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Geometric properties of the ridge function manifold

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Abstract We study geometrical properties of the ridge function manifold $\mathcal{R}_n$ consisting of all possible linear combinations of n functions of the form $g(a\cdot x)$, where $a \cdot x$ is the inner product in $\mathbb{R}^d$. We obtain an estimate for the ε-entropy numbers in terms of smaller ε-covering numbers of the compact class $G_{n,s}$ formed by the intersection of the class $\mathcal{R}_n$ with the unit ball $B\mathcal{P}_s^d$ in the space of polynomials on $\mathbb{R}^d$ of degree s. In particular we show that for $n \le s^{d-1}$ the ε-entropy number $H_\varepsilon(G_{n,s}, L_q)$ of the class $G_{n,s}$ in the space L_q is of order $ns \log 1/\varepsilon$ (modulo a logarithmic factor). Note that the ε-entropy number $H_\varepsilon(B\mathcal{P}_s^d, L_q)$ of the unit ball is of order $s^d \log 1/\varepsilon$. Moreover, we obtain an estimate for the pseudo-dimension of the ridge function class $G_{n,s}$.

Keywords ε-entropy · Pseudo-dimension · Growth number · Ridge function manifold

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1 Introduction and main results

Let $C(\mathbb{R}^d)$ be the space of all continuous functions on the space $\mathbb{R}^d$. Consider in $C(\mathbb{R}^d)$ the class of functions

$$\mathcal{R} = \text{span}\{g_i(a_i \cdot x) : \ g_i \in C(\mathbb{R}),\ a_i \in \mathbb{R}^d\},$$

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consisting of all possible linear combinations of continuous ridge functions of the form $g(a \cdot x)$, $a \in \mathbb{R}^d$, where $a \cdot x$ is the inner product of vectors a and x. Let n be any natural number. Denote by

$$\mathcal{R}_n = \left\{ \sum_{i=1}^{n} g_i(a_i \cdot x) : \quad g_i \in C(\mathbb{R}), \ a_i \in \mathbb{R}^d \right\},$$

the subclass in $\mathcal{R}$ formed by all possible linear combinations of n ridge functions.

The study of such a manifold $\mathcal{R}$ of ridge functions plays a central role in both pure and applied mathematics as is manifested in the series of works [3, 6–8, 10, 12] (Temlyakov, unpublished manuscript) that concern the density of $\mathcal{R}$ in the space of continuous functions and the approximation of function classes by $\mathcal{R}$ (see also the survey of [9]).

Let $L_q = L_q(\Omega)$, $1 \le q < \infty$, be the space of q-integrable functions on the unit cube $\Omega = [-1, 1]^d$ with the norm $\|f\|_{L_q} = \left(\int_\Omega |f(x)|^q \, dx\right)^{1/q}$. In the case $q = \infty$ the space $L_\infty(\Omega)$ consists of all function with the bounded norm $\|f\|_{L_\infty} = \operatorname{ess\,sup}\{|f(x)| : \ x \in \Omega\}$. Denote by $BL_q = \{f : \ \|f\|_{L_q} \le 1\}$ the unit ball in the space L_q. Given a natural number s we denote by $\mathcal{P}_s^d$ the space of all real polynomials of degree at most s on $\mathbb{R}^d$. Let $B_q\mathcal{P}_s^d = BL_q \cap \mathcal{P}_s^d$ be the unit ball in the space $\mathcal{P}_s^d$.

In the current work we study certain geometrical properties of the manifold $\mathcal{R}_n$, namely, we estimate the ε-entropy numbers in terms of smaller ε-covering numbers of the compact class $G_{n,s} = \mathcal{R}_n \cap B_q\mathcal{P}_s^d$ formed by the intersection of the class $\mathcal{R}_n$ with the unit ball in the space of polynomials of degree s on $\mathbb{R}^d$. In particular we show that for $n \le s^{d-1}$, the ε-entropy number $H_\varepsilon(G_{n,s}, L_q)$ of the class $G_{n,s}$ in the space L_q is of order $ns \log_2 1/\varepsilon$ (modulo a logarithmic factor). Note that the ε-entropy number $H_\varepsilon(B_q\mathcal{P}_s^d, L_q)$ of the unit ball $B_q\mathcal{P}_s^d$ is of order $s^d \log_2 1/\varepsilon$. This result answers the question posed by Allan Pinkus: what is cardinality of the intersection of the ridge function manifold with the unit ball in the space of polynomials of a given degree?

Let X be a Banach space and let $B = \{x \in X : \ \|x\| \le 1\}$ the unit ball in X. Denote by $B(x, r) = rB + x$ the ball in X of radius r centered at the point x. All logarithms henceforth are taken with respect to 2. Let F be some compact set in the space X. For any positive number ε the ε-entropy of a set F in the space X represents the quantity $H_\varepsilon(F, X) = \log N_\varepsilon(F, X)$, where $N_\varepsilon(F, X)$ is the minimal number of elements in F those forms an ε-net, i.e.

$$N_\varepsilon(F, X) = \min \left\{ N : \ \exists \ x_1, ..., x_N \in F \text{ such that } F \subset \bigcup_{k=1}^{N} B(x_k, \varepsilon) \right\}.$$

$N_\varepsilon(F, X)$ is called the ε-covering number of the set F. In the class $\mathcal{R}_n$ consider the intersection

$$\mathcal{R}_{n,s} = \mathcal{R}_n \cap \mathcal{P}_s^d$$

of the manifold $\mathcal{R}_n$ with the polynomial space $\mathcal{P}_s^d$.

Theorem 1.1 *Let $n, s \in \mathbb{N}$, $1 \le q \le \infty$ and $0 < \varepsilon < 1$. Then the ε-entropy number of the class $\mathcal{R}_{n,s} \cap BL_q$ in the space L_q satisfies the inequalities*

1. *if $n \le s^{d-1}$, then*

$$c_1 ns \le \frac{H_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q)}{\log \frac{1}{\varepsilon}} \le c_2 nsl_{n,s}, \tag{1}$$

   where $l_{n,s} = \log \frac{2es^{d-1}}{n}$,
2. *if $n > s^{d-1}$, then*

$$c_1' s^d \le \frac{H_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q)}{\log \frac{1}{\varepsilon}} \le c_2' s^d, \tag{2}$$

   where c_1, c_2, c_2, c_2' are constants depending only on d.

