

# General solution of the Kozai mechanism

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**Abstract** Kinoshita and Nakai (Celest. Mech. Dyn. Astr. 75, 125–147, 1999) gave the analytical solution of the Kozai mechanism. In this solution the eccentricity and the inclination of a disturbed body take any value, but the argument of the pericenter is restricted to take  $0^\circ$  or  $90^\circ$ . In this paper, we derive the general solution that can be applied for any value of the argument of the pericenter.

**Keywords** Kozai mechanism · Analytical solution · Asteroid Kozai (3040) · Natural satellite S2002N3

## 1 Introduction

Kozai (1962) found that if the  $z$ -component of the angular momentum of an asteroid disturbed by outer giant planets is small, the argument of the perihelion librates around  $90^\circ$  or  $270^\circ$  and both the eccentricity and the inclination largely change. This phenomenon is called the Kozai mechanism or the Kozai resonance. This phenomenon was already discussed by Lidov (1961, 1962) in the dynamics of an artificial satellite disturbed by external bodies. The dynamical system discussed by Lidov is mathematically equivalent with the system discussed by Kozai. In this sense, we may call this phenomenon the Lidov–Kozai mechanism rather than the Kozai mechanism. Kozai (1962) showed that the solution could be expressed with use of the Weierstrass elliptic function  $\wp$  and Kinoshita and Nakai (1991) and Kinoshita and Nakai (1999) gave the explicit analytical solutions with use of Jacobi elliptic functions. The first paper (1991) gave only the solution that the argument of the pericenter circulates with the initial condition  $\omega = 0$ . The second paper (1999) gave in addition to the circulation case, the solution in which the argument of the pericenter librates with the

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initial condition  $\omega = \pi/2$  and the solution for the ascending node. In this paper, we give the general solution with any initial values of the eccentricity, the inclination, and the argument of the pericenter. In the process of refereeing of this paper, the referee noticed that [Vashkovyuk \(1999\)](#) had derived the general solution of the Lidov–Kozai mechanism from the different approach.

In Sect. 3, we solve the equations of motion. In Sect. 4, we apply the obtained analytical solution to two dynamical systems—an asteroid Kozai (3040) and an outer satellite of Neptune, S2002N3.

## 2 Summary of the previous works

Here we shortly summarize the method of the treatment of the Lidov–Kozai mechanism. The Hamiltonian for a massless body disturbed by a massive body, which moves outside a disturbed body, in terms of Delaunay variables, is

$$F = F(L, G, H, l, \omega, \Omega, \lambda'), \quad (1)$$

where  $\lambda'$  is the mean longitude of the disturbing body. After the elimination of short periodic terms the new Hamiltonian takes the following form:

$$F^* = F^*(L, G, H, -, \omega, -, -). \quad (2)$$

Since the angle variables  $l$  and  $\Omega$  are cyclic, the corresponding momentums  $L$  and  $H$  are constant.

The number of degrees of freedom is reduced to one and the global feature of the solution can be obtained from the equi-energy curves defined by the Eq. (2). [Kozai \(1962\)](#) gave the equi-energy curves in the  $(2\omega, x)$  plane (Figs. 2–5 in his paper), where

$$x = 1 - e^2. \quad (3)$$

[Kinoshita and Nakai \(1999\)](#) gave the equi-energy curves in  $(\omega, e)$  plane (Fig. 1). The global feature of the Lidov–Kozai mechanism depends on the  $z$ -component of the angular momentum. We introduce a parameter  $h$  as

$$h = (H/L)^2 = (1 - e^2) \cos^2 I = \text{const}. \quad (4)$$

For  $h > 0.6$  the motion is circulation. For  $h < 0.6$  the libration region appears ([Kozai 1962](#)) and the separatrix separates the libration region from the circulation region. The separatrix goes toward  $e = 0$ .

The main part of  $F^*$  is

$$F^* = \gamma C + R, \quad (5)$$

$$C = (2 + 3e^2)(3 \cos^2 I - 1) + 15e^2 \sin^2 I \cos 2\omega, \quad (6)$$

$$\gamma = \frac{1}{16} \frac{m_d}{m_d + m_c} \frac{n_d^2}{(1 - e_d^2)^{3/2}} a_d^2, \quad (7)$$

where  $m_d, n_d, a_d$ , and  $e_d$  are the mass, the mean motion, the semi-major axis, and the eccentricity of a disturbing body,  $m_c$  is the mass of the central body. The first term in (5) is the main perturbation and the second term  $R$  will be neglected in this paper

and  $R/(\gamma C)$  is of order  $ae_d/a_d$  for  $e_d \neq 0$  and  $(a/a_d)^2$  for  $e_d = 0$ , where  $a$  is the semi-major axis of the disturbed body.

