, \quad \varepsilon \in \mathbb{C}. \tag{2.6}$$

In this way, a solution $(\varphi, \psi) = (\varphi(t), \psi(t))$ of (2.5) gives a solution $(\mathbf{q}(t), \mathbf{p}(t)) := (\varphi \mathbf{d}, \psi \mathbf{M}^{-1} \mathbf{d})$ of Eqs. (2.2) with the corresponding phase curve

$$\Gamma_{k,\varepsilon} := \{(\mathbf{q}, \mathbf{p}) \in \mathbb{C}^{2n} \mid (\mathbf{q}, \mathbf{p}) = (\varphi \mathbf{d}, \psi \mathbf{M}^{-1} \mathbf{d}), (\varphi, \psi) \in \Gamma_{k,\varepsilon}\} \subset \Pi(\mathbf{d}). \tag{2.7}$$

In order to find necessary conditions for the Liouville integrability we consider the variational equations along an arbitrary phase curve $\Gamma_{k,\varepsilon}$ with $\varepsilon \neq 0$. These variational equations have the form

$$\ddot{\mathbf{x}} = -\varphi(t)^{k-2} \mathbf{M}V''(\mathbf{d})\mathbf{x}, \tag{2.8}$$

where $V''(\mathbf{d})$ is the Hessian of V calculated at \mathbf{d} . A linear change of variables $\mathbf{x} = \mathbf{A}\xi$ transforms the above system into the following one

$$\ddot{\xi} = -\varphi(t)^{k-2} \mathbf{J}\xi, \tag{2.9}$$

where

$$\mathbf{J} = \mathbf{A}^{-1} \mathbf{M}V''(\mathbf{d})\mathbf{A} = \text{diag}(J_{i_1}(\lambda_1), \dots, (J_{i_p}(\lambda_p))),$$

is the Jordan normal form of matrix $\mathbf{M}V''(\mathbf{d})$; $\lambda_i \in \text{spectr}(\mathbf{M}V''(\mathbf{d}))$ for $1 \leq i \leq p$, and $J_d(\lambda)$ is the Jordan block of size d with the corresponding eigenvalue λ , i.e.,

$$J_d(\lambda) := \begin{bmatrix} \lambda & 0 & 0 & \dots & \dots & 0 \\ 1 & \lambda & 0 & \dots & \dots & 0 \\ 0 & 1 & \lambda & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \lambda \end{bmatrix} \in \mathbb{M}(d, \mathbb{C}), \tag{2.10}$$

where $\mathbb{M}(d, \mathbb{C})$ denotes the set of $d \times d$ complex matrices.

Remark 2.1 Components of $\mathbf{M}V'(\mathbf{q})$ are homogeneous functions of degree $(k - 1)$. Using the Euler identity for them one can prove that if \mathbf{d} is a proper Darboux point, then it is an eigenvector of matrix $\mathbf{M}V''(\mathbf{d})$ with the corresponding eigenvalue $\lambda = k - 1$. Moreover, if \mathbf{d} is not isolated proper Darboux point, then $\lambda = 1$ is an eigenvalue of matrix $\mathbf{M}V''(\mathbf{d})$. In fact, a proper Darboux point \mathbf{d} is a zero of the following map

$$\mathbb{C}^n \ni \mathbf{q} \mapsto \mathbf{F}(\mathbf{q}) := \mathbf{M}V'(\mathbf{q}) - \mathbf{q} \in \mathbb{C}^n. \tag{2.11}$$

If $\det \mathbf{F}'(\mathbf{d}) = \det(\mathbf{M}V''(\mathbf{d}) - \mathbf{E}_n) \neq 0$, then \mathbf{d} is an isolated zero of \mathbf{F} . Here \mathbf{E}_n is n -dimensional identity matrix. Thus, if \mathbf{d} is not isolated, then the Jacobi matrix $\mathbf{F}'(\mathbf{d})$ is singular. Equivalently, $\lambda = 1$ is an eigenvalue of matrix $\mathbf{M}V''(\mathbf{d})$ as we claimed.

Necessary conditions for the Liouville integrability of Hamiltonian systems of the form (2.2) which come from an analysis of the differential Galois group of the variational Eqs. (2.9) can be formulated in the following form.