Consider another property of the manifold $\mathcal{R}_{n,s}$. For a given vector $h = (h_1, ..., h_m) \in \mathbb{R}^m$ we define the sgn-valued vector $\operatorname{sgn} h = (\operatorname{sgn} h_1, ..., \operatorname{sgn} h_m)$, where $\operatorname{sgn} a = 1$ for $a \ge 0$, and $\operatorname{sgn} a = -1$ for $a < 0$. For a set $H \subset \mathbb{R}^m$, denote by $\operatorname{sgn} H$ the set of vectors $\{\operatorname{sgn} h : h \in H\}$. We denote by $H + a$ the set $\{h + a : h \in H\}$. For a finite set Q we denote by $|Q|$ the cardinality of the set Q.

Definition 1 Let $R = \{r\}$ be a set of functions defined on $\mathbb{R}^d$. The *Vapnik-Chervonenkis dimension* $\dim_{VC} R$ of the set R is defined as the maximal natural number m such that there exists a collection $\{\xi_1, ..., \xi_m\}$ in $\mathbb{R}^d$ for which the cardinality of the vector set $S = \{(\operatorname{sgn} r(\xi_1), ...\operatorname{sgn} r(\xi_m)) : r \in R\}$ equals 2^m. The quantity

$$\dim_p R = \max_f \dim_{VC} (R + f),$$

where f runs over all functions defined on $\mathbb{R}^d$, is called the *pseudo-dimension* of the set R.

Theorem 1.2 *Let $n, s \in \mathbb{N}$. Then there are a constants c_1, c_2, c_1', c_2' depending only on d, such that the pseudo-dimension of the set $\mathcal{R}_{n,s}$ satisfies the inequalities*

1. *if $n \le s^{d-1}$, then $c_1 ns \le \dim_p \mathcal{R}_{n,s} \le c_2 nsl_{n,s}$,*
2. *if $n > s^{d-1}$, then $c_1' s^d \le \dim_p \mathcal{R}_{n,s} \le c_2' s^d$.*

Henceforth we denote by c, c_i, c_i', $i = 0, 1, ...$ positive constants depending only on the parameter d. For two positive sequences a_n and b_n, $n = 0, 1, ...$ we write $a_n \asymp b_n$ if there exist positive constants c_1 and c_2 such that $c_1 \le a_n / b_n \le c_2$ for all $n = 0, 1, ...$

2 Estimating entropy by the growth number

Let B be any convex body in the m-dimensional vector space $\mathbb{R}^m$ which satisfies the symmetrical condition: if $x = (x_1, ..., x_m)$ belongs to B, then every point of the form $(\pm x_1, ..., \pm x_m)$ also belongs to B. We consider the linear normed space l_B^m consisting of vectors from $\mathbb{R}^m$ with the unit ball B and estimate the ε-entropy of a compact set D by the growth number of D which is defined as follows:

Definition 2 Given a set $D \subset \mathbb{R}^m$ define the number

$$\mathrm{Gr}(D) = \max\{|\mathrm{sgn}\,(D - a)| : \ a \in \mathbb{R}^m\},$$

which is called *the growth number* of the set D.

Theorem 2.1 *Let M be any set in the space $\mathbb{R}^m$ and $n = \log \mathrm{Gr}(M)$. Then for any $0 < \varepsilon < 1/2$ the ε-covering number of the intersection of the set M with the unit ball B satisfies the following inequality:*

$$N_\varepsilon(M \cap B, l_B^m) \leq \left(\frac{1}{\varepsilon}\right)^{c_0 n},$$

where c_0 is some absolute constant.

We start with proving a few auxiliary statements. Let F be a compact body in the space l_B^m. We define the ε-capacity of the set F as the quantity $H_\varepsilon^c(F, l_B^m) = \log N_\varepsilon^c(F, l_B^m)$, where $N_\varepsilon^c(F, l_B^m)$ is the maximal number of points such that their pairwise-distances are at least ε, that is,

$$N_\varepsilon^c(F, l_B^m) = \max\left\{N : \ \exists\ x_1, ..., x_N \in F \text{ such that } \|x_i - x_j\| \geq \varepsilon, \ \forall\ i \neq j\right\}.$$

$N_\varepsilon^c(F, X)$ is called the ε-capacity number of the set F.

Proposition 2.2 (see [4]) *The ε-covering number and ε-capacity number of F satisfy the following relationship:*

$$N_{2\varepsilon}^c(F, l_B^m) \leq N_\varepsilon(F, l_B^m) \leq N_\varepsilon^c(F, l_B^m). \tag{3}$$

We briefly denote $N_\varepsilon(D, l_B^m)$ by $N_\varepsilon(D)$.

Lemma 2.3 *Let M be any set in the space $\mathbb{R}^m$, $a \in \mathbb{R}^m$, and $\mathrm{Gr}(M) = 2^n$, r be a positive number and $B(a, r) = rB + a$ be a ball of radius r with center a. Then for any positive number ε there exists a point $a^* \in B(a, r)$ such that*

$$N_\varepsilon\left(M \cap B(a, r)\right) \leq 2^n N_\varepsilon\left(M \cap B\left(a^*, \frac{2}{3}r\right)\right).$$

