Erratum to: Sufficient conditions for Willmore immersions in $\mathbb{R}^3$ to be minimal surfaces

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We provide a correction of the proof of Corollary 2 in the original article. Thus, we give a corrected and rigorous proof of the assertion that any complete, closed Willmore surface in $\mathbb{R}^3$ which can be represented as a smooth graph over $\mathbb{S}^2$ has to be a round sphere.

For some immersion $X \in C^4(\Theta, \mathbb{R}^3)$ from some open set $\Theta \subset \mathbb{R}^2$ into Euclidean $\mathbb{R}^3$, its Willmore energy is defined by

$$W(X) := \int_{X(\Theta)} H^2 dA,$$

where the mean curvature H of X is defined as the mean value of the principal curvatures. Critical points of the Willmore energy w.r.t. compactly supported smooth variations, i.e. surfaces X that satisfy

$$\delta W(X, \varphi) := \frac{d}{dt} W(X + t\varphi) \Big|_{t=0} = 0 \quad \forall \varphi \in C_c^\infty(\Theta, \mathbb{R}^3),$$

are called “Willmore surfaces” and meet the following Euler–Lagrange equation:

$$\Delta_X H + 2H(H^2 - K) = 0 \quad \text{in } \Theta. \quad (1)$$

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Here, K denotes the Gaussian curvature of X , and Δ_X is the Laplace–Beltrami operator of the surface X . In Corollary 2 of the original article, we asserted that any complete, closed Willmore surface S in $\mathbb{R}^3$, which can be represented as a smooth graph over $\mathbb{S}^2$, has to be a round sphere. In the proof of that Corollary, we claimed to be able to derive this result by a combination of Theorem 1 (a) of that article with Hopf’s maximum principle applied to the mean curvature H of S and with several standard arguments of classical differential geometry. More precisely, using the fact that S is assumed to be a smooth graph over $\mathbb{S}^2$, we asserted that the boundary of some fixed connected component C of the set $[H_X > 0]$ or of the set $[H_X < 0]$ was of class C^4 , where H_X denotes the mean curvature of some certain parametrization X of S . But this assertion cannot be proven by the authors. Instead, one has to use that there is some conformally parametrized representation X of S on $\mathbb{R}^2$ which is thus a real analytic mapping, so that, in particular, its mean curvature H_X is a real analytic function. Hence, combining this with the simplified Willmore equation (3) for H_X together with Lemma 3 of the original article one obtains that the boundary of C is a union of finitely many real analytic Jordan arcs with a well-defined exterior unit normal and thus that the connected component C is a normal domain in the sense of Definition 1 of the original article, thus in the sense of Definition 1 below. After that, we can apply Lemma 4 of the original article, thus Lemma 1 below, to conclude that $H_X \equiv 0$ on C arrives at the desired contradiction, as C is a non-void connected component of either $[H_X > 0]$ or of $[H_X < 0]$. The rest of the proof can then be taken over from the original article without any changes.

Correction of the proof of Corollary 2 in the original article

First, for the convenience of the reader we repeat here:

Definition 1 (i) We call a bounded domain $\Omega \subset \mathbb{R}^2$ *finitely connected* if $\partial\Omega$ consists of finitely many connected components.

(ii) We call a bounded domain $\Omega \subset \mathbb{R}^2$ a *normal domain* if it is finitely connected, if each connected component β_j , $j = 1, \dots, p$, of $\partial\Omega$ is a closed, regular, piecewise C^1 –Jordan curve and if there exists a unique exterior unit normal field at $\partial\Omega$ in every point of $\partial\Omega \setminus \mathcal{E}$ for some at most finite set $\mathcal{E}$. The above condition on each β_j means precisely: for each j there has to exist some continuous function $c_j : [0, 1] \rightarrow \mathbb{R}^2$ whose image is β_j , which is injective on $[0, 1)$, satisfies $c_j(0) = c_j(1)$ and moreover that there exists some subdivision $0 = t^0 < t^1 < t^2 < \dots < t^{M_j} = 1$ of $[0, 1]$ into consecutive intervals $[t^l, t^{l+1}]$ such that each restriction $c_j|_{[t^l, t^{l+1}]}$ is of class $C^1([t^l, t^{l+1}], \mathbb{R}^2)$ and meets $\dot{c}_j(t) \neq 0$ for any $t \in [t^l, t^{l+1}]$.

