On some $\mathcal{AC}^\perp$ manifolds

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Abstract We give a description of all Gray AC manifolds (M, g) whose Ricci tensor has two eigenvalues of multiplicity 1 and $\dim M - 1$.

Keywords Gray manifold · Einstein–Weyl manifold

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1 Introduction

In this paper, we study complete Riemannian manifolds whose Ricci tensor ρ satisfies the relation

$$\nabla_X \rho(X, X) = \frac{2X \text{Scal}}{n+2} g(X, X) \quad (1.1)$$

where $\text{Scal} = \text{tr}_g \rho$ is the scalar curvature of (M, g) and $n = \dim M$. This property was studied by A. Gray [4] and by A. Besse [1]. A. Gray called Riemannian manifolds satisfying (1.1) $\mathcal{AC}^\perp$ manifolds; we shall also call them Gray manifolds. We describe all simply connected and real analytic Gray $\mathcal{AC}^\perp$ manifolds whose Ricci tensor has two eigenvalues λ, μ of multiplicity $\dim M - 1$ and 1.

If $([g], D)$ is an Einstein–Weyl structure on a compact manifold M , and g_0 is the Gauduchon metric in the conformal class $[g]$, then (M, g_0) is an $\mathcal{AC}^\perp$ manifold (see [6]) with the above property. If ω_0 is the Lee form of g_0 , i.e., $Dg_0 = \omega_0 \otimes g_0$, then the Weyl structure determined by the pair $(g_0, -\omega_0)$ is also Einstein–Weyl (see [6]). Hence if $([g], D)$ is an Einstein–Weyl structure on a compact manifold M , and g_0 is the Gauduchon metric of $([g], D)$, then on M there is another Einstein–Weyl structure $([g], D^-)$ such that $D^-g_0 = -\omega_0 \otimes g_0$. Thus, (M, g_0) is then a Riemannian manifold which admits a pair of

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Einstein–Weyl structures determined by the pairs (g_0, ω_0) and $(g_0, -\omega_0)$. More generally, a Riemannian manifold (M, g) which admits two Einstein–Weyl structures determined by pairs (g, ω) and $(g, -\omega)$ is an $\mathcal{AC}^\perp$ manifold whose Ricci tensor has two eigenvalues λ, μ of multiplicity $\dim M - 1$ and 1. We prove that Gray manifolds with two eigenvalues λ, μ of multiplicity $\dim M - 1$ and 1 always admit a conformal vector field ξ which is an eigenfield of the Ricci tensor of (M, g) . The field ξ is a Killing vector field or a conformal closed gradient vector field.

In particular, we give a description of all complete, real analytic, simply connected Riemannian manifolds (M, g) admitting a pair of Einstein–Weyl structures $(g, \omega), (g, -\omega)$. We show that on such manifolds $\omega(X) = g(\xi, X)$ where ξ is either Killing vector field or a closed gradient conformal field. In the second case, we give a complete classification of such manifolds. We prove that in this case the two Einstein–Weyl structures are conformally Einstein.

For general $\mathcal{AC}^\perp$ manifolds, we prove that the eigenvalues λ, μ of the Ricci tensor satisfy the equations: $n\mu - 2(n - 1)\lambda = C_0 = \text{const}$ in the case of a non-Killing conformal vector field and $(n - 4)\lambda + 2\mu = C_0 = \text{const}$ in the case of a Killing vector field.

We also give new examples of compact $\mathcal{AC}^\perp$ manifolds diffeomorphic to the sphere $S^n, n > 2$.

2 $\mathcal{AC}^\perp$ manifolds and Einstein–Weyl manifolds

By an $\mathcal{AC}^\perp$ manifold (see [1, 4]), we mean a Riemannian manifold (M, g) satisfying the condition

$$\mathfrak{C}_{XYZ} \nabla_X \rho(Y, Z) = \frac{2}{\dim M + 2} \mathfrak{C}_{XYZ} X \text{Scal} g(Y, Z), \quad (2.1)$$

where ρ is the Ricci tensor of (M, g) , Scal is the scalar curvature and $\mathfrak{C}$ means the cyclic sum.

In [6], it is proved that a Riemannian manifold (M, g) is an $\mathcal{AC}^\perp$ manifold if and only if the Ricci endomorphism Ric of (M, g) is of the form $\text{Ric} = S + \frac{2}{n+2} \text{Scal} Id$ where S is a Killing tensor and $n = \dim M$. Let us recall that a $(1, 1)$ tensor S on a Riemannian manifold (M, g) is called a *Killing tensor* if $g(\nabla S(X, X), X) = 0$ for all $X \in \text{TM}$.

Define the integer-valued function $E_S(x) = (\text{the number of distinct eigenvalues of } S_x)$ and set $M_S = \{x \in M : E_S \text{ is constant in a neighborhood of } x\}$. The set M_S is open and dense in M , and the eigenvalues λ_i of S are distinct and smooth in each component U of M_S . Let us denote by D_{λ_i} the eigendistribution corresponding to λ_i . We have (see [6]).

Proposition *Let S be a Killing tensor on M and U be a component of M_S and $\lambda_1, \dots, \lambda_k \in C^\infty(U)$ be eigenfunctions of S . Then for all $X \in D_{\lambda_i}$ we have*

$$\nabla S(X, X) = -\frac{1}{2} \nabla \lambda_i \|X\|^2 \quad (2.2)$$

and $D_{\lambda_i} \subset \ker d\lambda_i$. If $i \neq j$ and $X \in \Gamma(D_{\lambda_i}), Y \in \Gamma(D_{\lambda_j})$ then

$$g(\nabla_X X, Y) = \frac{1}{2} \frac{Y \lambda_i}{\lambda_j - \lambda_i} \|X\|^2. \quad (2.3)$$

If $T(X, Y) = g(SX, Y)$ is a Killing tensor on (M, g) and c is a geodesic on M , then the function $\phi(t) = T(\dot{c}(t), \dot{c}(t))$ is constant on the domain of c . In fact, $\phi'(t) = \nabla_{\dot{c}(t)} T((\dot{c}(t), \dot{c}(t))) = 0$.