Proof Introduce the set of vertices of the unit cube

$$\Delta^m = \{\delta = (\delta_1, ..., \delta_m) : \ \delta_i = \pm 1, \ i = 1, ..., m\}$$

in the space $\mathbb{R}^m$. Consider in Δ^m the subset $Q_a = \operatorname{sgn}(M - a)$. Since $\operatorname{Gr}(M) = 2^n$, then for every a we have

$$|Q_a| \leq 2^n. \tag{4}$$

For every point $\delta \in \Delta^m$ we introduce the subset in the ball $B(a, r)$

$$B(a, r, \delta) = \{x \in B(a, r) : \ \operatorname{sgn}(x - a) = \delta\}.$$

We have $B(a, r) = \bigcup_{\delta \in \Delta^m} B(a, r, \delta)$, and $B(a, r, \delta) \cap B(a, r, \delta') = \emptyset$ for any $\delta \neq \delta' \in \Delta^m$. From the definition of the set Q_a we obtain

$$M \cap B(a, r) = \bigcup_{\delta \in \Delta^m} [M \cap B(a, r, \delta)] = \bigcup_{\delta \in Q_a} [M \cap B(a, r, \delta)] .$$

Then

$$\begin{aligned} N_\varepsilon [M \cap B(a, r)] &\leq N_\varepsilon \left[\cup_{\delta \in Q_a} (M \cap B(a, r, \delta))\right] \\ &\leq \sum_{\delta \in Q_a} N_\varepsilon [M \cap B(a, r, \delta)] \leq |Q_a| \, N_\varepsilon \left[M \cap B(a, r, \delta^*)\right], \end{aligned} \tag{5}$$

where δ^* is some vector in Δ^m. Let $0 < \lambda < 1$ be some number and $a_1, ..., a_s$ a minimal λr-net of the set $B(a, r, \delta^*)$. Then $s = N_{\lambda r} [B(a, r, \delta^*)]$. Therefore, we have the inequality

$$N_\varepsilon \left[M \cap B(a, r, \delta^*)\right] \leq \sum_{i=1}^{s} N_\varepsilon [M \cap B(a_i, \lambda r)] \leq s N_\varepsilon [M \cap B(a_{i^*}, \lambda r)] \tag{6}$$

for some $1 \leq i^* \leq s$. Put $\lambda = 2/3$. By Proposition 2.2 we have

$$\begin{aligned} s &= N_{\lambda r} \left[B\left(a, r, \delta^*\right)\right] \leq N^c_{\lambda r} \left[B\left(a, r, \delta^*\right)\right] \leq \frac{V(B(a, r + \lambda r/2, \delta^*))}{V(B(0, \lambda r))} \\ &= \frac{2^{-m} V(B(a, r + \lambda r/2))}{V(B(0, \lambda r))} = 2^{-m} \left(\frac{r + \lambda r/2}{\lambda r}\right)^m = 1. \end{aligned}$$

Hence the inequality (6) implies

$$N_\varepsilon \left[M \cap B(a, r, \delta^*)\right] \leq N_\varepsilon \left[M \cap B(a_{i^*}, 2r/3)\right]. \tag{7}$$

By the union of the inequalities (5), (4), (6) and (7) we have

$$N_\varepsilon [M \cap B(a, r)] \leq 2^n N_\varepsilon \left[M \cap B(a, r, \delta^*)\right] \leq 2^n N_\varepsilon \left[M \cap B(a_{i^*}, 2r/3)\right]$$

which proves the lemma. □

Proof of Theorem 2.1 According to Lemma 2.3 there exists a sequence of points $a_1, ..., a_k$ belonging to the ball B which satisfies the inequalities

$$\begin{aligned} N_\varepsilon [M \cap B] &\leq 2^n N_\varepsilon \left[M \cap B(a_1, 2/3)\right] \\ &\leq 2^{2n} N_\varepsilon \left[M \cap B(a_2, (2/3)^2)\right] \leq ... \leq 2^{kn} N_\varepsilon \left[M \cap B(a_k, (2/3)^k)\right]. \end{aligned}$$

We choose k such that $(2/3)^k \le \varepsilon \le (2/3)^{k-1}$. Then

$$N_\varepsilon\,[M\cap B] \le 2^{kn} N_\varepsilon\,[M\cap B(a_k,\varepsilon)] = 2^{kn} \le (1/\varepsilon)^{c_0 n},$$

where $c_0 = 1 + 1/\log(3/2)$. The theorem is proved. □

3 The entropy of polynomial manifolds

Let α,β and ν be non-negative integers satisfying $\nu=\alpha+\beta$. Let $z=(z_1,...,z_\nu)$ be any point in the space $\mathbb{R}^\nu$. We represent the point z as $z=(u,v)$ where $u=(u_1,...,u_\alpha)$ and $v=(v_1,...,v_\beta)$ are the corresponding projections of z on the subspaces $\mathbb{R}^\alpha$ and $\mathbb{R}^\beta$.

Let s be any natural number. Consider the matrix $Q_{m,\alpha}=(q_{i,j})_{i=1,\ j=1}^{m,\ \alpha}$ which consists of polynomials $q_{i,j}=q_{i,j}(v)$ from the space $\mathcal{P}_s^\beta$. Construct the polynomials of degree $s+1$ on the space $\mathbb{R}^\nu$ as follows:

$$p_i(z) := p_i(u,v) := \sum_{j=1}^{\alpha} u_j q_{i,j}(v), \qquad i=1,...,m. \tag{8}$$

Introduce in the space $\mathbb{R}^m$ the polynomial manifold

$$M := M_{m,s,\nu,\alpha} := \{(p_1(z),...,p_m(z)) :\ z\in\mathbb{R}^\nu\}. \tag{9}$$

Theorem 3.1 *Let B be any convex body in the vector space $\mathbb{R}^m$ such that if $z=(z_1,...,z_m)$ belongs to B, then every point of the form $(\pm z_1,...,\pm z_m)$ also belongs to B. Let $\nu\le m$. Then for any positive ε the ε-covering number of the intersection of the manifold $M_{m,s,\nu,\alpha}$ with the body B satisfies the inequality*

$$N_\varepsilon(M_{m,s,\nu,\alpha}\cap B, l_B^m) \le \left(\frac{1}{\varepsilon}\right)^{c_0 n_{m,s,\nu,\alpha}},$$

where $n_{m,s,\nu,\alpha} = \log\left[(4s)^\beta(\nu+1)^{\beta+2}\left(\frac{2em}{\nu}\right)^\nu\right]$.

