Cyclic Contractions of Dimer Algebras Always Exist

Charlie Beil¹

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Abstract We show that every nondegenerate dimer algebra A on a torus admits a cyclic contraction to a cancellative dimer algebra. This implies, for example, that A is Calabi-Yau if and only if it is noetherian; and that the center of A has Krull dimension 3.

Keywords Dimer algebra · Dimer model · Noncommutative algebraic geometry · Non-noetherian ring

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1 Introduction

The main objective of this article is to show that every nondegenerate dimer algebra on a torus admits a cyclic contraction to a cancellative (i.e., consistent) dimer algebra. Dimer algebras were introduced in string theory [16, 17, 20], and have found wide application to many areas of mathematics, such as noncommutative resolutions [5, 7, 11, 13, 22], the McKay correspondence [14, 21], cluster algebras and categories [1, 19], number theory [12], and mirror symmetry [10, 15, 18].

A dimer algebra $A = kQ/I$ is a quiver algebra whose quiver Q embeds into a compact surface, with relations I defined by a potential (see Definition 2.1); in this article we will assume that the surface is a torus. A dimer algebra is said to be nondegenerate if each arrow is contained in a perfect matching.

Presented by: Henning Krause

✉ Charlie Beil
charles.beil@uni-graz.at

¹ Institut für Mathematik und Wissenschaftliches Rechnen, Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria

Let $A = kQ/I$ and $A' = kQ'/I'$ be nondegenerate dimer algebras, and suppose Q' is obtained from Q by contracting a set of arrows $Q_1^* \subset Q_1$ to vertices. This operation defines a k -linear map of path algebras

$$\psi : kQ \rightarrow kQ'.$$

If $\psi(I) \subseteq I'$, then ψ induces a k -linear map of dimer algebras, called a *contraction*,

$$\psi : A \rightarrow A'.$$

If, in addition, A' is cancellative and ψ preserves the so-called cycle algebra, then ψ is called a *cyclic contraction*. An example of a cyclic contraction is given in Fig. 1. Cyclic contractions were introduced in [2], and have been an essential tool in the study of non-cancellative dimer algebras. Our main result is the following.

Theorem 1.1 *Every nondegenerate dimer algebra on a torus admits a cyclic contraction.*

This theorem is important because it implies that the results of [2–4, 6, 8], which assume the existence of cyclic contractions, hold for every nondegenerate dimer algebra. For example, suppose A admits a cyclic contraction $\psi : A \rightarrow A'$; then A is cancellative if and only if A is noetherian, if and only if its center Z is noetherian, if and only if A is a finitely generated Z -module [4, Theorem 1.1]. Furthermore, if A is non-cancellative, then Z is non-noetherian of Krull dimension 3, generically Gorenstein, and contains precisely one closed point $\mathfrak{z}_0$ of positive geometric dimension [6, Theorem 1.1]. In this case, A is locally Morita equivalent to A' away from $\mathfrak{z}_0$, and the Azumaya locus of A coincides with the intersection of the Azumaya locus of A' and the noetherian locus of Z [5, Theorem 1.1].

We emphasize two points regarding the structure of the map $\psi : A \rightarrow A'$ and the cycle algebra S , assuming ψ is nontrivial.

- Consider the idempotent

$$\epsilon := 1_A - \sum_{\delta \in Q_1^*} e_{h(\delta)}.$$

Although ψ itself is only a k -linear map and not an algebra homomorphism, the restriction to the subalgebra

$$\psi : \epsilon A \epsilon \rightarrow A'$$

is an algebra homomorphism. This restriction becomes an algebra isomorphism under localizations away from $\mathfrak{z}_0$ [5, Proposition 2.12.1].

- The cycle algebra S is isomorphic to the center of A' , and is a depiction of the reduced center $Z/\text{nil } Z$ of A [6, Theorem 1.1]. Let

$$\mathbb{S}(A) \subset \text{Rep}_1 \mathcal{Q}_0(A) \quad \text{and} \quad \mathbb{S}(A') \subset \text{Rep}_1 \mathcal{Q}'_0(A')$$

Fig. 1 (Example 4.4.) A cyclic contraction $\psi : A \rightarrow A'$. Both quivers are drawn on a torus, and Q' is obtained from Q by contracting the arrow $\delta \in Q_1$. δ is the only nonrigid arrow in Q

be the open subvarieties consisting of simple modules over A and A' , respectively. Denote by $\overline{S}(A)$ and $\overline{S}(A')$ their Zariski closures. Then S is isomorphic to the GL-invariant rings [5, Theorem 3.14]

$$S \cong k[\overline{S}(A)]^{\text{GL}} \cong k[\overline{S}(A')]^{\text{GL}}. \quad (1)$$

Remark 1.2 In the context of a four-dimensional $\mathcal{N} = 1$ abelian quiver gauge theory with quiver Q , the mesonic chiral ring is a commutative algebra generated by all the cycles in Q modulo the superpotential relations I . Theorem 1.1 then states, loosely, that every low energy non-superconformal dimer theory can be Higgsed to a superconformal dimer theory with the same mesonic chiral ring. (The mesonic chiral ring is not quite the same as the cycle algebra, however; see [5, Remark 3.15].)

We give a brief outline of our proof of Theorem 1.1. To obtain a cyclic contraction of a dimer algebra $A = kQ/I$, we form a sequence of contractions starting with Q , where in each iteration a single arrow is contracted; an example is given in Fig. 2. Each contracted arrow δ has the property that each perfect matching D containing δ can be ‘moved off’ of δ and onto a different set of arrows D' , in such a way that D and D' are identical perfect matchings from the perspective of the cycles in the quiver. Since D can be transformed into D' in this way, we call D a ‘nonrigid perfect matching’. Furthermore, we say δ is a ‘nonrigid arrow’, since we may contract δ without changing the underlying cycle structure of A .

We then show that a perfect matching D is nonrigid if and only if it is simple, that is, $Q \setminus D$ supports a simple representation of dimension vector $(1, \dots, 1)$. Moreover, it was shown in [4] that if each arrow of a dimer algebra is contained in a simple matching, then it is cancellative (in fact, this is a necessary and sufficient condition for cancellativity). Therefore, by running the sequence of single arrow contractions until there are only rigid arrows remaining—in particular, until each arrow is contained in a simple matching—we end with a dimer algebra that is cancellative and has the same cycle structure as A .

2 Preliminary Definitions

Throughout, k is an algebraically closed field. Given a quiver Q , we denote by kQ the path algebra of Q , and by Q_ℓ the paths of length ℓ . The idempotent at vertex $i \in Q_0$ is denoted e_i , and the head and tail maps are denoted $h, t : Q_1 \rightarrow Q_0$. Multiplication of paths is read right to left, following the composition of maps.