A *conformal vector field* ξ on a Riemannian manifold satisfies the relation $L_\xi g = \alpha g$. This is equivalent to $\nabla_X \omega(Y) + \nabla_Y \omega(X) = \alpha g(X, Y)$ where $\omega = g(\xi, \cdot)$.

Finally, let us recall that a Riemannian manifold with Killing Ricci tensor is called after A. Gray an *$\mathcal{AC}^\perp$ manifold* (it is an $\mathcal{AC}^\perp$ manifold with constant scalar curvature).

We start with some basic facts concerning Einstein–Weyl geometry. For more details, see [12], [10], [11].

Let M be an n -dimensional manifold with a conformal structure $[g]$ and a torsion-free affine connection D . This defines an *Einstein–Weyl* (E–W) *structure* if D preserves the conformal structure, i.e., there exists a 1-form ω on M such that

$$Dg = \omega \otimes g \quad (2.4)$$

and the Ricci tensor $\rho^D(X, Y) = \text{tr}\{Z \rightarrow R^D(Z, X)Y\}$ of D , where R^D is the curvature tensor of (M, D) , satisfies the condition

$$\rho^D(X, Y) + \rho^D(Y, X) = \bar{\Lambda}g(X, Y) \text{ for every } X, Y \in TM$$

for some function $\bar{\Lambda} \in C^\infty(M)$. P. Gauduchon ([2]) proved the fundamental theorem that if M is compact, then there exists a Riemannian metric $g_0 \in [g]$ for which the Lee form ω_0 associated with g_0 is co-closed- $\delta\omega_0 = 0$ where $Dg_0 = \omega_0 \otimes g_0$, and g_0 is unique up to homothety. The metric g_0 is called *Gauduchon* or the *standard* metric of the E–W structure $([g], D)$ on M . Note that if M is not compact, a standard metric may not exist.

Let ρ be the Ricci tensor of (M, g) and let us denote by Ric the Ricci endomorphism of (M, g) , i.e., $\rho(X, Y) = g(X, RicY)$. We recall an important theorem (see [10]):

Theorem *A metric g and a 1-form ω determine an E–W structure if and only if there exists a function $\Lambda \in C^\infty(M)$ such that*

$$\rho + \frac{1}{4}(n-2)\mathcal{D}\omega = \Lambda g \quad (2.5)$$

where $\mathcal{D}\omega(X, Y) = (\nabla_X \omega)Y + (\nabla_Y \omega)X + \omega(X)\omega(Y)$ and $n = \dim M$. If (1.5) holds then

$$\bar{\Lambda} = 2\Lambda + \text{div}\omega - \frac{1}{2}(n-2) \|\omega^\sharp\|^2 \quad (2.6)$$

Compact Einstein–Weyl manifolds with the Gauduchon metric are Gray manifolds. By a *Riemannian manifold with a pair of Einstein–Weyl structures*, we mean a manifold (M, g) for which there exists a 1-form ω such that $(g, \omega), (g, -\omega)$ determine two Einstein–Weyl structures $([g], D), ([g], D^-)$ with $Dg = \omega \otimes g, D^-g = -\omega \otimes g$. Note that every Einstein–Weyl structure $(D, [g])$ on a compact manifold M determines a Riemannian manifold (M, g) with two Einstein–Weyl structures, namely $([g], D), ([g], D^-)$ where g is the Gauduchon metric for the conformal manifold $(M, [g])$ and $D^-g = -\omega_0 \otimes g$ where ω_0 is the Lee form of g with respect to D (see [6]).

It is not difficult to prove that any Riemannian manifold admitting a pair of E–W structures is an $\mathcal{AC}^\perp$ manifold with two eigenvalues λ, μ of the Ricci tensor of multiplicities $n-1, 1$ and such that $\lambda \geq \mu$ (see [3]). In this case, $\omega(X) = g(\xi, X)$ where ξ is a conformal vector field. We will prove the converse: if (M, g) is a simply connected, real analytic $\mathcal{AC}^\perp$ manifold with two eigenvalues λ, μ of multiplicities $n-1, 1$ and $\lambda > \mu$ at least at one point, then (M, g) admits a pair of Einstein–Weyl structures (g, ω) and $(g, -\omega)$.

3 $\mathcal{AC}^\perp$ manifolds (M, g) whose Ricci tensor has two eigenvalues of multiplicity $\dim M - 1$ and 1

We start with:

Theorem 1 *Let (M, g) be a real analytic complete, simply connected $\mathcal{AC}^\perp$ manifold whose Ricci tensor has two eigenvalues λ, μ of multiplicity $n - 1, 1$ in the set $V = \{x \in M : \lambda(x) \neq \mu(x)\}$. Then, the set $N = \{x \in M : \lambda(x) = \mu(x)\}$ has an empty interior and there exists a conformal vector field ξ such that $g(\xi, \xi) = |\lambda - \mu|$ and $(Ric - \mu Id)\xi = 0$. If ξ is a Killing vector field, then λ' is constant, while if ξ is conformal not Killing, then μ' is constant. Here λ', μ' are eigenvalues of the Killing tensor $S = Ric - \frac{2Scal}{n+2} Id$ corresponding to λ, μ respectively.*