Theorem 2.2 *Assume that the Hamiltonian system defined by Hamiltonian (2.1) with a homogeneous potential $V \in \mathbb{C}(\mathbf{q})$ of degree $k \in \mathbb{Z}^*$ satisfies the following conditions:*

1. *there exists a non-zero $\mathbf{d} \in \mathbb{C}^n$ such that $\mathbf{M}V'(\mathbf{d}) = \mathbf{d}$, and*
2. *the system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighbourhood U of phase curve $\Gamma_{k,\varepsilon}$ with $\varepsilon \neq 0$, and independent on $U \setminus \Gamma_{k,\varepsilon}$.*

Then for each eigenvalue $\lambda \in \text{spectr}(\mathbf{M}V''(\mathbf{d}))$, pair (k, λ) belongs to an item of the following table

case	k	λ
1.	± 2	arbitrary
2.	k	$p + \frac{k}{2}p(p - 1)$
3.	k	$\frac{1}{2} \left(\frac{k - 1}{k} + p(p + 1)k \right)$
4.	3	$-\frac{1}{24} + \frac{1}{6}(1 + 3p)^2, \quad -\frac{1}{24} + \frac{3}{32}(1 + 4p)^2$ $-\frac{1}{24} + \frac{3}{50}(1 + 5p)^2, \quad -\frac{1}{24} + \frac{3}{50}(2 + 5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1 + 3p)^2$ (2.12)
6.	5	$-\frac{9}{40} + \frac{5}{18}(1 + 3p)^2, \quad -\frac{9}{40} + \frac{1}{10}(2 + 5p)^2$
7.	-3	$\frac{25}{24} - \frac{1}{6}(1 + 3p)^2, \quad \frac{25}{24} - \frac{3}{32}(1 + 4p)^2$ $\frac{25}{24} - \frac{3}{50}(1 + 5p)^2, \quad \frac{25}{24} - \frac{3}{50}(2 + 5p)^2$
8.	-4	$\frac{9}{8} - \frac{2}{9}(1 + 3p)^2$
9.	-5	$\frac{49}{40} - \frac{5}{18}(1 + 3p)^2, \quad \frac{49}{40} - \frac{1}{10}(2 + 5p)^2$

where p is an integer. Moreover, for $|k| \neq 2$,

1. matrix $\mathbf{M}V''(\mathbf{d})$ does not have a Jordan block of size $d \geq 3$;
2. if matrix $\mathbf{M}V''(\mathbf{d})$ has a Jordan block $J_d(\lambda)$ of size $d = 2$, then the corresponding eigenvalue $\lambda \in \text{spectr}(\mathbf{M}V''(\mathbf{d}))$ satisfies the following conditions:
 - (a) if $|k| > 2$, then λ does not belong to the second item of table (2.12);
 - (b) if $k = -1$, then $\lambda = 1$;
 - (c) if $k = 1$, then $\lambda = 0$.

We denote by \mathcal{M}_k a subset of rational numbers λ specified by the table in the above theorem for a given k , e.g., for $|k| > 5$ we have

$$\mathcal{M}_k = \left\{ p + \frac{k}{2}p(p - 1) \mid p \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \left[\frac{k - 1}{k} + p(p + 1)k \right] \mid p \in \mathbb{Z} \right\}. \tag{2.13}$$

Let us remark that the above theorem does not give any obstruction for the integrability if $k = 2$ or $k = -2$.

Under assumption $\mathbf{M} = \mathbf{E}_n$, the first part of Theorem 2.2 coincides with Morales–Ramis theorem (Morales-Ruiz and Ramis 2001), and the second part coincides with Theorem 1.3 in Duval and Maciejewski (2009).

The generalisations of the above cited theorems to the case $\mathbf{M} \neq \mathbf{E}_n$ are straightforward. Since we assumed that matrix \mathbf{M} is non-singular and symmetric, there exists a non-singular matrix \mathbf{B} such that

$$\mathbf{B}^T \mathbf{M} \mathbf{B} = \mathbf{E}_n. \tag{2.14}$$

Thus, we can make the following canonical transformation

$$\mathbf{q} = \mathbf{C}\mathbf{Q}, \quad \mathbf{p} = \mathbf{B}\mathbf{P}, \quad \text{where } \mathbf{C} := (\mathbf{B}^{-1})^T,$$

which transforms the Hamiltonian (2.1) into

$$K(\mathbf{Q}, \mathbf{P}) := H(\mathbf{C}\mathbf{Q}, \mathbf{B}\mathbf{P}) = \frac{1}{2}\mathbf{P}^T\mathbf{P} + U(\mathbf{Q}), \quad \text{where } U(\mathbf{Q}) := V(\mathbf{C}\mathbf{Q}). \quad (2.15)$$