Using Theorem 2.1 we have

$$N_\varepsilon(M\cap B, l_B^m) \le \left(\frac{1}{\varepsilon}\right)^{c_0\log \mathrm{Gr}(M)}, \tag{10}$$

where $\mathrm{Gr}(M)$ is the growth number of the manifold M. Thus in order to prove Theorem 3.1 we need to estimate the growth number $\mathrm{Gr}(M)$. This is closely related to the estimation of the number of connected components of polynomial manifolds as is done for instance in [2, 7, 13, 14].

Let p be a real polynomial from the space $\mathcal{P}_s^\nu$, i.e., a polynomial on ν variables of degree at most s. The set of points $x\in\mathbb{R}^\nu$ on which a polynomial p vanishes will be denoted by $Z(p)$. Let D be a set in the space $\mathbb{R}^\nu$. We denote by $\mathrm{NCC}(D)$ the number of connected components of the set D. Let $p_1,...,p_m$

be a collection of m real polynomials from $\mathcal{P}_s^\nu$ and consider the finite set of sgn-valued vectors

$$E(p_1, ..., p_m) = \{(\operatorname{sgn} p_1(z), ..., \operatorname{sgn} p_m(z)) : \ z \in \mathbb{R}^\nu\}.$$

In order to estimate the growth number $\operatorname{Gr}(M)$ we need an estimate on the cardinality of $E(p_1, ..., p_m)$. Introduce in the space $\mathbb{R}^\nu$ the domains $D = \bigcup_{i=1}^m Z(p_i)$ and $D' = \mathbb{R}^\nu \setminus D$. We denote by $\operatorname{sgn} p(z) = (\operatorname{sgn} p_1(z), ..., \operatorname{sgn} p_m(z))$. The vector function $\operatorname{sgn} p(z)$ is constant on any connected component of the domain D'. Therefore the cardinality of the set $E(p_1, ..., p_m)$ does not exceed the number $NCC(D')$ of connected components of the domain D'. Hence we now estimate the number $\operatorname{NCC}(D')$.

Theorem 3.2 *Let $\nu \leq m$ and let $p_1, ..., p_m$ be the polynomials defined in* (8). *Then the number of connected components of the domain $D' = \mathbb{R}^\nu \setminus \bigcup_{i=1}^m Z(p_i)$ satisfies the inequality*

$$\operatorname{NCC}(D') \leq (4s)^\beta (\nu + 1)^{\beta+2} \left(\frac{2em}{\nu}\right)^\nu .$$

Proof We use the following inequality of Warren (see [14], Th. 1,2): there exist positive numbers $\delta_1, ..., \delta_m$ such that the number of connected components of the set D' satisfies the inequality

$$\operatorname{NCC}(D') \leq \sum_J \sum_{\varepsilon_j = \pm 1, \ j \in J} \operatorname{NCC}\left(\bigcap_{j \in J} Z(p_j + \varepsilon_j \delta_j)\right), \tag{11}$$

where J runs over all subsets of the set $\{1, ..., m\}$, and ε_j with $j \in J$ taking all possible values ± 1. □

Let $r_1(v), ..., r_n(v)$ be any n polynomials in the variable $v \in \mathbb{R}^\beta$ and let $\deg r_i$ be the degree of the polynomial r_i. We set $g = \deg r_1 + ... + \deg r_n$. Consider in the space $\mathbb{R}^\beta$ the manifold

$$R = \{v : \ r_1(v) \geq 0, ..., r_n(v) \geq 0\}.$$

Lemma 3.3 (Milnor [5], Th.3) *The number* $\operatorname{NCC}(R)$ *of connected components of the set R satisfies*

$$\operatorname{NCC}(R) \leq \frac{1}{2}(2 + g)(1 + g)^{\beta - 1}.$$

From Milnor [5] it directly follows that the number of connected components, $\operatorname{NCC}(R)$, is bounded from above by rank $H^*(R)$ which is the rank of full cohomology group $H^*(R)$. Milnor obtained the estimate: rank $H^*(R) \leq \frac{1}{2}(2 + g)(1 + g)^{\beta-1}$.

Fix a subset J from the set $\{1, ..., \nu\}$. Using Lemma 3.3 we estimate the numbers

$$\mathrm{NCC}\left(\bigcap_{j\in J} Z(p_j + \varepsilon_j\delta_j)\right).$$

Without loss of generality we may take $J = \{1, ..., k\}$, and $\varepsilon_1 = \cdots = \varepsilon_k = -1$, where k is the cardinality of the subset J.

Lemma 3.4 *For any* $1 \le k \le \nu$ *and positive numbers* $\delta_1, ..., \delta_k$*, the following inequality holds*

$$\mathrm{NCC}\left(\bigcap_{i=1}^{k} Z(p_i - \delta_i)\right) \le (4s)^{\beta}(k+1)^{\beta+1}.$$

Proof Consider the system of k linear equations

$$\begin{cases} q_{11}(v)u_1 + \ldots + q_{1\alpha}(v)u_\alpha = \delta_1 \\ \qquad\vdots \\ q_{k1}(v)u_1 + \ldots + q_{k\alpha}(v)u_\alpha = \delta_k \end{cases} \tag{12}$$

in the variables $u_1, ..., u_\alpha$, with coefficients $q_{ij}(v)$ and constants δ_i. For any fixed v, denote by $D_l(v)$ the sum of squares of all minors of order l of the matrix $\left(q_{ij}(z)\right)_{i=1\ j=1}^{k,\ \alpha}$, and by $\overline{D_{l+1}}(v)$ the sum of squares of all minors of order $l+1$ of the extended matrix

$$\begin{pmatrix} q_{11}(v) & \ldots & q_{1\alpha}(v) & \delta_1 \\ \vdots & \ddots & \vdots & \vdots \\ q_{k1}(v) & \ldots & q_{k\alpha}(v) & \delta_k \end{pmatrix}.$$