Proof Let $V_+ = \{x \in M : \lambda(x) > \mu(x)\}$, $V_- = \{x \in M : \lambda(x) < \mu(x)\}$. Note that $M = V_+ \cup V_- \cup N$. Let $\mathcal{D}$ be the distribution on V defined by $D_\mu = \mathcal{D} = \ker(Ric - \mu Id)$. Let us define on V the tensor $m = g(p_{\mathcal{D}}, p_{\mathcal{D}})$ where $p_{\mathcal{D}} : TM \rightarrow \mathcal{D}$ is the orthogonal projection onto $\mathcal{D}$. Then, $\rho = \lambda g + (\mu - \lambda)m$. Let us assume that $V_+ \neq \emptyset$. The distribution $\mathcal{D}|_{V_+}$ is locally spanned by a unit vector field ξ_0 . We have $Ric\xi_0 = \mu\xi_0$. Hence on V_+ ,

$$\rho = \lambda g - \omega \otimes \omega$$

where $\omega = g(\xi, \cdot)$ and $\xi = \sqrt{\lambda - \mu}\xi_0$.

Now we shall check when the tensor $S = Ric - \frac{2}{n+2}Scalg$ is a Killing tensor. Note that for $T(X, Y) = g(SX, Y)$ we have $T = \lambda'g - \omega \otimes \omega$ where $\lambda' = \lambda - \frac{2}{n+2}Scal$. Hence

$$\nabla_X T(X, X) = X\lambda'g(X, X) - 2\nabla_X \omega(X)\omega(X) = X\lambda'g(X, X) - h(X, X)\omega(X)$$

where h is the symmetric $(2, 0)$ tensor defined by $h(X, Y) = \nabla_X \omega(Y) + \nabla_Y \omega(X)$. Consequently, T is a Killing tensor if and only if

$$X\lambda'g(X, X) - h(X, X)\omega(X) = 0. \quad (3.1)$$

If $\omega(X) = 0$, then from (3.1) it follows that $d\lambda'(X) = 0$. Thus, $d\lambda' = \alpha\omega$ for some $\alpha \in C^\infty(V_+)$. Consequently

$$\omega(X)(\alpha g(X, X) - h(X, X)) = 0$$

for every $X \in TM|_{V_+}$.

Let $F(X, Y) = \alpha g(X, Y) - h(X, Y)$. Then, F is a symmetric $(2, 0)$ tensor and

$$\omega(X)F(X, X) = 0 \quad (3.2)$$

for every $X \in TM|_{V_+}$. Let us write $X = tY + \xi$ where $Y \in D_\lambda$, $\omega(Y) = 0$, $t \in \mathbb{R}$. Then, $\omega(X) = g(\xi, \xi) \neq 0$, and from (2.2) we get $F(tY + \xi, tY + \xi) = 0$. Consequently, $t^2 F(Y, Y) + 2tF(Y, \xi) + F(\xi, \xi) = 0$. Since $t \in \mathbb{R}$ is arbitrary, we obtain $F(Y, Y) = F(\xi, Y) = F(\xi, \xi) = 0$. Thus, $F = 0$. Hence, $h(X, Y) = \alpha g(X, Y)$ and ξ is a local conformal field. Since $h(X, Y) = \nabla_X \omega(Y) + \nabla_Y \omega(X)$, we get $n\alpha = -2\delta\omega$. We also have

$$d\lambda' = \alpha\omega \quad (3.3)$$

and $0 = d\alpha \wedge \omega + \alpha d\omega$. Hence in the set $U_+ = \{x \in V_+ : \alpha(x) \neq 0\}$, we obtain $d\omega = -d \ln |\alpha| \wedge \omega$. It follows that in U_+ the distribution $D_\lambda = \ker \omega$ is integrable. Thus for $X, Y \in D_\lambda$, we get $\nabla_X \omega(Y) = g(\nabla_X \xi, Y) = -g(\xi, \nabla_X Y) = -g(\xi, \nabla_Y X) = g(\nabla_Y \xi, X) =$

$\nabla_Y \omega(X)$. Consequently, $\nabla_X \omega(Y) = \frac{1}{2} \alpha g(X, Y)$ and $p_{D_\lambda}(\nabla_X \xi) = \frac{1}{2} \alpha X$. Hence for a vector field ξ and $X \in D_\lambda$, we get $\nabla_X \xi = \frac{\alpha}{2} X + \phi(X)\xi$ for $\phi = \frac{1}{2} d \ln g(\xi, \xi)$. Now for $X, Y \in D_\lambda$, we have

$$\begin{aligned} 2\nabla_X \nabla_Y \xi &= X\alpha Y + \alpha \nabla_X Y + 2X\phi(Y)\xi + \phi(Y)(\alpha X + 2\phi(X)\xi), \\ 2\nabla_Y \nabla_X \xi &= Y\alpha X + \alpha \nabla_Y X + 2Y\phi(X)\xi + \phi(X)(\alpha Y + 2\phi(Y)\xi), \\ 2\nabla_{[X, Y]}\xi &= \alpha[X, Y] + 2\phi([X, Y])\xi \end{aligned}$$

Thus, $2R(X, Y)\xi = X\alpha Y - Y\alpha X - \phi(X)\alpha Y + \phi(Y)\alpha X$ for $X, Y \in \ker \omega$. Let $\beta = d\alpha - \alpha\phi$. Consequently, taking the trace over vectors perpendicular to ξ in the previous formula we get $0 = \rho(Y, \xi) = \frac{1}{2}(\beta(Y) - (n-1)\beta(Y)) = -\frac{n-2}{2}\beta(Y)$. Hence, $\beta(X) = 0$ for $X \in D_\lambda$ and