Definition 2.1 • A dimer quiver Q is a quiver whose underlying graph $\overline{Q}$ embeds into a real two-torus T^2 such that each connected component of $T^2 \setminus \overline{Q}$ is simply connected

Fig. 2 (Example 4.6.) A maximal sequence of contractions as in (10). Each quiver is drawn on a torus. Q' is a cancellative dimer quiver with a length 1 unit cycle, and the two blue loops are redundant generators for the dimer algebra $A' = kQ'/I'$

and bounded by an oriented cycle, called a *unit cycle*.¹ The *dimer algebra* A of Q is the quotient kQ/I , where I is the ideal

$$I := \langle p - q \mid \exists a \in Q_1 \text{ s.t. } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ, \quad (2)$$

and p, q are (possibly trivial) paths.

- A and Q are *non-cancellative* if there are paths $p, q, r \in A$ for which $p \neq q$, and

$$pr = qr \neq 0 \quad \text{or} \quad rp = rq \neq 0;$$

otherwise A and Q are *cancellative*.

- Since I is generated by binomials in the paths of Q , we also refer to the equivalence class $p + I$ of a path p in Q as a *path* in A . If p and q are paths in Q (resp. A) that are equal modulo I , then we will write $p \equiv q$ (resp. $p = q$).

In the literature, unit cycles are typically required to have length at least 2 or 3. However, we allow unit cycles to have length 1 since such cycles may form under a cyclic contraction. An example of a cancellative dimer algebra with a length 1 unit cycle is given in Example 4.6.

If $a \in Q_1$ is a unit cycle and pa is the complementary unit cycle containing a , then p equals the vertex $e_{t(a)}$ modulo I . The case where p has length 1 leads us to introduce the following definition.

Definition 2.2 A length 1 path $a \in Q_1$ is an *arrow* if a is not equal to a vertex modulo I ; otherwise a is a *pseudo-arrow*.

The following well-known definitions are slightly modified under our distinction between arrows and length 1 paths.

Definition 2.3 • A *perfect matching* D of Q is a set of arrows such that each unit cycle contains precisely one arrow in D .

- A perfect matching D is *simple* if there is a cycle in $Q \setminus D$ that passes through each vertex of Q (that is, the subquiver with arrow set $Q_1 \setminus D$ supports a simple A -module of dimension $1^{|Q_0|}$).

For each perfect matching D , consider the map

$$n_D : Q_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$$

defined by sending a path p to the number of arrow subpaths of p that are contained in D . Note that if $p, q \in Q_{\geq 0}$ are paths satisfying $t(p) = h(q)$, then

$$n_D(pq) = n_D(p) + n_D(q).$$

Furthermore, if p, p' are paths satisfying $p \equiv p'$, then $n_D(p) = n_D(p')$. In particular, n_D induces a well-defined map on the paths of A .

¹In more general contexts, the two-torus may be replaced by a compact surface (e.g., [1, 12]). The dual graph of a dimer quiver is called a dimer model or brane tiling.

Now consider a contraction of dimer algebras $\psi : A \rightarrow A'$, with A' cancellative. Consider the polynomial ring $k[x_D \mid D \in \mathcal{S}']$ generated by the simple matchings $\mathcal{S}'$ of A' . To each path $p \in A'$, associate the monomial

$$\bar{\tau}(p) := \prod_{D \in \mathcal{S}'} x_D^{n_D(p)}.$$

The map ψ is called a *cyclic contraction* if

$$S := k[\cup_{i \in Q_0} \bar{\tau}(\psi(e_i A e_i))] = k[\cup_{i \in Q'_0} \bar{\tau}(e_i A' e_i)] =: S'.$$

In this case, we call S the *cycle algebra* of A and A' , and say ψ ‘preserves the cycle algebra’.²

Notation 2.4 Denote by $\mathcal{P}$ and $\mathcal{P}'$ the set of perfect matchings of A and A' , respectively. For each pair of vertices $i, j \in Q_0$ and $i', j' \in Q'_0$, consider the k -linear maps³

$$\bar{\eta} : e_j A e_i \rightarrow k[x_D \mid D \in \mathcal{P}] \quad \text{and} \quad \bar{\eta}' : e_{j'} A' e_{i'} \rightarrow k[y_D \mid D \in \mathcal{P}'],$$

defined on paths by

$$\bar{\eta}(p) := \prod_{D \in \mathcal{P}} x_D^{n_D(p)} \quad \text{and} \quad \bar{\eta}'(q) := \prod_{D \in \mathcal{P}'} y_D^{n_D(q)},$$

and extended k -linearly. Note that for a path p , we have $x_D \mid \bar{\eta}(p)$ if and only if p has an arrow subpath that is contained in D .

For $i \in Q_0$, denote by σ_i either a choice of unit cycle in $e_i k Q e_i$, or the unique unit cycle in $e_i A e_i$. Denote the $\bar{\eta}$, $\bar{\eta}'$, and $\bar{\tau}$ images of the unit cycles in Q and Q' by

$$\sigma_{\mathcal{P}} := \prod_{D \in \mathcal{P}} x_D, \quad \sigma_{\mathcal{P}'} := \prod_{D \in \mathcal{P}'} y_D, \quad \sigma_{\mathcal{S}'} := \prod_{D \in \mathcal{S}'} y_D.$$

Let $\pi : \mathbb{R}^2 \rightarrow T^2$ be a covering map such that for some vertex $i \in Q_0$, we have $\pi^{-1}(i) = \mathbb{Z}^2$. Denote by $Q^+ := \pi^{-1}(Q)$ the infinite covering quiver of Q . For each path p in Q , denote by p^+ the path in Q^+ with tail in $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.

Notation 2.5 By a *cyclic subpath*, we mean a cyclic subpath that is not equal to a vertex modulo I .

Consider the following sets of cycles in Q :

- Let $\mathcal{C}$ be the set of cycles in Q .⁴
- For $u \in \mathbb{Z}^2$, let $\mathcal{C}^u$ be the set of cycles $p \in \mathcal{C}$ such that

$$h(p^+) = t(p^+) + u \in Q_0^+.$$

- For $i \in Q_0$, let $\mathcal{C}_i$ be the set of cycles in the vertex corner ring $e_i k Q e_i$.
- Let $\hat{\mathcal{C}}$ be the set of cycles $p \in \mathcal{C}$ such that the lift of each cyclic permutation of each representative of $p + I$ does not have a cyclic subpath.

²The uniqueness of S follows from Eq. 1; see Corollary 4.16 below.