It follows from a theorem of Kronecker-Capelli that the set V of all vectors $v \in \mathbb{R}^{\beta}$ for which there exists a solution to the system (12) may be expressed as

$$V = \bigcup_{l=1}^{k-1} V_l, \qquad V_l = \{v \in \mathbb{R}^{\beta} :\ D_l(v) > 0,\ \overline{D_{l+1}}(v) = 0\}. \tag{13}$$

Since the $q_{ij}(v)$ are polynomials of degree s then $D_l(v)$ and $\overline{D_{l+1}}(v)$ are polynomials of degree $2sl$ and $2s(l+1)$, respectively. From the continuous dependence of the solutions of (12) on the coefficients $q_{ij}(v)$, when v runs over some connected component of V_l, it follows that $\mathrm{NCC}\left(\bigcap_{l=1}^{k} Z(p_l - \delta_l)\right) = \mathrm{NCC}(V)$. From (13) we have

$$\mathrm{NCC}\left(\bigcap_{l=1}^{k} Z(p_l - \delta_l)\right) = \mathrm{NCC}(V) \le \sum_{l=1}^{k-1} \mathrm{NCC}(V_l). \tag{14}$$

Note that for any l a set V_l may be represented as the set of solutions of the system of inequalities

$$D_l(v) > 0, \qquad -\overline{D_{l+1}}(v) \geq 0. \tag{15}$$

We claim that $\mathrm{NCC}(V_l) \leq (4s(k+1))^\beta$. Indeed, assume that $NCC(V_l) \geq \mu + 1$, where $\mu = (4s(k+1))^\beta$. Then there exist $\mu + 1$ disjoint components $Q_1, ..., Q_{\mu+1}$ of the set V_l. For each $1 \leq i \leq \mu + 1$, choose a point w_i^* in Q_i. Put $\gamma = \min_i D_l(w_i^*)$, and note that $\gamma > 0$. Consider the set in $\mathbb{R}^\beta$

$$V_l' = \{v : \ D_l(v) \geq \gamma, \ -\overline{D_{l+1}}(v) \geq 0\}.$$

From Lemma 3.3 it follows that the number of connected components of the set V_l' satisfies the inequality $\mathrm{NCC}(V_l') \leq \frac{1}{2}(2+g)(1+g)^{\beta-1}$, where $g = \deg D_l + \deg \overline{D_{l+1}}$. Since $g \leq 2sl + 2s(l+1) \leq 4s(k+1)$ we have

$$\mathrm{NCC}(V_l') \leq \frac{1}{2}(2 + 4s(k+1))(1 + 4s(k+1))^{\beta-1} \leq (4s(k+1))^\beta = \mu.$$

On the other hand, since $V_l' \cap Q_i \neq \emptyset$ for all $i = 1, ..., \mu + 1$, and $Q_1, ..., Q_{\mu+1}$ do not intersect, then

$$\mathrm{NCC}(V_l') \geq \mathrm{NCC}(V_l) \geq \mu + 1$$

yielding a contradiction. Hence $\mathrm{NCC}(V_l) \leq (4s(k+1))^\beta$. From here and (14) we obtain

$$\mathrm{NCC}\left(\bigcap_{l=1}^{k} Z(p_l - \delta_l)\right) \leq \sum_{l=1}^{k-1} \mathrm{NCC}(V_l) \leq \sum_{l=1}^{k-1} (4s(k+1))^\beta \leq (4s)^\beta (k+1)^{\beta+1}.$$

Lemma 3.4 is proved. □

We now continue the proof of Theorem 3.2. From the inequality of (11) and Lemma 3.4 one obtains the following estimate for the number of connected components of the set $D' = \mathbb{R}^\nu \setminus \bigcup_{i=1}^m Z(p_i)$

$$\begin{aligned}
\mathrm{NCC}(D') &\leq \sum_J \sum_{\{\varepsilon_j :\ j \in J\}} (4s)^\beta (k+1)^{\beta+1} \\
&\leq (4s)^\beta (\alpha + \beta + 1)^{\beta+1} \sum_{k=1}^{\nu} \binom{m}{k} 2^k \\
&\leq 2^\nu (4s)^\beta (\nu+1)^{\beta+1} \sum_{k=1}^{\nu} \frac{m^k}{k!} \leq (4s)^\beta (\nu+1)^{\beta+2} \left(\frac{2em}{\nu}\right)^\nu,
\end{aligned}$$

where we make use of the condition $\alpha + \beta = \nu \leq m$. Theorem 3.2 is proved.

Proof of Theorem 3.1 We estimate the growth number

$$\mathrm{Gr}\,(M) = \max\{|\mathrm{sgn}\,(M - a)| : \ a \in \mathbb{R}^m\}$$

of the manifold M. Let $a = (a_1, ..., a_m)$ be a fixed point in $\mathbb{R}^m$. Consider in the space $\mathbb{R}^\nu$ the domains $D_a = \bigcup_{i=1}^m Z(p_i - a_i)$ and $D'_a = \mathbb{R}^\nu \setminus D_a$. Since the cardinality of the set $\operatorname{sgn}(M - a)$ does not exceed the number of connected components of the set D'_a then

$$\mathrm{Gr}\,(M) \le \max_{a \in \mathbb{R}^\nu} \mathrm{NCC}(D'_a). \tag{16}$$

We have $\nu \le m$. Therefore, from Theorem 3.2 for any $a \in \mathbb{R}^\nu$ we obtain

$$\mathrm{NCC}(D'_a) \le (4s)^\beta (\nu + 1)^{\beta+2} \left(\frac{2em}{\nu}\right)^\nu. \tag{17}$$

Combining the inequalities (16), (17) and (10) we obtain the estimate for the ε-covering number

$$N_\varepsilon(M \cap B, l_B^m) \le \left(\frac{1}{\varepsilon}\right)^{c_0 n},$$

where $n = \log\left[(4s)^\beta (\nu + 1)^{\beta+2} \left(\frac{2em}{\nu}\right)^\nu\right]$. Theorem 3.1 is proved. □