$$R(X, Y)\xi = 0$$

for $X, Y \in D_\lambda$. We have $d \ln |\alpha|(X) = \phi(X)$ for $X \in D_\lambda$. Hence, for $X \in D_\lambda$, $\nabla_X \xi = \frac{\alpha}{2} X + d \ln |\alpha|(X)\xi$ and $\nabla_X \omega(\xi) = d \ln |\alpha|(X)\omega(\xi)$. Thus, $\nabla_\xi \omega(X) = -d \ln |\alpha|(X)\omega(\xi)$ and $d\omega(X, \xi) = 2d \ln |\alpha|(X)\omega(\xi)$. On the other hand, $d\omega = -d \ln |\alpha| \wedge \omega$ and $d\omega(X, \xi) = -d \ln |\alpha|(X)\omega(\xi)$. Consequently $d \ln |\alpha|(X) = 0$ for $X \in D_\lambda$. It follows that $d\omega = 0$ and $\nabla_X \xi = \frac{1}{2} \alpha X$ for all $X \in \text{TM}$. The formula $\nabla_X \xi = \frac{1}{2} \alpha X$ is clear if $g(X, \xi) = 0$. We will show that it also holds for $X = \xi$. In fact, if X is any vector perpendicular to ξ then $0 = \alpha g(X, \xi) = g(\nabla_X \xi, \xi) + g(\nabla_\xi \xi, X) = g(\nabla_\xi \xi, X)$. Hence $\nabla_\xi \xi = \gamma \xi$ for some function γ . On the other hand, we have $\alpha g(\xi, \xi) = g(\nabla_\xi \xi, \xi) + g(\nabla_\xi \xi, \xi) = 2\gamma g(\xi, \xi)$. It follows that $\gamma = \frac{1}{2} \alpha$. Thus, $R(X, Y)\xi = \frac{1}{2} d\alpha(X)Y - \frac{1}{2} d\alpha(Y)X$, and consequently, $\rho(Y, \xi) = -\frac{n-1}{2} d\alpha(Y)$. Hence, $d\alpha = -\frac{2\mu}{n-1} \omega$.

We also have $dg(\xi, \xi)(X) = 2g(\nabla_X \xi, \xi) = \alpha\omega(X) = d\lambda'(X)$. Note that $d\mu' = 0$ in $M - N$ where μ' is an eigenvalue of the Killing tensor associated with Ric corresponding to the one-dimensional distribution D_μ [see (2.3)]. Note that $g(\xi, \xi) = \lambda' - \mu'$.

Analogously, we construct a conformal local vector field ξ in V_- . In particular in V_- , we have $d\lambda' = -\alpha\omega$ where $n\alpha = -2\delta\omega$ and $\nabla_X \xi = \frac{\alpha}{2} X$. Note that if ξ, ξ' are two local fields constructed as above, then $\xi = \pm \xi'$ in the intersection of their domains. Now let us define $M_- = \text{int}\{x \in M - N : \alpha(x) = 0\}$ and $M_+ = \{x \in M - N : \alpha(x) \neq 0\}$.

We now show that if $M_- \neq \emptyset$, then $M_+ = \emptyset$ hence ξ is a Killing vector field (see [6]). Let $x_0 \in M_-$ and $x_1 \in M_+$. If there exists a smooth curve c joining x_0 and x_1 and such that $\text{im } c \cap N = \emptyset$, then by analytic continuation we see that λ' is constant along this curve, i.e., λ' is constant on a certain neighborhood U_x of any $x \in \text{im } c$. Analogously, μ' is constant, and hence, $\alpha(x_1) = 0$, a contradiction. A curve as above indeed exists: if U is a connected component of x_1 in M_+ , then there exists a geodesic c with $c(0) = x_0$ and $c(1) \in U$ which doesn't meet N (see [6]).

If $N = \emptyset$, then ξ exists and is smooth in a neighborhood of any point $x \in M$. Note that in the case of a Killing vector field the constructed vector field is smooth also if $N \neq \emptyset$, and then one of the sets V_+ , V_- vanishes. Next we show that a similar situation holds for a conformal vector field.

Let us assume now that $\text{int}\{x : \alpha(x) = 0\} = \emptyset$. In every component of the set $M - N$ and in the interior of N , the function μ' is constant. Since μ' is continuous, it is constant on M . The set $\{x : \alpha(x) \neq 0\}$ is dense in $M - N$. In this set, $\nabla_X \xi = \frac{\alpha}{2} X$. If $x_0 \in M - N$, then in a neighborhood of x_0 the local field ξ is smooth and $\nabla_X \xi = \frac{\alpha}{2} X$ also if $\alpha(x_0) = 0$ (we take the limit). Hence in $M - N$, we have $d\alpha = -\frac{2\mu}{n-1} \omega$. Note that the function α^2 is independent

of the choice of the local field ξ , and $\alpha^2 = \frac{||\nabla\lambda'||^2}{|\lambda-\mu|}$; hence, α is continuous in $M - N$. Note that $\alpha d\alpha = -\frac{2\mu}{n-1}\alpha\omega = -\frac{2\mu}{n-1}d\lambda'$. On the other hand, since μ' is constant, it follows that $n\mu - 2(n-1)\lambda = C_0 = \text{const}$ and $d\lambda = \frac{n}{2(n-1)}d\mu$. Since $\lambda - \mu = \lambda' - \mu'$ and μ' is constant, we get $d\lambda' = d\lambda - d\mu$ and $(n-1)\alpha d\alpha = -2\mu(d\lambda - d\mu) = \frac{2-n}{n-1}\mu d\mu$. Hence in any component of $M - N$, we have $\pm(n-1)^2\alpha^2 + (n-2)\mu^2 = C_1 = \text{const}$ (the minus sign is in the components contained in V_-). Since μ' is constant, it follows that λ', λ, μ are real analytic functions on the whole of M (we have $(n-1)\lambda + \mu = \text{Scal}$, $n\mu - 2(n-1)\lambda = C_0 = \text{const}$ and the scalar curvature Scal of (M, g) is real analytic).