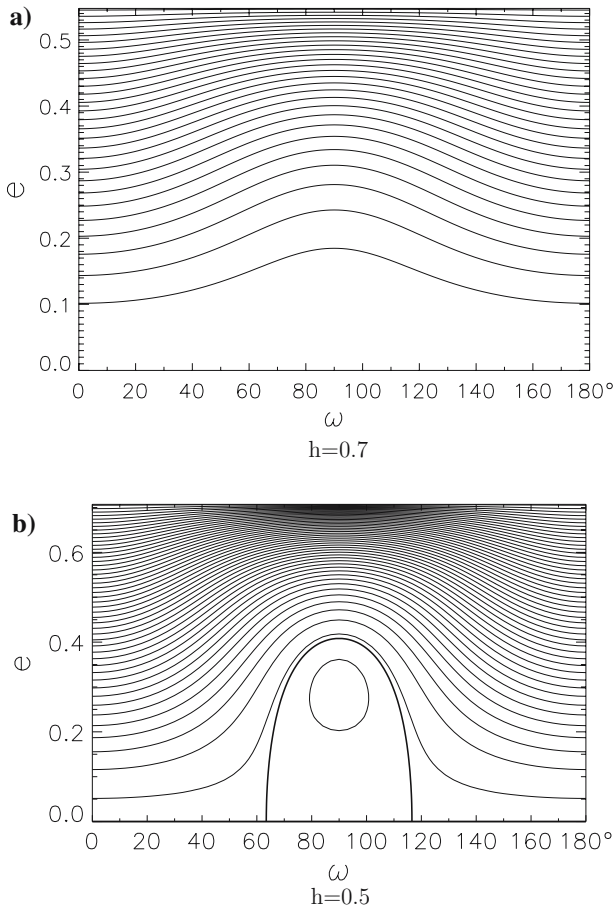
From (6) and (4), the value of the constant  $C$  for the separatrix is

$$C_{se} = 2(3h - 1), \tag{8}$$

which is obtained after substituting  $e = 0$ . Now we can classify the motions by the constants  $h$  and  $C$ , which are evaluated from the initial values of the eccentricity, the inclination, and the argument of the pericenter:

- (a) For  $h > 0.6$  the motion is circulation.
- (b) For  $h < 0.6$  and  $C < C_{se}$  the motion is libration.
- (c) For  $h < 0.6$  and  $C > C_{se}$  the motion is circulation.

Figure 1a, b shows the energy curves for  $h = 0.7$  and  $h = 0.5$ , respectively. As shown in Fig. 1b, when the libration region appears, the change of the eccentricity for the motions near the separatrix is large and the change of the eccentricity for the motions close to the libration center is small.



**Fig. 1** Equi-energy curves: the thick line in Fig. 1b is the separatrix

Now the equations of motion are obtained from the Hamiltonian (5):

$$\frac{d\sqrt{x}}{dt^*} = -\frac{15}{8}e^2 \sin^2 I \sin 2\omega, \quad (9)$$

$$\frac{d\omega}{dt^*} = -\frac{3}{8\sqrt{x}} \left( x - 5 \cos^2 I + 5(\cos^2 I - x) \cos 2\omega \right), \quad (10)$$

where

$$t^* = \gamma^* t, \gamma^* = \frac{m_d}{m_d + m_c} (1 - e_d^2)^{-3/2} \frac{n_d^2}{n}. \quad (11)$$

### 3 General solution

Since the number of degrees of freedom is one, the motion is integrable. With use of the energy integral (6) we eliminate  $\omega$  from (9) and derive the equation for  $x$ .