³The monomial labelings $\bar{\eta}$, $\bar{\eta}'$, and $\bar{\tau}$ define algebra homomorphisms on A and A' ; see [2, Section 2.1].

⁴We caution that in the companion articles [2–6, 8], the set $\mathcal{C}$ is defined to be the set of cycles in A , i.e., cycles in Q modulo I .

We denote the intersection $\hat{\mathcal{C}} \cap \mathcal{C}^u \cap \mathcal{C}_i$, for example, by $\hat{\mathcal{C}}_i^u$. Note that although the lift of a cycle p in $\hat{\mathcal{C}}$ has no cyclic subpaths, p itself may have cyclic subpaths. We similarly define the set of cycles $\mathcal{C}'$ in Q' .

3 Cycles That Avoid A Perfect Matching

Let $A = kQ/I$ be a dimer algebra. Throughout this section, set

$$\sigma := \sigma_P \quad \text{and} \quad \bar{p} := \bar{\eta}(p).$$

By $\bar{p} \mid \bar{q}$, we mean that $\bar{p}$ divides $\bar{q}$ in $k[x_D \mid D \in \mathcal{P}]$. We introduce the following.

Definition 3.1 We say A and Q are *cycle-nondegenerate* if each cycle of Q contains an arrow that is contained in a perfect matching.

Let p be a cycle. If $\sigma \nmid \bar{p}$, then $p \in \hat{\mathcal{C}}$ [2, Lemma 4.8.3]; and if $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $i \in Q_0$ and $u \in \mathbb{Z}^2$, then the converse holds [2, Proposition 4.20.1]. In this section, we show that if Q is cycle-nondegenerate, then for each $u \in \mathbb{Z}^2$ there is a cycle $p \in \hat{\mathcal{C}}^u$ such that $\sigma \nmid \bar{p}$.

Lemma 3.2 Suppose p and q are paths in Q for which

$$t(p^+) = t(q^+) \quad \text{and} \quad h(p^+) = h(q^+).$$

Then there is an $m, n \geq 0$ such that

$$p\sigma_{t(p)}^m \equiv q\sigma_{t(p)}^n \quad \text{and} \quad \bar{p} = \bar{q}\sigma^{n-m}.$$

Furthermore, if p^+ is a cycle in Q^+ , then there is an $m \geq 0$ such that

$$\bar{p} = \sigma^m.$$

Proof The claims hold respectively by [2, Lemmas 4.3.1, 4.3.2, and 4.8.1]. $\square$

For $u \in \mathbb{Z}^2$, set

$$\mathcal{C}^{u,n} := \left\{ p \in \mathcal{C}^u \mid \sigma^n \mid \bar{p} \text{ and } \sigma^{n+1} \nmid \bar{p} \right\}.$$

Let $\mu = \mu(u) \geq 0$ be the smallest integer for which $\mathcal{C}^{u,\mu} \neq \emptyset$. The main purpose of this section is to show that for every $u \in \mathbb{Z}^2$, we have $\mu(u) = 0$.

Fix $u \in \mathbb{Z}^2 \setminus 0$.

Lemma 3.3 Let p and q be subpaths of cycles in $\mathcal{C}^{u,\mu}$. If

$$t(p^+) = t(q^+) \quad \text{and} \quad h(p^+) = h(q^+), \quad (3)$$

then $\bar{p} = \bar{q}$.

Proof Set $p_1 := p$ and $q_1 := q$, and suppose (3) holds. Since p_1, q_1 are subpaths of cycles in $\mathcal{C}^{u,\mu}$, there are (possibly trivial) paths p_2, q_2 such that $p_2 p_1$ and $q_2 q_1$ are cycles in $\mathcal{C}^{u,\mu}$. Since $p_2 p_1$ is a cycle in $\mathcal{C}^u$, we have

$$t(p_1^+) + u = h(p_2^+) = t(q_1^+) + u = h(q_2^+), \quad h(p_1^+) = t(p_2^+) = h(q_1^+) = t(q_2^+).$$

Let n_1 and n_2 be the respective maximum powers of σ that divide $\bar{q}_1$ and $\bar{p}_1$. Assume to the contrary that $n_1 < n_2$. Set $s := n_2 - n_1 > 0$. By Lemma 3.2, we have

$$\bar{p}_1 = \bar{q}_1 \sigma^{n_2 - n_1} = \bar{q}_1 \sigma^s. \quad (4)$$

Since $p_2 p_1$ is in $\mathcal{C}^{u, \mu}$, there is a monomial $g \in k[x_D \mid D \in \mathcal{P}]$, not divisible by σ , such that

$$\overline{p_2 p_1} = g \sigma^\mu.$$

Whence

$$\overline{p_2 q_1} = \bar{p}_2 \bar{q}_1 = \bar{p}_2 \bar{p}_1 \sigma^{-s} = \overline{p_2 p_1} \sigma^{-s} = g \sigma^{\mu-s}.$$

Thus the maximum power of σ that divides $\overline{p_2 q_1}$ is $\mu - s < \mu$. But the cycle $p_2 q_1$ is in $\mathcal{C}^u$, contrary to the minimality of μ . Therefore $n_1 \geq n_2$. A similar argument shows that $n_1 \leq n_2$, and so $n_1 = n_2$. Consequently, $\bar{p}_1 = \bar{q}_1$ by Eq. 4. $\square$

Lemma 3.4 *Let $p = p_m \cdots p_2 p_1$ be a cycle in $\mathcal{C}^u$ such that each p_i is a subpath of a cycle in $\mathcal{C}^{u, \mu}$. Then p is in $\mathcal{C}^{u, \mu}$.*

Proof By induction, it suffices to suppose $p = p_2 p_1$. Since p_1, p_2 are subpaths of cycles in $\mathcal{C}^{u, \mu}$, there are paths q_1, q_2 such that $q_2 p_1$ and $p_2 q_1$ are cycles in $\mathcal{C}_{t(p_1)}^{u, \mu}$. By Lemma 3.3, we have $\bar{p}_2 = \bar{q}_2$ (and $\bar{p}_1 = \bar{q}_1$). Thus

$$\overline{p_2 p_1} = \bar{p}_2 \bar{p}_1 = \bar{q}_2 \bar{p}_1 = \overline{q_2 p_1}.$$

Therefore $p_2 p_1$ is in $\mathcal{C}^{u, \mu}$. $\square$

Lemma 3.5 *If Q has a perfect matching, then there is containment $\mathcal{C}^{u, \mu} \subseteq \hat{\mathcal{C}}^u$.*

Proof Suppose $p \in \mathcal{C}^u \setminus \hat{\mathcal{C}}^u$. Then there is a cyclic permutation q of a representative of $p + I$ such that q^+ has a cyclic subpath r^+ . Clearly, $\bar{q} = \bar{p}$.