Now we show that $\nabla\lambda' = 0$ in N . It is clear that $\nabla\lambda' = 0$ in the interior of N . Let $x_0 \in \partial N$. Then, there exists a sequence of points x_m from one component of $M - N$ such that $\lim x_m = x_0$. This component, for example, is contained in V_+ . In that component, $(n-1)^2\alpha^2 + (n-2)\mu^2 = C_0 = \text{const}$. In particular, $\lim_{m \rightarrow \infty} \alpha^2(x_m) = \frac{1}{(n-1)^2}(C_0 - (n-2)\mu(x_0)^2)$ is finite which means that $\nabla\lambda' = 0$ at x_0 .

Now let assume that $N \neq \emptyset$ and let $x_0 \in N$. Let $c(t)$ be a geodesic with $c(0) = x_0$, $\|\dot{c}(0)\| = 1$. If $S = \text{Ric} - \frac{2\text{Scal}}{n+2}g$, then $T(\cdot, \cdot) = g(S, \cdot)$ is a Killing tensor and $T - \lambda'g = \pm\omega \otimes \omega$. Note that $\lim_{t \rightarrow 0} T(\dot{c}(t), \dot{c}(t)) = \mu'$, and since $T(\dot{c}(t), \dot{c}(t))$ is constant, it follows that $g(\xi(c(t)), \dot{c}(t))^2 = \pm(\lambda' - \mu')$. Hence, $g(\frac{1}{\sqrt{|\lambda' - \mu'|}}\xi(c(t)), \dot{c}(t)) = \pm 1$.

Since both vectors have length 1, it follows that $\xi(c(t)) = \pm\sqrt{|\lambda' - \mu'|}\dot{c}(t)$. If we introduce geodesic polar normal coordinates centered at x_0 , then we can assume that $\xi = \sqrt{|\lambda' - \mu'|}\frac{\partial}{\partial t}$ where t is a radial coordinate. In particular, $\nabla\lambda' = \alpha\sqrt{|\lambda - \mu|}\frac{\partial}{\partial t}$ on $M - N$ and $\nabla\lambda' = 0$ on N . Hence, $d\lambda'(X) = 0$ for any X tangent to the geodesic sphere and λ' is constant on geodesic spheres with center x_0 . Thus, λ' is a function of the radial coordinate t only.

If there existed a hypersurface $\Sigma \subset B(x_0, r)$ with $x_0 \in \Sigma$ such that the intersection of Σ with every geodesic sphere $S(x_0, s)$, $s < r$, is not empty, and on which $\lambda' - \mu' = 0$, then we would have $\lambda' = \mu'$ on the whole of B . In fact, if for $x_1 \neq x_0$ we have $\lambda'(x_1) = \mu'$, then $\lambda'(x) = \mu'$ on the sphere $S(x_0, s)$ containing the point x_1 . Hence, the sphere $S(x_0, s)$ would be a hypersurface Σ in a ball $B(x_1, t)$, where t is the injectivity radius at the point x_1 , which has the above property. It follows that the set N has a non-empty interior U . Since λ, μ are real analytic and $\lambda = \mu$ on U , it follows that $\lambda = \mu$ on M , a contradiction. It follows that in a geodesic ball $B(x_0, r)$ where r is the injectivity radius of M at x_0 , the set $\{x \in B(x_0, r) : \lambda'(x) = \mu'\}$ consists of only the point x_0 . Hence, the set N consists of isolated points. This implies that $M - N$ has only one component. It follows that $\lambda' - \mu'$ has a constant sign and one of the sets V_+, V_- is empty.

For $x_0 \in N$, let us define a local field ξ on the geodesic ball $B(x_0, r)$ where r is the injectivity radius at x_0 by $\xi = \sqrt{|\lambda' - \mu'|}\frac{\partial}{\partial t}$ where t is the radial coordinate. Then since M is simply connected, there exists a continuous vector field ξ on M such that $\rho = \lambda g - \omega \otimes \omega$ if $V_- = \emptyset$ or $\rho = \lambda g + \omega \otimes \omega$ if $V_+ = \emptyset$ and $\omega = g(\xi, \cdot)$ (we take the germs of local fields ξ as points of a two-sheeted covering of M). The field ξ is smooth on $M - N$ and defines a function α on $M - N$. The function α is smooth on $M - N$. We shall show that ξ is in fact smooth.

First we show that the function α has a continuous extension to the points $x_0 \in N$. We assume, for example, that $V_- = \emptyset$. Then, the function α^2 is real analytic and $(n-1)^2\alpha^2 + (n-2)\mu^2 = C_1 = \text{const}$ in $M - N$; hence, α^2 has a real analytic extension on the whole of M . If $\alpha(x_0)^2 > 0$, then α has a constant sign in a certain neighborhood of x_0 and $\alpha = \pm\frac{1}{n-1}\sqrt{C_1 - (n-2)\mu^2}$ is a smooth extension of α on the neighborhood of x_0 . If an extension of α^2 satisfies $\alpha^2(x_0) = 0$, then certainly $\lim_{x \rightarrow x_0} \alpha(x) = 0$ and taking $\alpha(x_0) = 0$ we get a function continuous at x_0 .