The energy integral  $C$  is expressed in terms of initial values of the eccentricity, the inclination, and of the argument pericenter:

$$C = (5 - 3x_0) \left( \frac{3h}{x_0} - 1 \right) + 15(1 - x_0) \left( 1 - \frac{h}{x_0} \right) \cos 2\omega_0. \quad (12)$$

From Eqs. (6) and (12) we have

$$\cos 2\omega = \frac{Q(x)}{5(1-x)(x-h)}, \quad (13)$$

$$Q(x) = (x_0 - x) \left( x - \frac{5h}{x_0} \right) + 5(1 - x_0) \left( 1 - \frac{h}{x_0} \right) x \cos 2\omega_0. \quad (14)$$

From this the following equation is derived:

$$\sin^2 \omega = \frac{2x(x_0^* - x)}{5(1-x)(x-h)}, \quad (15)$$

where

$$x_0^* = \frac{1}{4}(C_1 - C_2), \quad (16)$$

$$C_1 = 5 + 5h, \quad (17)$$

$$C_2 = \frac{5h}{x_0} + x_0 + 5(1 - x_0) \left( 1 - \frac{h}{x_0} \right) \cos 2\omega_0. \quad (18)$$

Similarly we get

$$\cos^2 \omega = \frac{3(x_2^* - x)(x - x_1^*)}{5(1-x)(x-h)}, \quad (19)$$

where  $x_1^*, x_2^* (x_1^* < x_2^*)$  are the roots of the following equation:

$$-3x^2 + \alpha x - 5h = 0, \quad (20)$$

$$\alpha = \frac{1}{2}(C_1 + C_2). \tag{21}$$

From (9) with use of (4), (15), and (19) the differential equation for  $x$  is

$$\frac{dx}{dt^*} = -\frac{3\sqrt{6}}{2}\sqrt{(x - x_1^*)(x_0^* - x)(x_2^* - x)}. \tag{22}$$

The form of the solution of (22) depends on the arrangement of  $x_0^*, x_1^*, x_2^*$ . Here we define

$$\alpha_0 = \min\{x_0^*, x_1^*, x_2^*\}, \tag{23}$$

$$\alpha_1 = \text{med}\{x_0^*, x_1^*, x_2^*\}, \tag{24}$$

$$\alpha_2 = \max\{x_0^*, x_1^*, x_2^*\}. \tag{25}$$

Now the solution of (22) is expressed as

$$x = \alpha_1 + (\alpha_0 - \alpha_1)\text{cn}^2\theta, \tag{26}$$

where

$$\theta = \frac{2K}{\pi}\left(\omega^* + \frac{\pi}{2}\right), \tag{27}$$

$$\omega^* = n_{\omega^*}t + \omega_0^*, \tag{28}$$

$$n_{\omega^*} = \frac{3\sqrt{6}\pi}{8K}\sqrt{\alpha_2 - \alpha_0}\gamma^*. \tag{29}$$

In the above expressions  $K$  is the complete elliptic integral of the first kind with the modulus  $k$

$$k^2 = \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}. \tag{30}$$

From the solution (26) and the integral (4) we get the extreme values of the eccentricity and the inclination:

$$e_{\max} = \sqrt{1 - \alpha_0}, \quad e_{\min} = \sqrt{1 - \alpha_1}, \tag{31}$$

$$I_{\max} = \cos^{-1}\sqrt{h/\alpha_1}, \quad I_{\min} = \cos^{-1}\sqrt{h/\alpha_0}. \tag{32}$$

The period of the angle variable  $\omega^*$ , which corresponds to the argument of the pericenter  $\omega$  is

$$P_{\omega^*} = \frac{2\pi}{n_{\omega^*}}. \tag{33}$$

The period of the changes of the eccentricity and the inclination is  $P_{\omega^*}/2$ .

The equation for the ascending node is derived from the Hamiltonian (5):

$$\frac{d\Omega}{dt} = \frac{3}{4}\gamma^*\hat{h}\left(1 - 2\frac{x_0^* - h}{x - h}\right), \tag{34}$$

where

$$\hat{h} = H/L. \quad (35)$$

By the similar method in Kinoshita and Nakai (1999), the Fourier expansion of  $\Omega$  is obtained.

$$\Omega = \Omega^* + \sum_{m=1}^{\infty} b_m \sin 2m\omega^*, \quad (36)$$

$$\Omega^* = n_{\Omega^*}t + \Omega_0^*, \quad (37)$$

where  $\Omega^*$  is the angle variable corresponding to the longitude of node and  $\Omega_0^*$  is an integral constant. The secular motion of  $\Omega$  is expressed as

$$n_{\Omega^*} = -\frac{3}{4}\hat{h}\gamma^* \left(-1 + 2\frac{x_0^* - h}{\alpha_2 - h}\right) - \epsilon \Lambda_0(\xi, k)n_{\omega^*}, \quad (38)$$

where  $\Lambda_0$  is Heuman's Lambda function:

$$\Lambda_0(\xi, k) = \frac{2}{\pi} (EF(\xi, k') + KE(\xi, k') - KF(\xi, k')). \quad (39)$$