By Lemma 3.2, $\bar{r} = \sigma^n$ for some $n \geq 1$. Furthermore, $\sigma \neq 1$ since the set of perfect matchings $\mathcal{P}$ is nonempty. But then upon removing r from q we find that q , whence p , is not in $\mathcal{C}^{u, \mu}$. $\square$

Construction 3.6 Given a path p , denote by $[p]$ the subquiver whose vertex and arrow sets are the vertex and arrow subpaths of p . To show that $\mu = 0$, consider a maximal chain of subquivers

$$Q^0 \subseteq Q^1 \subseteq \cdots \subseteq Q^N \subseteq Q, \quad (5)$$

where

$$Q^0 := \bigcup_{p \in \mathcal{C}^{u, \mu}} [p],$$

and for $1 \leq m \leq N$, there is a particular path s_m such that

$$Q^m := Q^{m-1} \cup [s_m].$$

The sequence of paths $s_1, s_2, \dots, s_N$ is defined inductively as follows.

Fix $n \geq 1$, and suppose that the sequence $s_1, s_2, \dots, s_{n-1}$ has been defined. Consider the set $\mathcal{S}^n$ of paths $s \in Q_{\geq 1}$ for which

- (i) the endpoints of s are in Q_0^{n-1} , and no other subpath of s lies in Q^{n-1} ;
- (ii) if there is a path p^+ in $(Q^{n-1})^+$ with endpoints $t(s^+)$ and $h(s^+)$, such that the interior of the compact region in $\mathbb{R}^2$ bounded by p^+ and s^+ contains no arrows in $(Q^{n-1})^+$, then

$$t(s^+) = t(p^+) \quad \text{and} \quad h(s^+) = h(p^+).$$

(Condition (ii) says, loosely, that s^+ ‘flows in the same direction’ as all the paths in $(Q^{n-1})^+$.)

We now define an order $<$ on $\mathcal{S}^n$. We say $s \in \mathcal{S}^n$ satisfies $(\star)$ if there are paths $p_1, p_2, q_1, q_2 \in Q_{\geq 0}^{n-1}$ such that

$$t((p_2 s_{n-1} p_1)^+) = t((q_2 s q_1)^+), \quad h((p_2 s_{n-1} p_1)^+) = h((q_2 s q_1)^+),$$

and for some non-negative integer $\ell \geq 0$,

$$\overline{p_2 s_{n-1} p_1} = \overline{q_2 s q_1} \sigma^\ell.$$

Given $s, t \in \mathcal{S}^n$, we declare $s < t$ if

- (1) s satisfies $(\star)$ and t does not; or
- (2) s satisfies $(\star)$ iff t satisfies $(\star)$; and $\bar{s} \mid \bar{t}$.

Finally, let s_n be a minimal element of $\mathcal{S}^n$ with respect to $<$.

We now use the chain of subquivers (5) to construct a perfect matching D of Q that avoids the cycles in $\mathcal{C}^{u, \mu}$.

Construction 3.7 For $n \geq 0$, denote by D^n the set of arrows a in $Q_1 \setminus Q_1^n$ such that

- (1) the tail $t(a)$ is in Q_0^n ;
- (2) a is not in a unit cycle that contains an arrow in D^m for $0 \leq m < n$; and
- (3) a is the unique arrow in the unit cycles containing a that satisfies (1) and (2).

Note that D^n is empty for sufficiently large n . Set

$$D := \bigcup_{n \geq 0} D^n \subset Q_1.$$

Proposition 3.8 Suppose Q is cycle-nondegenerate. Then D is a perfect matching of Q .

Proof By conditions (2) and (3) in Construction 3.7, each unit cycle of Q contains at most one arrow in D . Thus, to show that D is a perfect matching of Q , it suffices to show that each unit cycle also contains at least one arrow in D .

Let $N \geq 1$ be the minimum integer for which $Q^N = Q^{N+1}$. Assume to the contrary that there is a unit cycle σ_i that does not have an arrow subpath in D . Then one of the following cases holds.

- (1) σ_i is a unit cycle of Q^0 .
- (2) There is an $n \geq 1$ such that σ_i is a unit cycle of Q^n that is not wholly contained in Q^{n-1} , and σ_i does not contain precisely one arrow subpath in $Q_1^n \setminus Q_1^{n-1}$ with tail in Q_0^{n-1} .
 - (a) σ_i has no arrow subpath $a \in Q_1^n \setminus Q_1^{n-1}$ for which $t(a) \in Q_0^{n-1}$.
 - (b) σ_i has at least two arrow subpaths $a, c \in Q_1^n \setminus Q_1^{n-1}$ with $t(a), t(c) \in Q_0^{n-1}$.
- (3) σ_i has at least two arrow subpaths $a, c \in Q_1 \setminus Q_1^N$ with $t(a), t(c) \in Q_0^N$.

We claim that each case is not possible.

Case (1): Q^0 cannot contain a unit cycle of Q by Lemmas 3.4 and 3.5.

Recall that s_n is a minimal path in $\mathcal{S}^n$, and $Q^n = Q^{n-1} \cup [s_n]$. The following two subcases are shown in Fig. 3.

Case (2.a): By assumption, σ_i is a unit cycle of Q^n that does not meet Q^{n-1} . Thus σ_i may be formed from subpaths of s_n . But then s_n^+ has a proper nontrivial cyclic subpath t^+ . Let r be the path obtained by omitting t from s_n . Since t^+ is a cycle and $s_n \in \mathcal{S}^n$, we have $r \in \mathcal{S}^n$.

Since r is obtained by omitting t from s_n , we have $\bar{r} \mid \bar{s}_n$. Thus, if s_n satisfies $(\star)$, then r satisfies $(\star)$. Whence $r < s_n$. Furthermore, since Q is cycle-nondegenerate, $\bar{t} \neq 1$. In particular, $\bar{s}_n \nmid \bar{r}$. Therefore $s_n \not\leq r$. But this contradicts the minimality of s_n in $\mathcal{S}^n$ with respect to $<$.

Case (2.b): Since a, c are arrows in $Q_1^n \setminus Q_1^{n-1}$, both a and c are subpaths of s_n . By assumption, $t(a), t(c) \in Q_0^{n-1}$, and so s_n meets Q^{n-1} at a trivial subpath other than its endpoints. But then s_n is not in $\mathcal{S}^n$, contrary to our choice of s_n .