Let $c(t)$ be a unit speed geodesic with $c(0) = x_0 \in N$. We take ξ which in geodesic coordinates around x_0 is $\xi = \sqrt{|\lambda' - \mu'|} \frac{\partial}{\partial t}$. Note that $\sqrt{|\lambda' - \mu'|} = \pm g(\dot{c}, \xi)$. The function $\phi(t) = \sqrt{|\lambda' - \mu'|} \circ c(t)$ if $t \geq 0$ and $\phi(t) = -\sqrt{|\lambda' - \mu'|} \circ c(t)$ if $t < 0$ is differentiable at 0 and smooth in a neighborhood of 0. Indeed note that $\phi(t) = g(\dot{c}(t), \xi(c(t)))$ and $\phi'(t) = g(\dot{c}, \nabla_{\dot{c}} \xi) = \frac{1}{2} \alpha(c(t))$ for $t \neq 0$. Since α is continuous, it follows that ϕ is differentiable at 0 and has continuous derivative. The function $\alpha(t) = \alpha(c(t))$ is differentiable at 0 and $\alpha'(0) = 0$. In fact, $d\alpha = -\frac{2}{n-1} \mu \omega$ and $d\alpha(\dot{c}) = -\frac{2}{n-1} \mu \phi(t)$. Hence, $\alpha'(t) = -\frac{2}{n-1} \mu \circ c(t) \phi(t)$. Note that $\frac{1}{2} \alpha(t) = \phi'(t)$. The last equation holds also for $t = 0$ since α is continuous. Hence, we get an equation $\phi''(t) = -\frac{\mu(c(t))}{n-1} \phi(t)$ with initial conditions $\phi(0) = 0$, $\phi'(0) = \frac{1}{2} \alpha(x_0)$ where the function $\mu(c(t))$ is real analytic. If $\alpha(x_0) = 0$, then we get $\phi(t) = 0$, a contradiction. Hence, $\alpha(x_0) \neq 0$ and $\xi = \frac{1}{\alpha} \nabla \lambda'$ is a smooth vector field in a neighborhood of x_0 . Hence, ξ is a smooth vector field. $\diamond$

Remark $\mathcal{AC}^\perp$ manifolds for which ξ is a Killing vector field are described in [6]. In the rest of the paper, we shall assume that $\dim M = n + 1$.

Theorem 2 *Let us assume that (M, g) is a real analytic, simply connected $\mathcal{AC}^\perp$ manifold, whose Ricci tensor has two eigenvalues λ, μ of multiplicities $n, 1$ respectively, and such that $\mu - \frac{2\text{Scal}}{n+3}$ is constant and $\lambda - \frac{2\text{Scal}}{n+3}$ is non-constant. Then, $\lambda - \mu \geq 0$ on the whole of M or $\lambda - \mu \leq 0$ on the whole of M . If $|\lambda - \mu| > 0$ on M , then M is a warped product $M = \mathbb{R} \times_{f^2} M_*$ where M_* is an Einstein manifold. If $N \neq \emptyset$, then $N = \{x_0\}$ or $N = \{x_0, x_1\}$ and $M = \mathbb{R}_+ \times_{f^2} M_*$ in the first case, while $M = (0, \epsilon) \times_{f^2} M_*$ in the second case, where $M_* = (S^n, g_{\text{can}})$ is a sphere with the standard metric of constant sectional curvature. In all cases, the function f satisfies an equation*

$$-(n-1) \left(\frac{f''}{f} - \frac{(f')^2}{f^2} \right) = \tau f^{-2} - C f^2 \quad (3.4)$$

where $C \in \mathbb{R} - \{0\}$ and $\text{Ric}_{M_*} = \tau g_{M_*}$. If $N \neq \emptyset$ and M_* is a sphere with sectional curvature 1, then the function f satisfies the initial conditions $f(0) = 0$, $f'(0) = 1$ in the first case and in the second case $f(0) = 0$, $f'(0) = 1$, $f(\epsilon) = 0$, $f'(\epsilon) = -1$.

Proof If $N = \emptyset$, the proof is given in [7] (see also [5]). Let us assume first that $\sharp N \geq 2$. Let $x_0 \in N$ and $r = d(x_0, N - \{x_0\}) = d(x_0, x_1)$ where $x_1 \in N$. We show that the map $\exp_{x_0} : B(0, r) \rightarrow M$, where $B(0, r) = \{X \in T_{x_0} M : \|X\| < r\}$, is a diffeomorphism onto its image.

If $\exp_{x_0} X = \exp_{x_0} Y$, then we consider two cases $X = 0$, $Y \neq 0$ and $X, Y \neq 0$. In the first case, the geodesic $c(t) = \exp_{x_0} t \frac{Y}{\|Y\|}$ is a loop of length $s = \|Y\| < r$. In the second case, let $c(t) = \exp_{x_0} t \frac{X}{\|X\|}$, $c_1(t) = \exp_{x_0} t \frac{Y}{\|Y\|}$. Let $t_0 = \|X\|$, $t_1 = \|Y\|$. Then, $c(t_0) = c_1(t_1)$ and $\dot{c}(t_0) = \dot{c}_1(t_1)$ since $\xi(c(t_0)) = \sqrt{|\lambda - \mu|} \dot{c}(t_0) = \sqrt{|\lambda - \mu|} \dot{c}_1(t_0)$. It follows that $c(t_0 + s) = c_1(t_1 + s)$. Let us assume that $t_1 > t_0$. Then, $c(0) = x_0 = c_1(t_1 - t_0)$. Hence, c_1 is a geodesic loop at x_0 of length $t_1 - t_0 < r$. This contradicts $\xi(c_1(t)) = \sqrt{|\lambda - \mu|} \dot{c}_1(t)$ which is valid up to the first value of $t > 0$ for which $\lambda(c_1(t)) - \mu(c_1(t)) = 0$, if we start from $t = 0$ (note that $|\lambda(c_1(t)) - \mu(c_1(t))| = 0$ only if $c_1(t) = x_0$ for $t \in (0, r)$). Thus, $t_0 = t_1$ and $c = c_1$, which means that $X = Y$. Similarly, we obtain $Y = 0$ if $X = 0$. The mapping $\exp_{x_0} : B(0, r) \rightarrow M$ is a diffeomorphism. In fact, otherwise we would have a conjugate point on a geodesic $c(t) = \exp_t X$. Hence, we would have two different geodesics starting at x_0 and joining x_0 to $x_1 = \exp_s X$, where $s > 1$, $s\|X\| < r$. It leads to a contradiction as above.