4 Proof of the main theorems

Let s be any natural number. Introduce in the interval $[-1, 1]$ the collection of $4s$ points $\xi_i = \cos\frac{(2i-1)\pi}{8s}$, $i = 1, ..., 4s$. Set $m = (4s)^d$. Consider in the cube $\Omega = [-1, 1]^d$ the finite lattice of points

$$\Xi_m = \{\xi = (\xi_{i_1}, ..., \xi_{i_d}) : \ i_1, ..., i_d = 1, ..., 4s\}.$$

Let l_q^m be the normed space of functions consisting of polynomials P of the space $\mathcal{P}_s^d$ with the norm

$$\|P\|_{l_q^m} = \left(\frac{1}{m} \sum_{\xi \in \Xi_m} |P(\xi)|^q\right)^{1/q}.$$

Denote by B_q^m the unit ball in the space l_q^m. Consider in $\mathbb{R}^m$ the subset

$$\mathcal{R}_{n,s}^m = \{\{P(\xi)\}_{\xi \in \Xi_m} : \ P \in \mathcal{R}_{n,s}\}$$

which is the restriction of functions of $\mathcal{R}_{n,s}$ to the lattice Ξ_m.

Proposition 4.1 *Let $1 \le q \le \infty$. Then there is a constant c depending only on d such that for any positive number*

$$N_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q) \le N_{c\varepsilon}(\mathcal{R}_{n,s}^m \cap Bl_q^m, l_q^m). \tag{18}$$

Proof Let P be any polynomial from the space $\mathcal{P}_s^d$. Then the following inequality is true

$$\|P\|_{L_q} \le c\|P\|_{l_q^m}. \tag{19}$$

Indeed, by a change of variables $x_k = \cos t_k$, $k = 1, ..., d$, we have

$$\|P\|_{L_q}^q = \int_{\Omega} |P(x)|^q \, dx = \int_{[0,\pi]^d} |P(\cos t)|^q \, w(t) \, dt \le \int_{[0,\pi]^d} |P(\cos t)|^q \, dt,$$

where $\cos t = (\cos t_1, ..., \cos t_d)$ and $w(t) = \prod_{k=1}^d \sin t_k$. From the known relation (see [15]) we have

$$\int_{[0,\pi]^d} |P(\cos t)|^q \, dt \le \frac{c}{m} \sum_{\xi \in \Xi_m} |P(\xi)|^q. \tag{20}$$

Thus, the inequality (19) is proved. Note that (19) directly implies the inequality of (18). □

Proposition 4.2 *Consider in the set $\mathcal{R}_{n,s}$ the subset*

$$\mathcal{R}_{n,s}^* = \left\{ \sum_{i=1}^n \pi_i(a_i \cdot x) : \quad \pi_i \in \mathcal{P}_s^1, \ a_i \in \mathbb{R}^d \right\},$$

where π_i are any univariate polynomials of degree at most s. Then $\mathcal{R}_{n,s} = \mathcal{R}_{n,s}^*$.

By the Weierstrass theorem we have $\mathcal{R}_n = \overline{\bigcup_{k \ge s} \mathcal{R}_{n,k}^*}$, where $\overline{A}$ is the closure of the function set A in the space of continuous functions on the ball B^d. Therefore,

$$\mathcal{R}_{n,s} = \overline{\bigcup_{k \ge s} \mathcal{R}_{n,k}^*} \cap \mathcal{P}_s^d.$$

Since the set $\left(\bigcup_{k \ge s} \mathcal{R}_{n,k}^*\right) \bigcap \mathcal{P}_s^d$ is not empty then

$$\overline{\bigcup_k \mathcal{R}_{n,k}^*} \bigcap \mathcal{P}_s^d = \overline{\left(\bigcup_k \mathcal{R}_{n,k}^*\right) \bigcap \mathcal{P}_s^d}.$$

Therefore,

$$\mathcal{R}_{n,s} = \overline{\left(\bigcup_k \mathcal{R}_{n,k}^*\right) \bigcap \mathcal{P}_s^d} = \overline{\bigcup_k \left(\mathcal{R}_{n,k}^* \bigcap \mathcal{P}_s^d\right)}.$$

We show that for every $k \ge s$ the equality $\mathcal{R}_{n,k}^* \cap \mathcal{P}_s^d = \mathcal{R}_{n,s}^*$ holds. Indeed, if $k = s$ then that is obvious. Let $k > s$ and let $\pi(x) = \sum_{i=1}^n \pi_i(a_i \cdot x)$ be any polynomial from the set $\mathcal{R}_{n,k}^* \cap \mathcal{P}_s^d$ where $\pi_i \in \mathcal{P}_k^1$ and $a_i \in \mathbb{R}^d$. For every i we can represent the polynomial by $\pi_i(a_i \cdot x) = \sum_{j=0}^k c_{ij}(a_i \cdot x)^j$. Since the degree of polynomial π is equal to s, then $c_{ij} = 0$ for any i and $s + 1 \le j \le k$. Therefore the polynomial π belongs to $\mathcal{R}_{n,s}^*$. Hence, $\mathcal{R}_{n,k}^* \cap \mathcal{P}_s^d \subseteq \mathcal{R}_{n,s}^*$. The inclusion $\mathcal{R}_{n,k}^* \cap \mathcal{P}_s^d \supseteq \mathcal{R}_{n,s}^*$ is obvious. Thus, $\mathcal{R}_{n,s} = \mathcal{R}_{n,s}^*$.