The above shows that if $\det \mathbf{M} \neq 0$, then, in the context considered here, we can assume that $\mathbf{M} = \mathbf{E}_n$. However, in practise, it is more convenient to avoid the described transformation, as even simple examples show that it introduces quite complicated expressions.

It is also instructive to notice the following fact. If \mathbf{d} is a proper Darboux point of the potential V , then $\mathbf{c} := \mathbf{B}^T\mathbf{d}$ is a proper Darboux point of the transformed potential $U(\mathbf{Q})$. Thus, applying Theorem 2.2 or Theorem 1.1 in the original variables, we use the eigenvalues of matrix $\mathbf{M}V''(\mathbf{d})$ which is, in general non-symmetric. On the other hand, while working with the transformed system, we use symmetric matrix $U''(\mathbf{c})$. However, in general matrix $U''(\mathbf{c})$ is a complex matrix, so it is not necessarily diagonalisable. Of course those two matrices are similar, as we have

$$U''(\mathbf{c}) = \mathbf{C}^T V''(\mathbf{d})\mathbf{C} = \mathbf{C}^{-1}(\mathbf{C}\mathbf{C}^T V''(\mathbf{d}))\mathbf{C} = \mathbf{C}^{-1}\mathbf{M}V''(\mathbf{d})\mathbf{C}, \quad (2.16)$$

because matrix \mathbf{M} can be written as $\mathbf{M} = \mathbf{C}\mathbf{C}^T$, see (2.14).

In the formulation of Theorem 2.2 we assumed for the simplicity that the considered potential is a rational homogeneous function. However, this theorem remains valid for a class of homogeneous potentials which are meromorphic in a neighbourhood of the considered particular solution.

3 Proof of Theorem 1.1

The Hamiltonian (1.8) has the form (2.1) with matrix $\mathbf{M} = \mathbf{M}_r$ given by

$$\mathbf{M}_r := \begin{bmatrix} \frac{1}{\mu_1} & 0 & \frac{1}{m_3} & 0 \\ \mu_1 & \frac{1}{\mu_1} & 0 & \frac{1}{m_3} \\ 0 & \mu_1 & 0 & \frac{1}{m_3} \\ \frac{1}{m_3} & 0 & \frac{1}{\mu_2} & 0 \\ m_3 & 0 & \mu_2 & 0 \\ 0 & \frac{1}{m_3} & 0 & \frac{1}{\mu_2} \end{bmatrix}. \quad (3.1)$$

In order to apply Theorem 2.2 we need the existence of a proper Darboux point. In fact such a point exists and it is related to the Euler configuration.

Proposition 3.1 *Assume that $2n$ is a positive integer and masses m_1, m_2 and m_3 are positive. Then equation $\mathbf{M}_r U'_r(\mathbf{e}) = \mathbf{e}$ has a solution \mathbf{e} of the form*

$$\mathbf{e} := (a, 0, a(1 + \rho), 0), \quad a > 0, \quad \rho > 0, \quad (3.2)$$

where ρ is a unique positive root of the following polynomial

$$P(\rho) := m_1(1 + \rho)^{2n+1} (1 - \rho^{2n+2}) + m_2 [(1 + \rho)^{2n+2} - \rho^{2n+2}] + m_3 \rho^{2n+1} [1 - (1 + \rho)^{2n+2}], \quad (3.3)$$

and $a = \alpha^{-\frac{1}{2(n+1)}}$ is given by

$$\alpha = \frac{\rho^{2n+1}(\rho + 1)^{2n+1} [m_3 + (m_1 + m_3)\rho]}{2nm [m_3\rho^{2m+1} + m_1(1 + \rho)^{2n+1}]} \tag{3.4}$$

Proof For vector e of the form (3.2) equation $U'_r(e) = M_r^{-1}e$ reduces to the following equations

$$\begin{aligned} -m_3 + m_2\rho + 2\alpha nm \left(m_3 - \frac{m_2}{\rho^{2n+1}} \right) &= 0, \\ -m_3(\rho + 1) - m_1\rho + 2\alpha nm \left(\frac{m_1}{\rho^{2n+1}} + \frac{m_3}{(\rho + 1)^{2n+1}} \right) &= 0. \end{aligned} \tag{3.5}$$

Eliminating α from the above equations we find that ρ is a root of polynomial (3.3), and that α is given by (3.4).