Case (3): Let $\mathcal{A}$ be the set of arrow subpaths of σ_i in $Q_1 \setminus Q_1^N$. For $i \in Q_0$, denote by $m(i)$ the minimum integer for which $i \in Q_0^{m(i)}$. Let $a \in \mathcal{A}$ be such that

$$m(t(a)) = \min\{m(t(c)) \mid c \in \mathcal{A}\}.$$

Since $|\mathcal{A}| \geq 2$, each arrow in $\mathcal{A}$ is not in D , by condition (3) of Construction 3.7. In particular, $a \notin D$. Thus there is an arrow $c \in \mathcal{A} \setminus \{a\}$ and a path t^+ with endpoints $t(a^+)$ and $t(c^+)$, such that t is a subpath of $s_{m(t(a))}$. Let $b, d \in Q_{\geq 0}$ be paths such that $\sigma_i = dcba$.

Without loss of generality, suppose

$$t(t^+) = t(a^+) \quad \text{and} \quad h(t^+) = t(c^+).$$

Then by Lemma 3.2, there is an $\ell \in \mathbb{Z}$ such that

$$\bar{t} = \overline{ba} \sigma^\ell. \quad (6)$$

Furthermore, $\ell \geq -1$ since ba is a subpath of a unit cycle.

If $\ell = -1$, then $\overline{dc} = 1$ and $\bar{t} = 1$, since $dcba$ is a unit cycle. In particular, the cycle tdc satisfies

$$\overline{tdc} = \bar{t} \overline{dc} = 1.$$

Fig. 3 Cases for Proposition 3.8. Each quiver is drawn on the torus, the unit cycle σ_i is drawn in brown, and the red arrows are paths in Q^{n-1} . In (2.a), r factors into paths $r = r_2 r_1$, with $r_1, r_2 \in Q_{\geq 1}$. In (2.b), σ_i factors into paths $dcba$, with $b, d \in Q_{\geq 0}$. Irrelevant vertices and arrows are omitted

But then Q is cycle-degenerate, contrary to assumption. Thus $\ell \geq 0$. Therefore (6) implies that ba is in Q^N since t is in Q^N , by property $(\star)$. However, it then follows that a is in Q^N , contrary to assumption.

We have shown that each case is not possible. Therefore each unit cycle contains precisely one arrow in D , and so D is a perfect matching of Q . $\square$

Corollary 3.9 *Suppose Q is cycle-nondegenerate. Then for each $u \in \mathbb{Z}^2$, the set of cycles $\mathcal{C}^{u,0}$ is nonempty.*

Proof If $u = 0 \in \mathbb{Z}^2$, then each vertex of Q is in $\mathcal{C}^{u,0}$.

So let $u \in \mathbb{Z}^2 \setminus 0$, and set $\mu := \mu(u)$. By Proposition 3.8, there is a perfect matching D such that each cycle p in $\mathcal{C}^{u,\mu}$ is supported on $Q \setminus D$. In particular, $x_D \nmid \bar{p}$. Whence, $\sigma \nmid \bar{p}$. Therefore $\mu = 0$. $\square$

Example 3.10 We demonstrate the construction of a perfect matching D that avoids the cycles in $\mathcal{C}^{(0,1)}$. Consider the subquiver Q^+ of a dimer quiver shown in Fig. 4.

- (i) Q^0 is formed from the red arrows, and D^0 consists of the three brown arrows.
- (ii) We may choose s_1 to be the path formed from the red arrows, and s_2 to be the path formed from the blue arrows. Then $D^1 = \emptyset$, and D^2 consists of the single brown arrow.
- (iii) Alternatively, we may choose s_1 to be the cycle at i formed from the red arrows; s_2 to be the purple arrow; and s_3 to be the path formed from the blue arrows. Then $D^1 = D^2 = \emptyset$, and D^3 consists of the single brown arrow. Note that there are other choices for the paths s_1, s_2, s_3 as well.
- (iv) With any choice of paths, the union $D = \cup D^n$ consists of the four brown arrows, and is a perfect matching of Q .

4 Proof of Main Theorem

Let $A = kQ/I$ be a dimer algebra. To prove our main theorem, we introduce the following.

Fig. 4 (Example 3.10.) A subquiver demonstrating the construction of a perfect matching D that avoids the cycles in $\mathcal{C}^{u,0}$

Definition 4.1

- We say two perfect matchings D, D' of Q are *equivalent* if for each cycle p , we have

$$n_D(p) = n_{D'}(p).$$

- A perfect matching is *rigid* if it is not equivalent to another perfect matching.
- An arrow a is *nonrigid* if every perfect matching that contains a is equivalent to a perfect matching that does not contain a ; otherwise a is *rigid*.

Note that an arrow is nonrigid if it is not contained in any perfect matching, and it is rigid if it is contained in a rigid perfect matching. In Proposition 4.8, we will show that rigidity characterizes simple matchings.

Example 4.2 We give an example of equivalent perfect matchings. Suppose D is a perfect matching of Q and $Q \setminus D$ has a source at vertex i . Let α and β be the set of arrows of Q with head at i and tail at i , respectively. Then $\alpha \subseteq D$. Whence $\beta \cap D = \emptyset$, since each unit cycle of Q contains precisely one arrow in D . Let D' be the perfect matching obtained from D by replacing the subset α with the set β . Then D and D' are equivalent perfect matchings.

Proposition 4.3 Suppose Q is cycle-nondegenerate. Let Q' be the quiver obtained from Q by contracting a single nonrigid arrow $\delta \in Q_1$. Then

- (1) no cycle of Q is contracted to a vertex; and
- (2) Q' is a cycle-nondegenerate dimer quiver.

Proof In the following, set $\sigma := \sigma_{\mathcal{P}}$ and $\sigma' := \sigma_{\mathcal{P}'}$. Denote by $\psi : kQ \rightarrow kQ'$ the k -linear map defined by contracting δ . For paths $s \in Q_{\geq 0}$ and $t \in Q'_{\geq 0}$, set

$$\bar{s} := \bar{\eta}(s) \quad \text{and} \quad \bar{t} := \bar{\eta}'(t).$$

- (i) We first claim that Q' is a dimer quiver. It suffices to show that no cycle in Q contracts to a vertex under ψ . Assume to the contrary that there is a (nontrivial) cycle p for which $\psi(p)$ is a vertex.

Since δ is the only contracted arrow, we have $p = \delta$. Furthermore, δ is not equal to a vertex modulo I since it is an arrow, rather than a pseudo-arrow. Thus δ is contained in a perfect matching of Q , since Q is cycle-nondegenerate. But δ is a cycle of length 1. Therefore δ is rigid, contrary to assumption.