Moreover, Lemma 4 of the original article reads as

Lemma 1 Let $X \in C^4(\overline{\Omega}, \mathbb{R}^3)$ be some immersed Willmore surface and $\Omega \subset \mathbb{R}^2$ some bounded C^4 -domain. Assume there exists some normal domain $G \subset \Omega$ (in the sense of Definition 1) with $H \leq 0$ (or $H \geq 0$) in $G \cap O$, $H = 0$ on ∂G , where O is some open neighbourhood of ∂G . Furthermore, there have to exist $c, d \in \mathbb{R}$, $V \in \mathbb{S}^2$ such that $\tilde{X} := cX + dV$ satisfies

$$\inf_{\partial G} \langle \tilde{X}, N \rangle \geq 0 \quad \text{as well as} \quad \sup_{\partial G} \langle \tilde{X}, N \rangle > 0.$$

Then, $H \equiv 0$ holds on $\overline{\Omega}$, i.e. X is a minimal surface on $\overline{\Omega}$.

Definition 2 We call a closed surface S in $\mathbb{R}^3$ a *smooth graph over $\mathbb{S}^2$* if there is some C^∞ -function $r : \mathbb{S}^2 \rightarrow (0, +\infty)$ such that $S = \{r(z)z \mid z \in \mathbb{S}^2\}$. Equivalently, for each point $z \in \mathbb{S}^2$ the intersection $\{tz \mid t \geq 0\} \cap S$ consists of exactly one point p_z , with $r(z) = |p_z| > 0$, and such that the function $z \mapsto r(z)$ is smooth.

Lemma 2 Let S be some closed surface which is a smooth graph over $\mathbb{S}^2$, and let $N(p)$ denote the exterior unit normal of S in some arbitrary point $p \in S$. Then there holds $\langle N(p), p \rangle > 0$ in each point $p \in S$.

Proof By assumption, there exists some smooth function $r : \mathbb{S}^2 \rightarrow (0, +\infty)$ such that $S = \{r(z)z : z \in \mathbb{S}^2\}$. We now define the function

$$f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, (0, +\infty)) \quad \text{by} \quad f(x) := \frac{|x|}{r\left(\frac{x}{|x|}\right)} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

and note for $x \neq 0$ the equivalences

$$f(x) = 1 \Leftrightarrow |x| = r\left(\frac{x}{|x|}\right) \Leftrightarrow x = r\left(\frac{x}{|x|}\right) \frac{x}{|x|} \Leftrightarrow x \in S.$$

In particular, this shows that $S = f^{-1}(1)$, i.e. S is given as a level set of the function f . Differentiating the 1-homogeneity relation $f(\lambda x) = \lambda f(x)$ w.r.t. λ and evaluating at $\lambda = 1$ yields

$$\langle \nabla f(x), x \rangle = f(x). \quad (2)$$

From $f(x) > 0$ we conclude that $\nabla f(x) \neq 0 \forall x \in \mathbb{R}^3 \setminus \{0\}$, and in particular that the exterior unit normal to S is given by $N(x) = \frac{\nabla f(x)}{|\nabla f(x)|}$ for $x \in S$. Inserting this into (2) yields the claim. $\square$

Now we are able to give the corrected proof of Corollary 2 in the original article:

Theorem 1 Suppose S is a complete, closed Willmore surface in $\mathbb{R}^3$ which can be represented as a smooth graph over $\mathbb{S}^2$. Then S has to be a round sphere.