Clearly $\deg P(\rho) = 4n + 3$. Moreover, $P(0) > 0$, and $P(\rho) \rightarrow -\infty$ as $\rho \rightarrow \infty$. We write $P(\rho)$ in the following form

$$P(\rho) = \sum_{i=0}^{4n+3} P_i \rho^i.$$

From (3.3) it is easy to deduce that $P_i < 0$ for $i > 2n + 1$, and $P_i > 0$ for $i \leq 2n + 1$. Hence we have exactly one change of sign of the coefficients, and this proves that $P(\rho)$ has one positive root. □

Remark 3.2 It can be easily shown that if a and ρ are such that e given (3.2) is a proper Darboux point, then also

$$e_s := (a \cos s, a \sin s, a(1 + \rho) \cos s, a(1 + \rho) \sin s), \tag{3.6}$$

is a proper Darboux point for an arbitrary $s \in \mathbb{R}$. Hence, $e = e_0$ is not isolated.

Now, our proof of Theorem 1.1 follows the following steps:

1. We show the matrix $M_r U''_r(e)$ has eigenvalues $(1, -(2n + 1), \lambda, -(2n + 1)\lambda)$, where $\lambda > 0$.
2. If the system is integrable, then $\lambda_1 = \lambda$, and $\lambda_2 = -(2n + 1)\lambda$ satisfy conditions of Theorem 2.2. We show that this implies that $\lambda = 1$.
3. Finally, we prove that for positive masses m_1, m_2 and m_3 the equality $\lambda = 1$ is impossible.

The next three propositions prove the statements formulated in the above steps.

Proposition 3.3 *Assume that $2n$ is a positive integer, and e is the solution of $M_r U'_r(e) = e$ given by Proposition 3.1. Then matrix $M_r U''_r(e)$ has eigenvalues $(1, -(2n + 1), \lambda, -(2n + 1)\lambda)$, where $\lambda > 0$.*

Proof Potential U_r is homogeneous of degree $k = -2n$. This is why $k - 1 = -(2n + 1)$ is an eigenvalue of $M_r U''_r(e)$. Moreover, by Remark 3.2, e is not isolated proper Darboux point. So, by Remark 2.1, $\lambda = 1$ is an eigenvalue of $M_r U''_r(e)$.

Let $p(z) := \det(M_r U''_r(e) - zE_4)$ with e given by (3.2) be the characteristic polynomial of matrix $M_r U''_r(e)$. The coefficients of this polynomial depend rationally on α and ρ . Using (3.4) we eliminate from them α . Next we notice that the coefficients of polynomial $P(\rho)$

given by (3.3) are linear with respect to the masses. Thus, assuming that ρ is the positive root of $P(\rho)$, from $P(\rho) = 0$, we obtain

$$m_2 = m_1 r_1(\rho) + m_3 r_3(\rho),$$

where

$$r_1(\rho) := \frac{(1 + \rho)^{2n+1}(\rho^{2n+2} - 1)}{(1 + \rho)^{2n+2} - \rho^{2n+2}}, \quad r_3(\rho) := \frac{\rho^{2n+1} [(1 + \rho)^{2n+2} - 1]}{(1 + \rho)^{2n+2} - \rho^{2n+2}}.$$

Using the above relation we eliminate m_2 from the coefficients of polynomial $p(z)$. After this operation it factors

$$p(z) = (z - 1)(z + 2n + 1)p_2(z), \quad \text{where } p_2(z) = z^2 + bz + c. \tag{3.7}$$