- (ii) We claim that if D is a perfect matching of Q and $x_D \nmid \bar{\delta}$ (that is, δ is not contained in D), then $\psi(D)$ is a perfect matching of Q' .

Consider a unit cycle σ'_j of Q' . Each unit cycle of Q' admits a ψ -preimage that is a unit cycle of Q .⁵ Thus, there is a unit cycle σ_i of Q such that $\psi(\sigma_i) = \sigma'_j$. Since D is a perfect matching, there is precisely one arrow subpath a of σ_i that is in D . Furthermore, since D does not contain δ , we have $a \neq \delta$. Whence $\psi(a)$ is an arrow in σ'_j . Thus there is precisely one arrow subpath of σ'_j , namely $\psi(a)$, that is in $\psi(D)$.

⁵We note that a unit 2-cycle may form from contracting δ , and the two arrows it is composed of may be redundant generators of A' [2, Lemma 4.6]. If these arrows are removed, then there would be a unit cycle of Q' that does not have ψ -preimage which is a unit cycle of Q .

Therefore $\psi(D)$ is a perfect matching of Q' .

(iii) We now claim that Q' is cycle-nondegenerate. Let q be a cycle in Q' . We want to show that $\bar{q} \neq 1$.

(iii.a) First suppose that q has a ψ -preimage p which is a cycle. Since Q is cycle-nondegenerate, there is a perfect matching $D \in \mathcal{P}$ for which $x_D \mid \bar{p}$.

If $x_D \nmid \bar{\delta}$, then $\psi(D)$ is a perfect matching of Q' by Claim (ii). Thus $y_{\psi(D)} \mid \bar{q}$.

So suppose $x_D \mid \bar{\delta}$. Since δ is nonrigid, there is a perfect matching $D' \in \mathcal{P}$ equivalent to D that does not contain δ . In particular, $x_{D'} \mid \bar{p}$. Furthermore, $\psi(D')$ is a perfect matching of Q' , again by Claim (ii). Thus $y_{\psi(D')} \mid \bar{q}$.

Therefore in either case, $\bar{q} \neq 1$.

(iii.b) Finally, suppose that q does not have a ψ -preimage which is a cycle (in particular, q may not admit any ψ -preimage). Assume to the contrary that $\bar{q} = 1$.

Since ψ contracts a single arrow, there is a cyclic permutation q' of q , and a path s in Q , such that $\psi(s) = q'$; see Fig. 5. Then

$$\overline{\psi(s)} = \bar{q}' = \bar{q} = 1.$$

Let t be a path for which δt is a unit cycle; then ts is a cycle in Q . Let $u \in \mathbb{Z}^2$ be such that $ts \in \mathcal{C}^u$. By Corollary 3.9, there is a cycle r in $\mathcal{C}^{u,0}$. By Lemma 3.2, there is an $m \in \mathbb{Z}$ such that

$$\sigma^m \bar{r} = \bar{ts}. \quad (7)$$

Since $r \in \mathcal{C}^{u,0}$, we have $\sigma \nmid \bar{r}$. Whence $m \geq 0$. Furthermore,

$$\sigma^m \overline{\psi(r)} \stackrel{(I)}{=} \overline{\psi(ts)} \stackrel{(II)}{=} \overline{\psi(t)\psi(s)} = \overline{\psi(t)} \overline{\psi(s)} \stackrel{(III)}{=} \overline{\psi(t)} \stackrel{(IV)}{=} \sigma', \quad (8)$$

where (I) holds by Eq. 7; (II) holds since $ts \neq 0$; (III) holds since $\overline{\psi(s)} = 1$; and (IV) holds since $\psi(t)$ is a unit cycle.

Now $\bar{r} \neq 1$ since r is a cycle and Q is cycle-nondegenerate. Thus $\overline{\psi(r)} \neq 1$ by Claim (iii.a), with r and $\psi(r)$ in place of p and q , respectively. But then (8) implies $m = 0$. Therefore (7) implies

$$\bar{r} = \bar{ts}. \quad (9)$$

Since $r \in \mathcal{C}^{u,0}$, there is a perfect matching $D \in \mathcal{P}$ such that $x_D \nmid \bar{r}$. Thus $x_D \nmid \bar{ts}$ by Eq. 9. Whence $x_D \nmid \bar{t}$. Consequently, $x_D \mid \bar{\delta}$ since δt is a unit cycle. Furthermore, δ is nonrigid, and so there is a perfect matching $D' \in \mathcal{P}$ equivalent to D such that $x_{D'} \nmid \bar{\delta}$. Thus $x_{D'} \mid \bar{t}$, again since δt is a unit cycle. Hence, $n_{D'}(t) \geq 1$. Therefore, since D and D' are equivalent, we have

$$0 = n_D(r) = n_D(ts) = n_{D'}(ts) = n_{D'}(t) + n_{D'}(s) \geq 1.$$

Fig. 5 Setup for Claim (iii.b) in the proof of Proposition 4.3, drawn on the cover Q^+ . The path δ is an arrow, $t\delta$ is a unit cycle, and r is a cycle in $\mathcal{C}^{u,0}$

But this is not possible, proving our claim. $\square$

Let Q be a cycle-nondegenerate dimer quiver. By Proposition 4.3, we may consider a maximal sequence of k -linear maps of dimer path algebras

$$kQ \xrightarrow{\psi_0} kQ^1 \xrightarrow{\psi_1} kQ^2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_m} kQ', \quad (10)$$

where ψ_j contracts one nonrigid arrow δ_j of Q^j . We claim that the composition

$$\psi := \psi_m \cdots \psi_0 : kQ \rightarrow kQ'$$

induces a cyclic contraction of dimer algebras

$$\psi : A = kQ/I \rightarrow A' = kQ'/I'.$$

In particular, A' is cancellative and the cycle algebra is preserved.

Example 4.4 Consider the cyclic contraction $\psi : A \rightarrow A'$ given in Fig. 1. The arrow δ , contracted under ψ , is nonrigid since it belongs to the perfect matching $D = \{c, \delta\}$, and D is equivalent to the perfect matching $D' = \{a, b\}$ not containing δ . It is straightforward to verify that all the other arrows of A belong to rigid perfect matchings.

The following example demonstrates why it is necessary to define ψ by a sequence of contractions where only one nonrigid arrow is contracted at a time.

