ERRATUM TO: ON BASE RADICAL AND SEMISIMPLE CLASSES DEFINED BY CLASS OPERATORS

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Abstract. Regarding the article of the title, which appeared in *Acta Math. Hungar.*, **138** (2013), 307–328, we regret to note that some of the arguments given in Section 6 are faulty. In particular, the proof of Theorem 6.3 is flawed (even if one corrects its statement to the more plausible claim that for every $a \in A$, $\mathcal{X}_b(a) \in \mathbf{S}(\mathcal{R})$). Hence also Corollaries 6.4 and 6.5 as well as Proposition 6.7 are in doubt as stated.

However, the main claim of the section, that every base radical/semisimple pair arises from a Hoehnke radical is still correct, although a different proof is required. In particular the stated definition of $\mathcal{X}_b$ must be altered in order that the arguments work. (Perhaps the earlier definition can be made to work, but we do not see how.)

We repeat the general definition of a Hoehnke radical operation on a universal class below, and follow it with the needed new arguments.

1. The new proof

In [2], which was reference [11] in the original article, the notion of a Hoehnke radical operation is defined. Adapted to the current setting, this can be expressed as follows: ρ is a Hoehnke radical operation on the universal class $\mathcal{A}$ if for each $a \in \mathcal{A}$, $\rho(a) \triangleleft a$, and the following are satisfied: for every $a \in \mathcal{A}$,

1. for $i \triangleleft a$, $(i \vee \rho(a)) / i \leq \rho(a/i)$, and

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2. $\rho(a/\rho(a)) = 0$.

The Hoehnke radical and semisimple classes corresponding to ρ are then all $a \in \mathcal{A}$ such that $\rho(a) = a$ and all $a \in \mathcal{A}$ for which $\rho(a) = 0$ respectively; let us call these a *Hoehnke pair*.

For any class $\mathcal{X}$ in $\mathcal{A}$, and for any $a \in \mathcal{A}$, define

$$\overline{\mathcal{X}}(a) = \bigwedge \{n \triangleleft a \mid a/n \in \mathcal{X}\}.$$

Theorem 3 of [1] applies to general algebras A in some suitable universal class and defines $\overline{\mathcal{X}}(A)$ as an intersection of congruences, and states that for any class $\mathcal{X}$ in $\mathcal{A}$, $\overline{\mathcal{X}}$ is a Hoehnke radical operation. The argument is similar in the current abstract setting, but we give it for completeness.

LEMMA 1.1. $\overline{\mathcal{X}}$ is a Hoehnke radical operation.

PROOF. Let $i \triangleleft a$. To show that $(i \vee \overline{\mathcal{X}}(a))/i \leq \overline{\mathcal{X}}(a/i)$, it suffices to show that if $k \triangleleft (a/i)$ and $(a/i)/k \in \mathcal{X}$, then $(i \vee \overline{\mathcal{X}}(a))/i \leq k$. From axiom 3 it follows that $k = j/i$ for some $j \triangleleft a$ such that $i \leq j$. Since $(a/i)/(j/i) \cong a/j$, we get $a/j \in \mathcal{X}$. Hence from the definition of $\overline{\mathcal{X}}(a)$, it follows that $\overline{\mathcal{X}}(a) \leq j$. Consequently $i \vee \overline{\mathcal{X}}(a) \leq j$. This implies that $(i \vee \overline{\mathcal{X}}(a))/i \leq j/i = k$, and we are done.

For the second property, let $i = \overline{\mathcal{X}}(a/\overline{\mathcal{X}}(a))$ and assume $i \neq 0$. By axiom 3, $i = j/\overline{\mathcal{X}}(a)$ for some $j \triangleleft a$ where $\overline{\mathcal{X}}(a) < j$. So if $n \triangleleft a$ is such that $a/n \cong (a/\overline{\mathcal{X}}(a))/(n/\overline{\mathcal{X}}(a)) \in \mathcal{X}$, then $i = j/\overline{\mathcal{X}}(a)$ is below $n/\overline{\mathcal{X}}(a) \triangleleft a/\overline{\mathcal{X}}(a)$, and so $j < n$. Hence $j \leq \overline{\mathcal{X}}(a)$ by the definition of $\overline{\mathcal{X}}(a)$, which is a contradiction. So $i = 0$. $\square$

Denote by $R_{\overline{\mathcal{X}}}$ the Hoehnke radical class determined by $\overline{\mathcal{X}}$, and $\mathcal{S}_{\overline{\mathcal{X}}}$ the corresponding Hoehnke semisimple class. Now note that the following are equivalent: $a \in R_{\overline{\mathcal{X}}}$; $\overline{\mathcal{X}}(a) = a$; for $i \triangleleft a$, $i \neq a$, $a/i \notin \mathcal{X}$; $a \in \mathbf{U}(\mathcal{X})$. So in general, $R_{\overline{\mathcal{X}}} = \mathbf{U}(\mathcal{X})$. Hence if $\mathcal{X}$ is a base semisimple class, $R_{\overline{\mathcal{X}}}$ will be the corresponding base radical class, but it remains to show that $\mathcal{S}_{\overline{\mathcal{X}}} = \mathcal{X}$.

THEOREM 1.2. The base radical and semisimple pair $\mathcal{R}, \mathbf{S}(\mathcal{R})$ is determined by the Hoehnke radical operation $\overline{\mathbf{S}(\mathcal{R})}$.

PROOF. As noted above, it suffices to show that $\mathcal{S}_{\overline{\mathbf{S}(\mathcal{R})}} = \mathbf{S}(\mathcal{R})$. It is clear enough that $\mathbf{S}(\mathcal{R}) \subseteq \mathcal{S}_{\overline{\mathbf{S}(\mathcal{R})}}$, so we show the converse.

Let $a \in \mathcal{S}_{\overline{\mathbf{S}(\mathcal{R})}}$. Suppose there exists non-zero $k \in \mathcal{R}$ that is accessible in a . Suppose that for all $j \triangleleft a$, if $a/j \in \mathbf{S}(\mathcal{R})$ then $k \leq j$. Then $k \leq \overline{\mathbf{S}(\mathcal{R})}(a) = 0$, a contradiction. So there is $j \triangleleft a$ for which $a/j \in \mathbf{S}(\mathcal{R})$ and $k \not\leq j$.

Now $k \vee j \triangleleft^* a$ by the second part of Lemma 6.2, so $(k \vee j)/j \triangleleft^* a/j$ by correspondence (axiom 3). But

$$0 \neq (k \vee j)/j \cong k/(k \wedge j) \in \mathcal{R}$$

since $\mathcal{R}$ is factor closed. Hence a/j has a non-zero accessible in $\mathcal{R}$, contradicting the fact that $a/j \in \mathbf{S}(\mathcal{R})$. So no such radical accessible k of a exists, and so $a \in \mathbf{S}(\mathcal{R})$. $\square$

References

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