By a direct calculation we found that

$$b = 2n\lambda, \quad \text{and } \frac{c}{b^2} = -\frac{2n + 1}{4n^2}, \tag{3.8}$$

where

$$\lambda := \frac{m_3 + (m_1 + m_3)\rho}{m_1(1 + \rho)^{2n+1} + m_3\rho^{2n+1}} R(\rho), \tag{3.9}$$

and

$$R(\rho) := \frac{(\rho(1 + \rho))^{2n} [(1 + \rho)^{2n+3} - 1 - \rho^{2n+3}]}{(1 + \rho)^{2n+1}(1 + \rho^{2n+1}) - \rho^{2n+1}}. \tag{3.10}$$

Evidently, we have $\lambda > 0$, and this finishes our proof. □

Proposition 3.4 *Assume that $k = -2n$ is a negative integer and $k \neq -2$. Then $\lambda \in \mathcal{M}_k$ and $(k - 1)\lambda \in \mathcal{M}_k$ if and only if $\lambda = 1$.*

Proof We know that for an arbitrary $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ we have

$$\mathcal{M}_k^{(1)} \cup \mathcal{M}_k^{(2)} \subset \mathcal{M}_k,$$

where

$$\mathcal{M}_k^{(1)} := \left\{ p + \frac{k}{2}p(p - 1) \mid p \in \mathbb{Z} \right\}, \tag{3.11}$$

and

$$\mathcal{M}_k^{(2)} := \left\{ \frac{1}{2} \left(\frac{k - 1}{k} + p(p + 1) \right) k \mid p \in \mathbb{Z} \right\}. \tag{3.12}$$

Moreover, for an arbitrary $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ numbers $\lambda_1 = 1$ and $\lambda_2 = k - 1$ are elements of $\mathcal{M}_k^{(1)}$.

Now we assume that $k = -2n$ is a negative integer, and that $k \neq -2$. We have to show that if $\lambda_1 = \lambda > 0$, and $\lambda_2 = -(2n + 1)\lambda < 0$, then $\lambda = 1$.

For $k = -1$ we have

$$\mathcal{M}_{-1} := \left\{ -\frac{1}{2}p(p - 3) \mid p > 1, \quad p \in \mathbb{N} \right\}. \tag{3.13}$$

This set contains just one positive element that is equal to one. Thus, $\lambda_1, \lambda_2 \in \mathcal{M}_{-1}$, iff $(\lambda_1, \lambda_2) = (1, -2)$. This ends the proof for $k = -1$.

For $k = -2n = -3$, we have

$$\mathcal{M}_{-3} = \bigcup_{i=1}^6 \mathcal{M}_{-3}^{(i)},$$

where

$$\begin{aligned} \mathcal{M}_{-3}^{(3)} &:= \left\{ \frac{25}{24} - \frac{1}{6} (1 + 3p)^2 \mid p \in \mathbb{Z} \right\}, \\ \mathcal{M}_{-3}^{(4)} &:= \left\{ \frac{25}{24} - \frac{3}{32} (1 + 4p)^2 \mid p \in \mathbb{Z} \right\}, \\ \mathcal{M}_{-3}^{(5)} &:= \left\{ \frac{25}{24} - \frac{3}{50} (1 + 5p)^2 \mid p \in \mathbb{Z} \right\}, \\ \mathcal{M}_{-3}^{(6)} &:= \left\{ \frac{25}{24} - \frac{3}{50} (2 + 5p)^2 \mid p \in \mathbb{Z} \right\}. \end{aligned} \tag{3.14}$$

Elements of sets $\mathcal{M}_{-3}^{(i)}$ have the following properties

$$\begin{aligned} \lambda \in \mathcal{M}_{-3}^{(1)} &\implies \lambda \in \mathbb{Z}, \\ \lambda \in \mathcal{M}_{-3}^{(2)} &\implies \lambda = \frac{s}{3}, \quad (s, 3) = 1, \\ \lambda \in \mathcal{M}_{-3}^{(3)} &\implies \lambda = \frac{s}{8}, \quad (s, 8) = 1, \\ \lambda \in \mathcal{M}_{-3}^{(4)} &\implies \lambda = \frac{s}{96}, \quad (s, 96) = 1, \\ \lambda \in \mathcal{M}_{-3}^{(5)} \cup \mathcal{M}_{-3}^{(6)} &\implies \lambda = \frac{s}{600}, \quad (s, 600) = 1. \end{aligned} \tag{3.15}$$