Example 4.5 Consider the two contractions of the non-cancellative dimer quiver Q given in Fig. 6. In each case, the contracted quiver Q' is cancellative, and the arrows in Q and Q' are labeled by their respective $\bar{\tau}\psi$ - and $\bar{\tau}$ -images. Furthermore, the arrows $a, b \in Q_1$ are both nonrigid. The cycle algebra is preserved in case (i),

$$S = k[xz, xw, yz, yw] = S'.$$

In contrast, the cycle algebra is not preserved in case (ii),

$$S = k[x, y, xz, yz] \subsetneq k[x, y, z] = S'.$$

This shows that, in general, the cycle algebra will not be preserved if more than one nonrigid arrow is contracted at a time. (In both cases, S is isomorphic to the conifold coordinate ring $k[s, t, u, v]/(st - uv)$.)

Example 4.6 Consider the cyclic contraction $\psi : A \rightarrow A'$ defined by the maximal sequence of contractions given in Fig. 2. Q' is a cancellative dimer quiver with a length 1 unit cycle. Observe that both loops, drawn in blue, are redundant generators for the dimer algebra $A' = kQ'/I'$; however, there is no (well-defined) contraction from A to A' with the loops removed from Q' .

Proposition 4.7 *If a perfect matching is rigid, then it is simple.*

Proof Let A be a dimer algebra, and let D be a perfect matching of Q which is not simple. We want to show that D is not rigid.

Let V be an A -module of dimension 1^{Q_0} with support $Q \setminus D$. Fix a simple submodule S of V . Denote by $Q^S \subset Q$ the supporting subquiver of S , and by $Q \setminus Q^S$ the subquiver of Q obtained by removing the arrows in Q^S .

Let α be the set of arrows in $Q \setminus Q^S$ whose tails lie in Q^S , and let β be the set of arrows in $Q \setminus Q^S$ whose heads lie in Q^S . (The sets α and β need not be disjoint.)

Fig. 6 (Example 4.5.) Two contractions of the non-cancellative dimer quiver Q . Each quiver is drawn on a torus. In case (i) ψ is cyclic, and in case (ii) ψ is not cyclic since the cycle algebra is not preserved

- (i) We claim that $\alpha \subseteq D$. Indeed, let $a \in \alpha$. Then $t(a) \in Q_0^S$ and $a \notin Q_1^S$. Thus, since S is a simple submodule of V , we have $aV = 0$. Whence $a \in D$, proving our claim.

Now consider the set of arrows

$$D' := (D \setminus \alpha) \cup \beta \subset Q_1. \quad (11)$$

- (ii) We claim that D' is a perfect matching of Q . Let $[\sigma]$ be a unit cycle subquiver of Q . It suffices to show that $[\sigma]$ contains precisely one arrow in D' .

First suppose $[\sigma]$ does not intersect Q^S . Then by Eq. 11, the unique arrow in $[\sigma]$ which belongs to D is the unique arrow in $[\sigma]$ which belongs to D' .

So suppose $[\sigma]$ intersects Q^S in a (possibly trivial) path; let p be such a path of maximal length. Then the head of p is the tail of an arrow a in $[\sigma]$ which belongs to α . Whence a belongs to D by Claim (i). Thus p is unique since D is a perfect matching.

Let b be the arrow in $[\sigma]$ whose head is the tail of p . Then b is in β . Thus b belongs to D' by Eq. 11. Furthermore, b is the unique arrow in $[\sigma]$ which belongs to D' since p is unique.

Therefore in either case, $[\sigma]$ contains precisely one arrow in D' .

- (iii) We claim that D and D' are equivalent perfect matchings. Let p be a cycle in A . If p is contained in Q^S , then

$$n_D(p) = 0 = n_{D'}(p).$$

So suppose p is a cycle in Q that is not wholly contained in Q^S . Then p must contain an arrow in β for each instance it enters the subquiver Q^S , and must contain an arrow in α for each instance it exits Q^S . Since p is a cycle, the number of times p enters Q^S equals the number of times p exits Q^S . It follows that

$$n_D(p) = n_{D'}(p).$$

Therefore D and D' are equivalent.

- (iv) Finally, we claim that D is not rigid. By Claim (iii) it suffices to show that $D' \neq D$. Since D is not simple, we have $S \neq V$. Whence $\alpha \neq \beta$. Therefore $D' \neq D$. $\square$

Proposition 4.8 *If a perfect matching is simple, then it is rigid. Consequently, a perfect matching is simple if and only if it is rigid.*

Proof Let A be a dimer algebra, and let D be a perfect matching of Q which is not rigid. We want to show that D is not simple.

Assume to the contrary that D is simple. Then there is a cycle p that contains each arrow in $Q_1 \setminus D$. Whence, $n_D(p) = 0$. Furthermore, since D is nonrigid, D is equivalent to a perfect matching $D' \neq D$. In particular,

$$n_{D'}(p) = n_D(p).$$

Let $a \in D' \setminus D$. Since $a \notin D$, a is a subpath of p . Thus, since $a \in D'$, we have $n_{D'}(p) \geq 1$. But then

$$0 = n_D(p) = n_{D'}(p) \geq 1,$$

a contradiction. Therefore D is not simple.

The equivalence of simple and rigid then follows from Proposition 4.7. $\square$

Proposition 4.9 *If a dimer algebra is non-cancellative, then it has an arrow that is not contained in any simple matching.*

Proof See [4, Corollary 3.5]. $\square$

Theorem 4.10 *The dimer algebra A' , defined by the sequence (10), is cancellative.*

Proof Recall that an arrow is nonrigid if it is not contained in any perfect matching; and rigidity is not defined for pseudo-arrows. Thus, since the sequence (10) is maximal, each arrow of Q' is contained in a rigid perfect matching (though Q' may contain pseudo-arrows, and therefore length 1 paths that do not belong to any perfect matching). But then each arrow of Q' is contained in a simple matching, by Proposition 4.7. Therefore A' is cancellative, by Proposition 4.9 matching. $\square$

If $\psi : A \rightarrow A'$ is a contraction of dimer algebras and A' has a perfect matching, then ψ does not contract an unoriented cycle of Q to a vertex [4, Lemma 3.9]. In the following, we prove the converse.

Lemma 4.11 *Consider the k -linear map of dimer path algebras $\psi : kQ \rightarrow kQ'$ defined by contracting a set of arrows in Q to vertices. If no unoriented cycle in Q is contracted to a vertex, then ψ induces a k -linear map of dimer algebras*

$$\psi : A = kQ/I \rightarrow A' = kQ'/I'.$$

Proof Factor $\psi : kQ \rightarrow kQ'$ into a sequence of k -linear maps of dimer path algebras

$$kQ \xrightarrow{\psi_0} kQ^1 \xrightarrow{\psi_1} kQ^2 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_m} kQ',$$

where each ψ_j contracts a single arrow of Q^j . To show that ψ induces a k -linear map $\psi : A \rightarrow A'$, that is, $\psi(I) \subseteq I'$, it suffices to show that for each $0 \leq j \leq m$, we have

$$\psi_j(I_j) \subseteq I_{j+1}.$$

We may therefore assume that $\psi : kQ \rightarrow kQ'$ contracts a single arrow δ .