Note that $\lambda - \mu$ depends only on the distance from x_0 . Hence, $\exp_{x_0} rS \subset N$ where S is a unit sphere in $T_{x_0}M$. Since $\exp_{x_0}$ is a continuous function, $\exp_{x_0} rS$ is connected and N consists of isolated points, it follows that $\exp_{x_0} rS = \{x_1\}$. Hence, $N = \{x_0, x_1\}$ and every unit geodesic c starting from x_0 satisfies $c(r) = x_1$. It follows that $\exp_{x_0} : B(0, r) \rightarrow M - \{x_1\}$ is a diffeomorphism. In the same way, $\exp_{x_1} : B(0, r) \rightarrow M - \{x_0\}$ is a diffeomorphism onto $M - \{x_0\}$.

Note that by the Gauss lemma, the leaves of D_λ are compact spheres S^n via the diffeomorphism $\exp_{x_0} : S(0, t) \rightarrow M$ where $S(0, t) = \{X \in T_{x_0}M : \|X\| = t \text{ where } t \in (0, r)\}$. Hence, $M = [0, r] \times_{f^2} S^n$. From [B, Lemma 9.114, page 269], it follows that on S^n there is a metric of constant sectional curvature $\lambda^2 g_{\text{can}}$ and $f(0) = 0 = f(r)$, $f'(0) = \frac{1}{\lambda} = -f'(r)$. Replacing f by a function λf , we see that we can assume that $f(0) = 0$, $f'(0) = 1$, $f(r) = 0$, $f'(r) = -1$.

Now consider the case $N = \{x_0\}$. Then, as above one can prove that $\exp_{x_0} : T_{x_0}M \rightarrow M$ is a diffeomorphism. Hence in this case, $M = \mathbb{R}_+ \times_{f^2} S^n$ where S^n has the a canonical metric. The function f satisfies the initial conditions $f(0) = 0$, $f'(0) = 1$. $\diamond$

Now we consider the equation (see [7, p.27])

$$-(n-1)\left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right) = \tau f^{-2} - C f^2$$

with initial conditions $f(0) = 0$, $f'(0) = 1$; hence, we consider the warped product $M = (0, \epsilon) \times_f S^n$ or $M = \mathbb{R}_+ \times_f S^n$ and $\tau = n - 1$.

If we write $g = \ln f$, then $(g)'' = -\frac{\tau}{n-1}e^{-2g} + \frac{C}{n-1}e^{2g}$. We get $\frac{1}{2}((g')^2)' = (\frac{\tau}{2(n-1)}e^{-2g} + \frac{C}{2(n-1)}e^{2g})'$ and $(g')^2 = (\frac{\tau}{(n-1)}e^{-2g} + \frac{C}{(n-1)}e^{2g} + A$ for some constant $A \in \mathbb{R}$. Consequently, if we assume that $\tau = n - 1$, we get an equation for f

$$(f')^2 = 1 + A f^2 + \frac{C}{n-1} f^4.$$

Using homothety, we can assume that $C = \pm(n-1)$. Thus, we get $(f')^2 = 1 + A f^2 + f^4$ or $(f')^2 = 1 + A f^2 - f^4$.

First let us assume that the quadratic polynomial $1 + At + \epsilon t^2$ where $\epsilon \in \{-1, 1\}$ has real roots. This happens for $\epsilon = -1$, and for $|A| \geq 2$ if $\epsilon = 1$. Hence, if $\epsilon = 1$ and $A < -2$, we get $(f')^2 = (a^2 - f^2)(\frac{1}{a^2} - f^2)$ for some $a \in (0, 1)$. Let f be the solution of this equation with initial conditions $f(0) = 0$, $f'(0) = 1$.

We shall show that f increases until it attains its maximum $a = f(t_0)$ and then decreases, and $f(2t_0) = 0$, $f'(2t_0) = -1$. In fact, f satisfies an equation $f' = \sqrt{(a^2 - f^2)\left(\frac{1}{a^2} - f^2\right)}$ if $f < a$. It follows that $\arcsin(\frac{f}{a})' = \sqrt{\frac{1}{a^2} - f^2} \geq \sqrt{\frac{1}{a^2} - a^2}$ if $f < a$. Consequently, $\arcsin(\frac{f}{a}) \geq t\sqrt{\frac{1}{a^2} - a^2}$ and f increases until it attains the value $f(t_0) = a$. The solution exists on an interval $[0, t_0]$. Now we define $f(t_0 + s) = f(t_0 - s)$ for $s \in (0, t_0]$. Then, f is defined on an interval $[0, 2t_0]$ and is a smooth function on an open neighborhood of this interval.

Similarly if $\epsilon = -1$, then we get an equation $f' = \sqrt{(a^2 - f^2)\left(\frac{1}{a^2} + f^2\right)}$ where $a > 0$.

Thus, $\arcsin(\frac{f}{a}) \geq t\sqrt{\frac{1}{a^2}}$, and again there exists $t_0 > 0$ such that $f(t_0) = a$. We extend the solution on the interval $[0, 2t_0]$ and obtain a solution satisfying $f(2t_0) = 0$, $f'(2t_0) = -1$.