Let \mathcal{M}_{-3}^+ denote the subset of elements of \mathcal{M}_{-3} which are positive and different from one. It has following elements

$$\mathcal{M}_{-3}^+ := \left\{ \frac{49}{600}, \frac{19}{96}, \frac{3}{8}, \frac{301}{600}, \frac{2}{3}, \frac{481}{600}, \frac{7}{8}, \frac{91}{96}, \frac{589}{600} \right\}.$$

We show that if $\lambda_1 = \lambda \in \mathcal{M}_{-3}^+$, then $\lambda_2 = (k - 1)\lambda = -4\lambda$ is not an element of \mathcal{M}_{-3} . In fact using properties (3.15) we easily show that there is only one possibility $\lambda_1 = 2/3$. However, this gives $\lambda_2 = -8/3$. Thus, by properties (3.15), if $\lambda_2 \in \mathcal{M}_{-3}$, then $\lambda_2 = -8/3 \in \mathcal{M}_{-3}^{(2)}$ but it is not true, as one can check directly. This ends the proof for $k = -3$.

For $k = -2n = -4$ we have

$$\mathcal{M}_{-4} = \bigcup_{i=1}^3 \mathcal{M}_{-4}^{(i)},$$

where

$$\mathcal{M}_{-4}^{(3)} := \left\{ \frac{9}{8} - \frac{2}{9} (1 + 3p)^2 \mid p \in \mathbb{Z} \right\}. \tag{3.16}$$

We can sort all elements of the sets $\mathcal{M}_{-3}^{(i)}$ into descending order

$$\begin{aligned} \mathcal{M}_{-4}^{(1)} &:= \{1, 0, -2, -5, -9, \dots\}, \\ \mathcal{M}_{-4}^{(2)} &:= \left\{ \frac{5}{8}, -\frac{27}{8}, -\frac{91}{8}, \dots \right\}, \\ \mathcal{M}_{-4}^{(3)} &:= \left\{ \frac{65}{72}, \frac{17}{72}, -\frac{175}{72}, -\frac{319}{72}, -\frac{703}{72}, \dots \right\}. \end{aligned} \tag{3.17}$$

Now the set \mathcal{M}_{-4}^+ of elements of \mathcal{M}_{-4} which are positive and different from one contains three elements

$$\mathcal{M}_{-4}^+ := \left\{ \frac{17}{72}, \frac{5}{8}, \frac{65}{72} \right\}.$$

Thus we have

$$\lambda_1 = \lambda \in \mathcal{M}_{-4}^+ \implies \lambda_2 = -5\lambda \in \left\{ -\frac{85}{72}, -\frac{25}{8}, -\frac{325}{72} \right\}.$$

Just a direct inspection of (3.17) shows that $\lambda_2 = -5\lambda \notin \mathcal{M}_{-4}$ and this ends the proof for $k = -4$.

For $k = -2n = -5$ we have

$$\mathcal{M}_{-5} = \bigcup_{i=1}^4 \mathcal{M}_{-5}^{(i)},$$

where

$$\mathcal{M}_{-5}^{(3)} := \left\{ \frac{49}{40} - \frac{5}{18}(1 + 3p)^2 \mid p \in \mathbb{Z} \right\}, \quad \mathcal{M}_{-5}^{(4)} := \left\{ \frac{49}{40} - \frac{1}{10}(2 + 5p)^2 \mid p \in \mathbb{Z} \right\}. \tag{3.18}$$

Proceeding in the way similar to the previous case we prove the statement for $k = -5$.

For $k = -2n < -5$, we have

$$\mathcal{M}_k := \mathcal{M}_k^{(1)} \cup \mathcal{M}_k^{(2)}. \tag{3.19}$$

If $\lambda \in \mathcal{M}_k^{(1)}$ is a positive number, then

$$\lambda = p(1 + n - np) > 0.$$

Hence, either

$$p > 0 \quad \text{and} \quad p < 1 + \frac{1}{n},$$

or

$$p < 0 \quad \text{and} \quad p > 1 + \frac{1}{n}.$$

As $n > 2$, the only possibility is $p = 1$, and this gives $\lambda = 1$.