Here  $E$  is the complete elliptic integral of the second kind,  $F(\xi, k')$  is the normal elliptic integral of the first kind and  $E(\xi, k')$  is the normal elliptic integral of the second kind. The arguments  $\xi, k'$  in  $F(\xi, k')$  and  $E(\xi, k')$  are defined as

$$\sin \xi = \sqrt{\frac{\alpha_2 - \alpha_0}{\alpha_2 - h}}, \quad (40)$$

$$k' = \sqrt{1 - k^2}. \quad (41)$$

The coefficient  $\epsilon$  of  $\Lambda_0$  in (38) is 1 for  $I < \pi/2$  and  $-1$  for  $I > \pi/2$ . The period of the ascending node is

$$P_{\Omega} = \frac{2\pi}{n_{\Omega}}. \quad (42)$$

The amplitudes of periodic terms with arguments  $2\omega^*$  in the expansion (36) are

$$b_m = -\frac{2(-q)^m}{m(1 - q^{2m})} \sinh m \frac{\pi c}{K}, \quad (43)$$

where

$$c = F(\xi, k') \quad (44)$$

and  $q$  is Jacobi's nome, which is a function of  $k^2$ .

#### 4 Application

We applied the analytical solutions given in the previous section to two solar system bodies—an asteroid, Kozai (3040), disturbed by Jupiter and a natural satellite, S2002N3, disturbed by the Sun. Kozai (3040) moves in the libration region and

**Table 1** Initial elements of disturbed bodies and disturbing bodies

	Kozai (3040)	Jupiter	S2002N3	Sun
$a$ (AU)	1.841	5.20	0.157	30.1104
$e$	0.2005	0.049	0.4237	0.009
$I$ (degree)	46.64	0.0	34.71	0.0
$\omega$ (degree)	290.2	10.0	142.4	10.0
$\Omega$ (degree)	10.0	10.0	10.0	10.0
$M$ (degree)	10.0	10.0	10.0	10.0

**Table 2** Comparison between Analytical Solution (A.S.) and Numerical Solution (N.S.)

	Kozai (3040)		S2002N3	
	A.S.	N.S.	A.S.	N.S.
$e_{\max}$	0.481	0.557	0.534	0.53
$e_{\min}$	0.138	0.147	0.354	0.36
$I_{\max}$ (degree)	47.23	47.0	37.23	37.2
$I_{\min}$ (degree)	39.90	36.0	28.21	28.6
$P_{\omega^*}$ (y.)	106100	85700	2440	2160
$P_{\Omega}$ (y.)	75700	68200	3150	3090

S2002N3 moves in the circulation region. We numerically integrated their orbits, of which initial elements are given in the Table 1. The reference plane is the orbital plane of the disturbing body. Since, we discussed the long periodic perturbations after the elimination of short periodic terms and the longitude of the node  $\Omega$  is cyclic, we adopted  $10^\circ$  as the initial values for  $\Omega$  and the mean anomaly  $M$  of the disturbed body, and  $\omega_d, \Omega_d,$  and  $M_d$  of the disturbing body. The numerical solutions do include short periodic terms and the numerical values of  $e_{\max}, e_{\min}, I_{\max}, I_{\min}, P_{\omega^*},$  and  $P_{\Omega}$  do depend on the initial values of  $\Omega, M, \omega_d, \Omega_d,$  and  $M_d$ . In Table 2, we give  $e_{\max}, e_{\min}, I_{\max}, I_{\min}, P_{\omega^*},$  and  $P_{\Omega}$  obtained from the analytical solution and the numerically integrated orbits. In the evaluation of the analytical solution we adopted the osculating elements (in Table 1) as the mean elements. Since the numerically obtained elements include short periodic terms, the numerical values for the integrated orbits are the averaged values. The differences between the analytical solution and the numerical solution for Kozai (3040) is mainly caused by the neglected disturbing function  $R$  (5) since the neglected term  $R$  is not small since  $a/a_d \simeq 0.35$ . The neglected  $R$  for S2002N3 is small because of  $a/a_d \simeq 0.005$ . The differences for S2002N3 are mainly caused by the difference between the osculating elements and the mean elements. The elements in the analytical solution are mean elements and the elements used in the numerical integration are osculating elements. In the evaluation of the analytical solution we adopted the osculating elements (in Table 1) as the mean elements.

If we want to make a precise ephemeris of a disturbed body, we have to take into account the short periodic terms, the neglected term  $R$  as the perturbation, and distinguish the mean elements from the osculating elements.

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