We have to show that f is an odd function at $t = 0$ and $t = 2t_0$. Note that f satisfies the equation $f'' = Af + 2\epsilon f^3$. Since f is real analytic, it suffices to show that $f^{(2k)}(0) = 0$ for

$k \in \mathbb{N}$. We use induction. We have $f(0) = 0$, $f''(0) = 0$. Let us assume that $f^{(2k)}(0) = 0$ for $k \leq n$ where $n \geq 1$. We will show that $f^{(2n+2)}(0) = 0$. Note that

$$f^{(2n+2)}(0) = Af^{(2n)}(0) + 2\epsilon \sum_{r=0}^{2n} \sum_{s=0}^r C_r^{2n} C_s^r f^{(s)} f^{(r-s)} f^{(2n-r)}, \quad (3.5)$$

where $C_k^n = \frac{n!}{(n-k)!k!}$. If s and $r-s$ are odd numbers, then $r = r-s+s$ is an even number and $f^{(2n-r)}(0) = 0$. Hence in all cases, the left-hand side of (3.5) equals 0. It follows that in both cases considered above we get a compact $\mathcal{AC}^\perp$ manifold diffeomorphic to S^{n+1} . If $\epsilon = 1$ and $A = -2$, then one can easily check that $f(t) = \tanh t$. In that case, we obtain a complete non-compact $\mathcal{AC}^\perp$ manifold diffeomorphic to $\mathbb{R}^{n+1}$ (see [7]). If $A > -2$, then we can obtain a solution which tends to infinity which may not be defined on the whole of $\mathbb{R}_+$. For example, for $A = 2$ we get $f(t) = \tan t$. Summarizing we have

Theorem 3 *On the sphere S^{n+1} , there exists a one-parameter family of $\mathcal{AC}^\perp$ metrics g_A such that $g_A = dt^2 + f^2 g_{\text{can}}$ and $(S^{n+1}, g_A) = [0, 2t_0] \times_{f^2} S^n$ where f satisfies the equation*

$$(f')^2 = 1 + Af^2 + f^4 \quad (3.6)$$

with $A \in (-\infty, -2)$, or

$$(f')^2 = 1 + Af^2 - f^4 \quad (3.7)$$

for $A \in \mathbb{R}$. In the case of Eq. (3.6), the corresponding $\mathcal{AC}^\perp$ manifold admits a pair of Einstein–Weyl structures both conformally Einstein and conformal to the standard sphere with constant sectional curvature.

Proof Note that in the case (3.6) the corresponding $\mathcal{AC}^\perp$ manifold has a Ricci tensor with eigenvalues λ, μ such that $\lambda - \mu = Cf^2 \geq 0$. Hence from (2.5), it determines a pair of Einstein–Weyl structures (g, ω') , $(g, -\omega')$ where $\omega' = \frac{2}{\sqrt{n-1}}\omega$ and $\omega(X) = g(\xi, X)$ was defined in the first part of the paper. Since $d\omega = 0$, both Einstein–Weyl structures are conformally Einstein. Let $\omega = d\phi$, then the metrics $g_1 = \exp(\phi)g$, $g_2 = \exp(-\phi)g$ are Einstein metrics on S^{n+1} . They are conformally equivalent to each other $g_1 = \exp(2\phi)g_2$. Thus, it follows from [8, 9] that g_i are standard metrics of constant sectional curvature on S^{n+1} . $\square$

Remark In the case of Eq. (3.6) with $A = -2$, we obtain a solution $f(t) = \tanh t$. The corresponding $\mathcal{AC}^\perp$ manifold $M = \mathbb{R}_+ \times_{f^2} S^n$ is complete (see [7]) and diffeomorphic to $\mathbb{R}^{n+1}$. It also admits a pair of Einstein–Weyl structures which are conformally Einstein. At least one of them has to be non-complete since any complete Einstein manifold which is conformal to a complete Einstein manifold has to be a sphere with standard metric of constant sectional curvature (see [8]).

Theorem 4 *On a manifold $M = \mathbb{R} \times_{f^2} M_*$ where (M_*, g_{M_*}) is an Einstein manifold such that $\text{Ric}_{M_*} = \tau g_{M_*}$ with $\tau < 0$ there exists a one-parameter family of complete $\mathcal{AC}^\perp$ metrics g_A such that $g_A = dt^2 + f^2 g_{M_*}$, where f satisfies the equation*

$$(f')^2 = \frac{\tau}{n-1} + Af^2 - f^4 \quad (3.8)$$

with $A \in \mathbb{R}$, $A^2 > -4\tau$, $\tau < 0$. The eigenvalues λ, μ of the Ricci tensor of (M, g_A) satisfy $\lambda \leq \mu$; hence, these manifolds do not admit a pair of Einstein–Weyl structures.

Proof Equation (3.8) is equivalent to the equation

$$f'' = Af - 2f^3 \quad (3.9)$$

with appropriate initial conditions. Hence, we can apply [B, Lemma 16.37 p.445]. Note that $(f')^2 = -(f^2 - a^2)(f^2 + \frac{\tau'}{a^2})$ if $A^2 > -4\tau'$, $A > 0$, $\tau' < 0$ where $\tau' = \frac{\tau}{n-1}$, $a > 0$. We can assume that $a < \frac{\sqrt{-\tau'}}{a}$. Hence for $P(f) = -(f^2 - a^2)(f^2 + \frac{\tau'}{a^2})$, we have $P(a) = P(\frac{\sqrt{-\tau'}}{a}) = 0$ and $P'(a)P'(\frac{\sqrt{-\tau'}}{a}) \neq 0$. Consequently, the equation $(f')^2 = P(f)$ admits a periodic solution $f : \mathbb{R} \rightarrow [a, \frac{\sqrt{-\tau'}}{a}]$. In that way, we get complete manifolds (M, g_A) . Note that these manifolds admit compact quotients. $\diamond$

Remark Note that an equation of type (3.9) (for a function $\frac{1}{f}$) was studied in [1] and complete and compact manifolds constructed in Theorem 4 are known (as manifolds with $Dr \in C(Q)$ in [1]). However, the $AC^\perp$ metrics on spheres S^{n+1} were not found by A. Besse.

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