If $\lambda \in \mathcal{M}_k^{(2)}$ is a positive number, then

$$4n\lambda = 2n + 1 - 4n^2p(p + 1) > 0,$$

for a non-negative integer p . The only possibility is $p = 0$, and this gives

$$\lambda = \frac{2n + 1}{4n}.$$

We show that

$$\lambda_2 = -(2n + 1)\lambda = -\frac{(2n + 1)^2}{4n} \notin \mathcal{M}_k.$$

In fact, $\lambda_2 \notin \mathcal{M}_k^{(1)}$ because λ_2 is not an integer. We show that $\lambda_2 \notin \mathcal{M}_k^{(2)}$. If $\lambda_2 \in \mathcal{M}_k^{(2)}$, then there exists a non-negative integer p such that

$$-(2n + 1)^2 = 2n + 1 - 4n^2 p(p + 1).$$

From the above equation we find that

$$n = \frac{2}{\sqrt{1 + 8p(1 + p)} - 3}.$$

Now, assumption that $n > 2$ implies that

$$\sqrt{1 + 8p(1 + p)} < 4,$$

and this in turn forces that $p = 0$. However it is impossible because it gives a negative n . A contradiction proves our claim. \square

Proposition 3.5 *For positive masses m_1, m_2 and m_3 , the eigenvalue λ given by (3.9) cannot be equal to one.*

Proof Let us assume that $\lambda = 1$, and that masses m_1, m_2 and m_3 are positive. Then from (3.9) we can determine m_3 and calculate the ratio m_2/m_3 . After some simple algebra we obtain

$$\frac{m_2}{m_3} = \frac{(\rho(1 + \rho))^{2n+1} - R(\rho)}{(1 + \rho)^{2n+1} - \rho R(\rho)}.$$

Taking into account the explicit form of $R(\rho)$ given by (3.10) we obtain

$$\frac{m_2}{m_3} = -\frac{r_3(\rho)}{\rho} = -\rho^{2n} \frac{(1 + \rho)^{2n+2} - 1}{(1 + \rho)^{2n+2} - \rho^{2n+2}}.$$

The above ratio is negative for a positive ρ . We have a contradiction with assumption that all masses are positive and this ends our proof. \square

Remark 3.6 Instead of the last Proposition we can use the following reasoning to conclude the proof of Theorem 1.1. If $\lambda = 1$, then matrix $\mathbf{M}_r U_r''(\mathbf{e})$ has double eigenvalue 1, and double eigenvalue $-(2n + 1)$. Moreover, it is not diagonalisable, and it has two Jordan blocks of dimension two. It means that those eigenvalues correspond to a non-real Darboux point \mathbf{e} . In this case, by the second part of Theorem 2.2, the system is not integrable.

4 Discussion and comments

For the partially reduced problem we can find other particular solutions. Namely, one can easily prove the following

Proposition 4.1 *Vector*

$$c := \alpha(a_1, b_1, a_2, b_2)^T, \quad \alpha^{2(n+1)} = 2nm, \tag{4.1}$$

satisfies

$$M_r U'_r(c) = c, \tag{4.2}$$

provided that

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 1. \tag{4.3}$$

Moreover, matrix $M_r U''_r(c)$ has eigenvalues $(-2n - 1, 1, \lambda_-, \lambda_+)$, where

$$\lambda_{\pm} := -n \pm (n + 1)\sqrt{1 - 3Q}, \quad Q := \frac{1}{m^2}(m_1 m_2 + m_2 m_3 + m_3 m_1). \tag{4.4}$$

Darboux point c gives a particular solution of the three-body problem corresponding to the triangular Lagrange solution. There are two good properties of this solution: we know it explicitly, and moreover we know explicitly the corresponding eigenvalues. Probably this is why it was used in [Morales-Ruiz and Simon \(2009\)](#), [Simon \(2007\)](#).

However, we cannot prove the non-integrability using only this solution. Let us consider as example the classical case $2n = 1$. Conditions $\lambda_{\pm} \in \mathcal{M}_{-1}$ can be written in the following form

$$-27Q = (p + 1)(p - 1)(p - 2)(p - 4), \tag{4.5}$$

for a certain integer p greater than one. As $Q > 0$, we have only one choice for p , namely $p = 3$, and this gives $Q = 8/27$. We can assume that the sum of masses is one. Then, m_1 and m_2 are independent parameters of the problem and condition $Q = 8/27$ defines a curve on the plane (m_1, m_2) . For all points lying on this curve the necessary conditions for the integrability are satisfied.

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