Let $p - q$ be a generator for I given in Eq. 2; that is, p, q are paths and there is an $a \in Q_1$ such that pa and qa are unit cycles. We claim that $\psi(p - q)$ is in $\psi(I)$.

If $\delta \neq a$, then $\psi(pa) = \psi(p)\psi(a)$ and $\psi(qa) = \psi(q)\psi(a)$ are unit cycles, and $\psi(a) \in Q'_1$ has length 1. Thus

$$\psi(p - q) = \psi(p) - \psi(q) \in I'. \quad (12)$$

So suppose that $\delta = a$, and no cycle in Q is contracted to a vertex under ψ . Then δ is not a loop. Whence, $\psi(p)$ and $\psi(q)$ are unit cycles. But all unit cycles at a fixed vertex are equal, modulo I' . Therefore (12) holds in this case as well. $\square$

Lemma 4.12 *If A is cancellative, then $C^{u,0} = \hat{C}^u$.*

Proof We have $C^{u,0} \subseteq \hat{C}^u$, by Lemma 3.5 (or alternatively, [2, Lemma 4.8.1]). Conversely, since A is cancellative, we have $C^{u,0} \supseteq \hat{C}^u$ by [2, Proposition 4.20.1]. $\square$

Proposition 4.13 *Let $\psi : kQ \rightarrow kQ'$ be the k -linear map defined by the sequence (10). If p is a cycle in $C^{u,0}$, then $\psi(p)$ is a cycle in $C'^{u,0}$.*

Proof Let $p \in C^{u,0}$. Then there is a perfect matching $D_1 \in \mathcal{P}$ such that

$$x_{D_1} \nmid \bar{\eta}(p). \quad (13)$$

Since $\psi : kQ \rightarrow kQ'$ is defined by the sequence (10), where each ψ_j contracts a single nonrigid arrow of Q^j , there is a perfect matching $D_2 \in \mathcal{P}$ equivalent to D_1 such that $\psi(D_2)$ is a perfect matching of Q' .⁶ Whence

$$n_{D_2}(p) = n_{D_1}(p) \stackrel{(1)}{=} 0,$$

where (1) holds by Eq. 13. That is, p does not have an arrow subpath that belongs to D_2 . Thus $\psi(p)$ does not have an arrow subpath that belongs to $\psi(D_2)$:

$$n_{\psi(D_2)}(\psi(p)) = 0.$$

Consequently, $\sigma_{\mathcal{P}'} \nmid \bar{\eta}'\psi(p)$. Hence $\psi(p)$ is in $C'^{v,0}$ for some $v \in \mathbb{Z}^2$. But $v = u$ since no cycle in Q contracts to a vertex under ψ , by Proposition 4.3.1. Therefore $\psi(p)$ is in $C'^{u,0}$. $\square$

Theorem 4.14 *Let $\psi : kQ \rightarrow kQ'$ be the k -linear map defined by the sequence (10). Then ψ induces a contraction of dimer algebras $\psi : A \rightarrow A'$, and $S = S'$.*

Proof (i) No cycle in Q contracts to a vertex under ψ , by Proposition 4.3.1. Therefore $\psi : kQ \rightarrow kQ'$ induces a contraction of dimer algebras $\psi : A \rightarrow A'$, by Lemma 4.11.

(ii) We claim that $S = S'$. Set $\sigma := \sigma_{S'}$.

The inclusion $S \subseteq S'$ holds since the ψ -image of a cycle in Q is a cycle in Q' .

To show that reverse inclusion, let $g \in S'$. By Theorem 4.10, A' is cancellative. Thus S' is generated over k by σ and a set of monomials in the polynomial ring $k[x_D \mid D \in S']$ that are not divisible by σ , by [2, Theorem 5.9, Proposition 5.14]. Furthermore, σ is in S since σ is the $\bar{\tau}\psi$ -image of each unit cycle in Q . Therefore it suffices to assume that g is a monomial that is not divisible by σ .

⁶The set $\psi(D_1)$, which may consist of both arrows and vertices, may not be a perfect matching of Q' .

Fig. 7 The quiver of a degenerate dimer algebra, drawn on a torus

Since g is a monomial in S' , there is some $u \in \mathbb{Z}^2$ such that g is the $\bar{\tau}$ -image of a cycle q in $\mathcal{C}^u$. Furthermore, since $\sigma \nmid g$, we have $q \in \hat{\mathcal{C}}^u$ by Lemma 4.12.

Now there exists a cycle p in $\mathcal{C}^{u,0}$, by Corollary 3.9. Whence $\psi(p)$ is in $\mathcal{C}^{u,0}$, by Proposition 4.13. Thus, since A' is cancellative, $\psi(p)$ is in $\hat{\mathcal{C}}^u$, by Lemma 4.12. Furthermore, since A' is cancellative, any two cycles in $\hat{\mathcal{C}}^u$ have the same $\bar{\tau}$ -image, by [2, Proposition 4.20.2]. Hence

$$g = \bar{\tau}(q) = \bar{\tau}\psi(p) \in S.$$

Therefore $S' \subseteq S$, and so $S = S'$. □

Theorems 4.10 and 4.14 together imply that every cycle-nondegenerate, hence nondegenerate, dimer algebra admits a cyclic contraction.

Example 4.15 A dimer algebra for which Theorem 1.1 *does not* apply is given in Fig. 7. Its quiver contains no perfect matchings, and is thus degenerate.

The following corollary allows us to refer to *the* cycle algebra of a dimer algebra.

Corollary 4.16 *Every cycle-nondegenerate, hence nondegenerate, dimer algebra A has a cycle algebra S , and S is independent of the choice of cyclic contraction, up to isomorphism.*

Proof Let A be cycle-nondegenerate. We have shown that A admits a cyclic contraction $\psi : A \rightarrow A'$, and thus A has a cycle algebra S . Recall the isomorphism in Eq. 1,

$$S \cong k[\overline{S(A)}]^{\text{GL}}.$$

Since the right-hand side is independent of A' , S does not depend on ψ . □

It was shown in [9, Theorem 1.3] that if Q is a nondegenerate dimer quiver, then a set of its arrows may be contracted to produce a cancellative dimer quiver Q' with the same characteristic polygon. In future work, we hope to determine how this theorem is related to Theorem 1.1.

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