Proof First of all, there must exist some point $p^* \in S$ with $H(p^*) \neq 0$, since otherwise S was a closed, complete minimal surface, which cannot exist by the Theorem of Gauss–Bonnet. We may assume $H(p^*) > 0$, as the other case $H(p^*) < 0$ can be treated in the same way. After a suitable rotation, we may also assume that $p^* = (0, 0, \lambda)$ for some $\lambda > 0$. By assumption, the representation $S = \{r(z)z \mid z \in \mathbb{S}^2\}$ holds for some smooth function $r : \mathbb{S}^2 \rightarrow (0, +\infty)$. First, let $pr : \mathbb{S}^2 \setminus \{p_*\} \rightarrow \mathbb{R}^2$ be the stereographic projection and $\rho := pr^{-1}$ its inverse diffeomorphism, where $p_* := (0, 0, 1) = \frac{p^*}{|p^*|}$ denotes the north pole of $\mathbb{S}^2$. Then $X(x_1, x_2) := r(\rho(x_1, x_2))\rho(x_1, x_2)$ is a parametrization of $S \setminus \{p^*\}$ over $\mathbb{R}^2$ whose mean curvature H_X satisfies the Willmore equation (1) on $\mathbb{R}^2$ since ρ is a conformal diffeomorphism. Let us now define

$$G = \{x \in \mathbb{R}^2 : H_X(x) < 0\} \subset \mathbb{R}^2,$$

which is open by the continuity of H . Moreover, $H_X = 0$ on ∂G is satisfied. We now show that G must be empty by contradiction, thus assuming that $G \neq \emptyset$. The continuity of H and the assumption $H(p^*) > 0$ imply that G must be a bounded subset of $\mathbb{R}^2$. In particular, there exists some $R > 0$ such that $\overline{G} \subset B_R$, where B_R denotes the open ball of radius R centred at 0. Having introduced conformal parameters on the disc $\overline{B_R}$ —being equipped

with the pull-back metric of X —by means of the general, Riemannian version of Koebe's Mapping Theorem in [1], we may assume that $X|_{\overline{B_R}}$ is already conformally parametrized. Then, Riviere's article [2] shows that X is in fact a real analytic mapping. Moreover, if X is conformally parametrized, then the Willmore equation satisfied by X simplifies to

$$\Delta H_X + c H_X = 0 \quad \text{on } \mathbb{R}^2, \quad (3)$$

where Δ now is the ordinary Laplace operator and $c := 2|\partial_1 X|^2(H_X^2 - K_X)$ is some non-negative, real analytic factor. Thus, Lemma 3 of the original article guarantees that the boundary ∂G is a finite union of real analytic curves. Due to the same Lemma, there exists some finite subset $F \subset \partial G$ such that $\nabla H_X \neq 0$ on $\partial G \setminus F$. Thus, $n(x) := \frac{\nabla H_X(x)}{|\nabla H_X(x)|}$ defines the unique exterior unit normal to ∂G at all points $x \in \partial G \setminus F$. In particular, each connected component of G is a normal domain in the sense of Definition 1. Furthermore, Lemma 2 shows the sign condition $\langle X, N \rangle > 0$ on $\mathbb{R}^2$, N denoting the exterior unit normal to S . Thus, Lemma 1 applied to $c = 1$, $d = 0$ and $\Omega = B_R$ yields $H_X \equiv 0$ on $\Omega = B_R$, contradicting the definition of G . Thus, $G = \emptyset$ and hence $H_X \geq 0$ on $\Omega = B_R$ is satisfied. As $R > 0$ can be chosen arbitrarily large, this in fact shows that $H_X \geq 0$ on $\mathbb{R}^2$ and hence that $H \geq 0$ on the original surface S . Now, the continuous function H must achieve its minimal value on the compact surface S . Noting the differential inequality

$$\Delta_S H = -2H(H^2 - K) \leq 0 \quad \text{on } S,$$

the strong maximum principle shows that $H \equiv \text{const.} \geq 0$ on S . On account of the Theorem of Gauss–Bonnet this constant must be positive. Now, any Willmore surface with constant, positive mean curvature H must satisfy $H^2 - K \equiv 0$, which implies that any point of such a surface is umbilic. On the other hand, it is well known that planes and spheres are the only complete, totally umbilic surfaces in $\mathbb{R}^3$. Noting that S has positive mean curvature, it cannot be a plane and thus must be a sphere. $\square$

References

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