Proposition 4.3 *Let l_B^m be the m-dimensional normed space $\mathbb{R}^m$ with a unit ball B. Then the ε-covering number $N_\varepsilon(B, l_B^m)$ of the ball B satisfies*

$$(4\varepsilon)^{-m} \le N_\varepsilon(B, l_B^m) \le (\varepsilon/3)^{-m}.$$

It follows from Proposition 2.2 that

$$N_{2\varepsilon}^c(B, l_B^m) \le N_\varepsilon(B, l_B^m) \le N_\varepsilon^c(B, l_B^m), \tag{21}$$

where $N_\varepsilon^c(B, l_B^m)$ is the ε-capacity number of the ball B. Denote by $V(D)$ the Lebesgue volume of a set D. Then we have the obvious inequalities:

$$N_\varepsilon^c(B, l_B^m) \le \max\{n : \; V((1+\varepsilon)B) > nV\left(\frac{\varepsilon}{2}B\right)\}$$

and

$$N_\varepsilon^c(B, l_B^m) \ge \max\{n : \; V((1-\varepsilon)B) > nV(\varepsilon B)\}.$$

Therefore, since $V(\varepsilon B) = \varepsilon^m V(B)$ we obtain

$$(2\varepsilon)^{-m} \le N_\varepsilon^c(B, l_B^m) \le (\varepsilon/3)^{-m}.$$

From here and (21) the statement of Proposition 4.3 directly follows.

Proof of Theorem 1.1 First we prove the right-hand inequalities in (1) and (2). Assume that $n \le s^{d-1}$. From Proposition 4.1 we have

$$N_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q) \le N_{c\varepsilon}(\mathcal{R}_{n,s}^m \cap Bl_q^m, l_q^m). \tag{22}$$

Using Theorem 2.1 we obtain

$$N_{c\varepsilon}(\mathcal{R}_{n,s}^m \cap Bl_q^m, l_q^m) \le \left(\frac{1}{\varepsilon}\right)^{c_0 \log \mathrm{Gr}(\mathcal{R}_{n,s}^m)}, \tag{23}$$

where $\mathrm{Gr}(\mathcal{R}_{n,s}^m)$ is the growth number of the manifold $\mathcal{R}_{n,s}^m$ and c_0 is some absolute constant.

Given s, n and d we define $m = (4s)^d$, $\alpha = (s+1)n$, $\beta = dn$ and $\nu = \alpha + \beta$. We show that the manifold $\mathcal{R}_{n,s}^m$ may be represent by the form (9), (8). By Proposition 4.2 we have

$$\mathcal{R}_{n,s}^m = \left\{ \left\{ P(\xi) = \sum_{k=1}^n \pi_k(a_k \cdot \xi) \right\}_{\xi \in \Xi_m} : \; \pi_k \in \mathcal{P}_s^1, \; a_k \in \mathbb{R}^d \right\}.$$

We enumerate all points ξ from the lattice Ξ_m by $\xi_1, ..., \xi_m$. For fixed i we consider the value

$$P(\xi_i) = \sum_{k=1}^n \pi_k(a_k \cdot \xi_i) = \sum_{k=1}^n \sum_{l=0}^s c_{k,l}(a_k \cdot \xi_i)^l.$$

of the polynomial P in the point ξ_i and denote $P(\xi_i)$ by $Q_i(c, a)$, that is consider the polynomial Q_i in variables c and a.

Enumerate the variables $\{c_{k,l}\}$, where $k = 1, ..., n$, $l = 0, ..., s$ by $\{u_j\}$, $j = 1, ..., \alpha$, $\alpha = n(s+1)$.

Let $a_{k,1}, \dots, a_{k,d}$ be the coordinates of vector a_k. We enumerate the variables $\{a_{k,r}\}$, where $k = 1, \dots, n, r = 1, \dots, d$ by $\{v_j\}$, $j = 1, \dots, \beta$, $\beta = dn$.

Let i is a fixed index. If the index j correspond to given indexes k and l then we denote the functions $(a_k \cdot \xi_i)^l$ by $q_{i,j}(v)$. Then we can rewrite the polynomial $Q_i(c, a)$ by the form

$$Q_i(c, a) := p_i(u, v) := \sum_{j=1}^{\alpha} u_j q_{i,j}(v).$$

Thus, we represent (see (9), (8)) the manifold $\mathcal{R}^m_{n,s}$ by the form

$$M := M_{m,s,\nu,\alpha} := \{(p_1(u, v), \dots, p_m(u, v)) : \ (u, v) \in \mathbb{R}^\nu\}.$$

Since $n \leq s^{d-1}$ then it is easy to see that $\nu \leq m$. Then according to inequalities (16) and (17) we have the estimate for the growth number

$$\mathrm{Gr}(\mathcal{R}^m_{n,s}) = \mathrm{Gr}(M) \leq (4s)^\beta (\nu + 1)^{\beta+2} \left(\frac{2em}{\nu}\right)^\nu.$$

By a straightforward computation we obtain

$$\mathrm{Gr}(\mathcal{R}^m_{n,s}) \leq \left(\frac{2es^{d-1}}{n}\right)^{c_1 sn}, \tag{24}$$

where $c_1 = 4d(2d + 1)$. Set $c = c_0 c_1$. Using the inequalities (22), (23) and (24) we then obtain

$$N_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q) \leq \left(\frac{1}{\varepsilon}\right)^{c_0 \log \mathrm{Gr}(\mathcal{R}^m_{n,s})} \leq \left(\frac{1}{\varepsilon}\right)^{cns \log(\frac{2es^{d-1}}{n})}. \tag{25}$$

Thus the right-hand inequality in (1) is proved.

We prove the right-hand inequality in (2). Let $n > s^{d-1}$. Denote by $\mathcal{Q}^d_s$ the subspace in $\mathcal{P}^d_s$ consisting of all homogeneous polynomials of degree s. We know (see [7]) that

$$\mathcal{P}^d_s = \mathcal{R}_{n,s} \tag{26}$$

for any $n \geq \dim \mathcal{Q}^d_s$. Since (see [11]) $\dim \mathcal{Q}^d_s = \binom{s+d-1}{s} \asymp s^{d-1}$, then

$$N_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q) \leq N_\varepsilon(\mathcal{P}^d_s \cap BL_q, L_q).$$

Let $\mathcal{P}^d_s(\Xi_m)$ be the restriction of functions of the space $\mathcal{P}^d_s$ to the lattice Ξ_m. We have $\dim \mathcal{P}^d_s \asymp s^d$. Applying the inequality (19) and Proposition 4.3 we have

$$N_\varepsilon(\mathcal{P}^d_s \cap BL_q, L_q) \leq N_{c\varepsilon}\left(\mathcal{P}^d_s(\Xi_m) \cap Bl^m_q, l^m_q\right) \leq \left(\frac{1}{\varepsilon}\right)^{c' s^d}. \tag{27}$$

Thus, from (25) and (27) the right-hand inequalities in (2) directly follow.

