Improved stability estimates and a characterization of the native space for matrix-valued RBFs

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Published online: 7 August 2008
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Erratum to: Adv Comput Math DOI 10.1007/s10444-007-9046-3

Unfortunately, there was an error in the proof of Theorem 6 of [2]. In this paper a corrected proof is presented.

We will use $\|\alpha\|_2$ to denote the standard Euclidean norm of $\alpha \in \mathbb{R}^m$, and write α^T for the transpose of α . We will use the following convention for the Fourier transform and inverse Fourier transform of a function:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix^T \xi} d\xi, \quad \check{f}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi^T x} d\xi.$$

Let $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ be a set of distinct points. We let q_X denote the separation radius of X , which is defined by

$$q_X := \frac{1}{2} \min_{i \neq j} \|x_i - x_j\|.$$

We will say an $m \times m$ matrix-valued function Φ is *positive definite* on $\mathbb{R}^n$ if given any finite, distinct set of points X , we have

$$\sum_{j,k=1}^N \alpha_j^T \Phi(x_j - x_k) \alpha_k \geq 0$$

for all $\alpha_1, \dots, \alpha_N$ in $\mathbb{R}^m$. This is equivalent to saying that the $mN \times mN$ matrix whose $(j, k)^{th}$ $m \times m$ block is given by $\Phi(x_j - x_k)$ is positive definite. This matrix

The online version of the original article can be found under doi:10.1007/s10444-007-9046-3.

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will be denoted by $A_{X,\phi}$. If a positive definite function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is radial, i.e. $\phi(x) = \phi(\|x\|_2)$, then we call ϕ a *radial basis function* (RBF).

Given an RBF ϕ defined on $\mathbb{R}^n$, one can construct the matrix-valued function $\Phi_{div} = (-\Delta + \nabla \nabla^T)\phi$. This function is positive definite, and its columns are divergence-free [1]. The stability of the matrix $A_{X,\Phi_{div}}$ has been studied, and recently the following result was presented [2, Theorem 6].

Theorem 6 *Let ϕ be an even positive definite function, which possesses a positive Fourier transform $\widehat{\phi} \in C(\mathbb{R}^n \setminus \{0\})$. With the function*

$$M(\sigma) := \inf_{\|\xi\|_2 \leq \sigma} \widehat{\phi}(\xi)$$

a lower bound on the smallest eigenvalue of the interpolation matrix is given by

$$\lambda_{\min}(A_{X,\Phi_{div}}) \geq \left(\frac{\sigma^2}{16\pi} \right)^{(n+2)/2} \frac{M(\sigma)\pi}{(4\pi)^n \Gamma((n+2)/2)}$$

for any $\sigma > 0$ satisfying

$$\sigma \geq \widetilde{C}/q_X,$$

where $\widetilde{C}$ is a constant independent of ϕ and X .

It has come to the attention of the author that there was an error in the proof. Below is a corrected version.

Proof To condense our notation, we define the constant

$$C_\sigma := \frac{M(\sigma)\Gamma((n+2)/2)}{\sigma^n \pi^{n/2}}$$

Let $\chi_{\frac{\sigma}{2},n}(\xi)$ be the characteristic function for the ball $B(0, \sigma/2) \subset \mathbb{R}^n$, and define ψ by $\psi := C_\sigma \chi_{\frac{\sigma}{2},n}^2$. Note that ψ is positive definite and the support of $\widehat{\psi}$ is in $B(0, \sigma)$. It follows then that $\widehat{\phi}(\xi) \geq \widehat{\psi}(\xi)$ for $\|\xi\|_2 > \sigma$. If $\|\xi\|_2 \leq \sigma$, we have

$$\widehat{\psi}(\xi) \leq C_\sigma \vee 1(B(0, \sigma)) \leq M(\sigma) \leq \widehat{\phi}(\xi).$$

This shows us that

$$\begin{aligned} \sum_{j,k=1}^N \alpha_j^T \Phi_{div}(x_j - x_k) \alpha_k &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T) \left(\sum_{j=1}^N \alpha_j e^{i\xi^T x_j} \right) \right\|_2^2 \|\xi\|_2^2 \widehat{\phi}(\xi) d\xi \\ &\geq (2\pi)^{-n} \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T) \left(\sum_{j=1}^N \alpha_j e^{i\xi^T x_j} \right) \right\|_2^2 \|\xi\|_2^2 \widehat{\psi}(\xi) d\xi \\ &= \sum_{j,k=1}^N \alpha_j^T \Psi_{div}(x_j - x_k) \alpha_k, \end{aligned}$$

where $\alpha_j \in \mathbb{R}^n$ and Ψ_{div} is defined by $\Psi_{div} = (-\Delta + \nabla \nabla^T)\psi$.

From the calculations done preceding equation (11) in [2, Section 4], we see that

$$\Psi_{div}(0) = C_\sigma(n-1)A(0)I,$$

where $A(0)$ is given by

$$A(0) = \left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{8\pi}{(n+2)\Gamma^2((n+2)/2)}.$$

Now we use [2, Lemma 3] to get:

$$\begin{aligned} \sum_{j,k=1}^N \alpha_j^T \Psi_{div}(x_j - x_k) \alpha_k &\geq \|\alpha\|_2^2 C_\sigma (n-1) A(0) - \sum_{j \neq k} |\alpha_j^T \Psi_{div}(x_j - x_k) \alpha_k| \\ &\geq \|\alpha\|_2^2 C_\sigma (n-1) A(0) - n \|\alpha\|_2^2 \max_k \sum_{j \neq k} C_\sigma \Lambda(x_j - x_k) \\ &= \|\alpha\|_2^2 \left(C_\sigma (n-1) A(0) - C_\sigma \frac{n-1}{2} A(0) \right) \\ &= \|\alpha\|_2^2 \frac{1}{2} C_\sigma (n-1) A(0), \end{aligned}$$

where Λ is defined in [2, Equation (13)]. The result now follows by rewriting C_σ and $A(0)$ to get

$$\begin{aligned} \lambda_{\min}(A_{X, \Phi_{div}}) &\geq \frac{1}{2} C_\sigma (n-1) A(0) \\ &= \frac{1}{2} \left(\frac{M(\sigma) \Gamma((n+2)/2)}{\sigma^n \pi^{n/2}} \right) (n-1) \left(\left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{8\pi}{(n+2)\Gamma^2((n+2)/2)} \right) \\ &= \left(\frac{(n-1)}{2(n+2)} \right) \frac{M(\sigma)}{\sigma^n \pi^{n/2}} \left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{8\pi}{\Gamma((n+2)/2)} \\ &\geq \frac{M(\sigma)}{\sigma^n \pi^{n/2}} \left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{\pi}{\Gamma((n+2)/2)} \\ &= \left(\frac{\sigma^2}{16\pi} \right)^{(n+2)/2} \frac{M(\sigma) \pi}{(4\pi)^n \Gamma((n+2)/2)}. \end{aligned}$$

Here we have used the fact that $(n-1)/(n+2) \geq 1/4$ for $n \geq 2$. $\square$

Acknowledgements The author would like to thank Stefan Schrödl for kindly pointing out the mistake.

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