Now we prove the left-hand inequalities in (1) and (2). Let $A = \{a_1, \dots, a_n\}$ be a fixed collection of points from $\mathbb{R}^d$. Consider the linear subset in the set $\mathcal{R}_n$,

$$\mathcal{R}_n(A) = \left\{\sum_{i=1}^{n} g_i(a_i \cdot x) : \ g_i \in C(\mathbb{R})\right\},$$

which is a linear space. Let $n \le s^{d-1}$. Then $\frac{1}{d!}ns \le \frac{s^d}{d!} \le \dim \mathcal{P}_s^d$. Construct the sets $\mathcal{R}_{n,s}(A) = \mathcal{R}_n(A) \cap \mathcal{P}_s^d$ and

$$\mathcal{R}_{n,s}^*(A) = \left\{ \sum_{i=1}^n \pi_i(a_i \cdot x) : \quad \pi_i \in \mathcal{P}_s^1 \right\}. \tag{28}$$

We can assume that the collection of points $a_1, ..., a_n$ is chosen such that the dimension $\mu = \dim \mathcal{R}_{n,s}^*(A)$ of the linear space $\mathcal{R}_{n,s}^*(A)$ belongs to the interval $[\frac{1}{d!}ns, \dim \mathcal{P}_s^d]$.

By analogy with Proposition 4.2 we can show that $\mathcal{R}_{n,s}^*(A) = \mathcal{R}_{n,s}(A)$. Also we have $\mathcal{R}_{n,s}^*(A) \subset \mathcal{R}_{n,s}$. Hence

$$N_\varepsilon(\mathcal{R}_{n,s} \cap BL_q, L_q) \ge N_\varepsilon(\mathcal{R}_{n,s}^*(A) \cap BL_q, L_q).$$

Using Propositions 4.2, 4.3 and 2.2 we obtain

$$N_\varepsilon(\mathcal{R}_{n,s}^*(A) \cap BL_q, L_q) \ge \left(\frac{1}{\varepsilon}\right)^{c\mu}.$$

Since $\mu \ge c_1 ns$, where $c_1 = \frac{1}{d!}$, then the left-hand inequality in (1) is proved.

If $n > s^{d-1}$ then using the embedding (26) we have $\mathcal{P}_{cs}^d \subset \mathcal{R}_{n,s}^*(A)$ with some absolute constant c. Thus, we obtain $\mu \ge \dim \mathcal{P}_{cs}^d \asymp s^d$. Therefore,

$$N_\varepsilon(\mathcal{R}_{n,s}^*(A) \cap BL_q, L_q) \ge \left(\frac{1}{\varepsilon}\right)^{c_1 s^d},$$

that is the left-hand inequality in (2) is also proved. Theorem 1.1 has now been completely proved. □

Proof of Theorem 1.2 Assume that $n \le s^{d-1}$. Set $\mu = \dim \mathcal{P}_s^d$. Let Ξ be any finite set of points in $\mathbb{R}^d$. Denote by $\mathcal{R}_{n,s}^\Xi$ the restriction of the set $\mathcal{R}_{n,s}$ on Ξ. Then the following inequality holds:

$$\begin{aligned}\dim_p \mathcal{R}_{n,s} &= \max \left\{k : \ \exists\, \Xi, a \text{ with } |\Xi| = k \text{ and } a \in \mathbb{R}^k \text{ s.t. } |\mathrm{sgn}(\mathcal{R}_{n,s}^\Xi - a)| = 2^k\right\} \\ &\le \log \max_{\Xi:\ |\Xi| \le \mu} \mathrm{Gr}(\mathcal{R}_{n,s}^\Xi).\end{aligned}$$

In analogy to the inequality of (24) we can show that

$$\mathrm{Gr}(\mathcal{R}_{n,s}^\Xi) \le \left(\frac{2es^{d-1}}{n}\right)^{c_1 sn},$$

for any Ξ with $|\Xi| \le \mu$. Thus,

$$\dim_p \mathcal{R}_{n,s} \le c_1 ns \log_2 \frac{2es^{d-1}}{n}. \tag{29}$$

Let $n > s^{d-1}$. We know (see [1]) that the pseudo-dimension $\dim_p L$ of a linear finite-dimensional subspace L coincides with the dimension $\dim L$ of L. Therefore,

$$\dim_p \mathcal{R}_{n,s} \leq \dim_p \mathcal{P}_s^d = \dim \mathcal{P}_s^d \asymp s^d. \quad (30)$$

From (29) and (30). the right-hand inequalities of Theorem 1.2 are proved.

Prove the left-hand inequalities. Let $n \leq s^{d-1}$. Let $A = \{a_1, ..., a_n\}$ be a collection of points from $\mathbb{R}^d$ such that $\dim \mathcal{R}_{n,s}(A) \geq cns$ with some constant c. Since $\mathcal{R}_{n,s}(A)$ is a linear subspace, then

$$\dim_p \mathcal{R}_{n,s} \geq \dim_p \mathcal{R}_{n,s}(A) = \dim \mathcal{R}_{n,s}(A) \geq cns.$$

If $n > s^{d-1}$ then for some set $A = \{a_1, ..., a_n\}$ the polynomial space $\mathcal{P}_s^d$ belongs to $\mathcal{R}_{n,s}$. Thus,

$$\dim_p \mathcal{R}_{n,s} \geq \dim_p \mathcal{R}_{n,s}(A) = \dim \mathcal{P}_s^d \asymp s^d.$$

Theorem 1.2 has been completely proved. □

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