Asynchronous Exponential Growth of a General Structured Population Model

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Abstract We consider a class of structured cell population models described by a first order partial differential equation perturbed by a general birth operator which describes in a unified way a wide class of birth phenomena ranging from cell division to the McKendrick model. Using the theory of positive stochastic semigroups we establish new criteria for an asynchronous exponential growth of solutions to such equations.

1 Introduction

Many physical and biological processes can be described in the following way. We have collection of objects called *individuals* which are characterized by some parameter m which

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ranges from $m = a$ to $m = 1$. The parameter m grows in time according to the equation

$$\frac{dm}{dt} = g(m). \tag{1}$$

Moreover, an individual can divide into, or give a birth to, new individuals. We denote by $b_k(m)$, $k = 1, 2, \ldots$, the rates at which an individual with a parameter m produces k individuals. If such an individual produces k individuals, then by $\mathcal{P}_k(m, A)$ we denote the probability that its descendant has the parameter located in the set A. Indeed, $\mathcal{P}_k$ is a transition probability function, i.e. for each $m \in [a, 1]$ the function $A \mapsto \mathcal{P}_k(m, A)$ defined on the σ-algebra Σ of Borel subsets of $[a, 1]$ is a probabilistic measure and for each $A \in \Sigma$ the function $m \mapsto \mathcal{P}_k(m, A)$ is Σ-measurable. Then the individual with the parameter m has

$$\mathcal{P}(m, A) = \sum_{k=1}^{\infty} k b_k(m) \mathcal{P}_k(m, A) \tag{2}$$

descendants with parameters in the set A in a unit of time and

$$b(m) = \sum_{k=0}^{\infty} k b_k(m) \tag{3}$$

is the mean number of its descendants in a unit of time. It is obvious that $\mathcal{P}(m, A) \leq b(m)$ for each m and each $A \in \Sigma$. Further, by $\mu(m)$ we denote the rate of loss of individuals with parameter m and we include in this rate the loss by death or by division. We mention that the technical assumptions, summarized in Sect. 2, ensure that the rates b and $\mathcal{P}$ are finite apart from possibly at $m = 1$ and thus the operators introduced below are well defined.

This general approach allows us to investigate different processes usually treated separately. For example, if we consider the process of cell division into two daughter cells, then $\mathcal{P}(m, A) = 2b_2(m)\mathcal{P}_2(m, A)$ but the transition function $\mathcal{P}_2(m, A)$ depends on the choice of the parameter m even in the case of equal division. If m is the volume of a cell, then

$$\mathcal{P}_2\left(m, \left\{\frac{m}{2}\right\}\right) = 1, \tag{4}$$

but if m is the age of a cell, then

$$\mathcal{P}_2(m, \{0\}) = 1. \tag{5}$$

In the case of unequal division the transition function can be given by a stochastic kernel, i.e.,

$$\mathcal{P}_2(m, A) = \int_A k(y, m) \, dy, \tag{6}$$

where k is a nonnegative measurable function such that $\int_a^1 k(y, m) \, dy = 1$ for all m.

One of the areas in which such models have been used is that of cell replication and maturation and these applications have been around for over 50 years, [6, 21, 22, 27, 31, 32]. A survey of many applications is given in the book edited by Metz and Diekmann [20]. Our model includes models of a population of organisms reproducing by binary fission with equal [8, 10, 12, 16, 17] and unequal division [2, 11, 13, 15]. Both types of binary fission models were considered in [29]. Moreover, our general scheme includes the classical age structured Sharpe-Lotka-McKendrick model [18, 30].

Our aim is to study the function $N(t, m)$ which describes the distribution of the population with respect to m; in other words, the number of individuals with the parameter m

between m_1 and m_2 at a time t is given by the formula $\int_{m_1}^{m_2} N(t, m)\, dm$. The equation satisfied by the function N depends on the choice of the transition function $\mathcal{P}$. For instance, if $\mathcal{P}$ is of the form (5), then $N(t, m)$ satisfies a first order partial differential equation with an integral boundary condition while in the cases (4) and (6) we obtain also a first order partial differential equation but with an integral perturbation.

In order to unify the description of our process it is better to begin with the dual (backward) equation

$$\frac{\partial u}{\partial t} = g(m)\frac{\partial u}{\partial m} - \mu(m)u(t, m) + \int_a^1 u(t, y)\mathcal{P}(m, dy). \tag{7}$$

Let $b^a(m) = \mathcal{P}(m, \{a\})$, i.e. $b^a(m)$ describes the rate at which the individuals with the parameter m produce individuals with the minimal parameter a, like in the age-structured population model. We denote by $\mathcal{P}^r$ the remaining part of the transition function $\mathcal{P}$, i.e. $\mathcal{P}^r(m, A) = \mathcal{P}(m, A \setminus \{a\})$ and we assume that for each m the measure $\mathcal{P}^r(m, \cdot)$ is absolutely continuous with respect to the Lebesgue measure for a.e. m. Further, let $b^r(m) = \mathcal{P}^r(m, [a, 1])$. Then, by the Radon-Nikodym theorem, there exists an operator P defined on the space

$$L_b^1[a, 1] = \left\{ f \in L^1[a, 1] : \int_a^1 |f(m)|b^r(m)\, dm < \infty \right\}$$

and with values in the space $L^1[a, 1]$ such that for each nonnegative function $f \in L_b^1[a, 1]$ and each set $A \in \Sigma$ we have

$$\int_A Pf(y)\, dy = \int_a^1 \mathcal{P}^r(m, A)f(m)\, dm. \tag{8}$$

If the distribution function $N(t, m)$ is sufficiently regular, then it satisfies the following equation

$$\frac{\partial N}{\partial t} + \frac{\partial(g(m)N)}{\partial m} = -\mu(m)N(t, m) + PN(t, m), \tag{9}$$

with the boundary condition

$$g(a)N(t, a) = \int_a^1 b^a(m)N(t, m)\, dm \tag{10}$$

and with the initial condition

$$N(0, m) = N_0(m) \quad \text{for } m \in [a, 1]. \tag{11}$$

The main aim of this paper is to reduce the problem to a Markov one by using the Perron eigenvectors of (7) and (9) and then to apply the techniques of stochastic semigroups [19, 25, 26, 28, 29] to study asynchronous exponential growth (AEG) property of the system (9)–(11). The idea of using Perron eigenvectors is not new and has been successfully used for instance in [23, 24], where it was combined with the relative entropy technique to prove the exponential decay of solutions. The stochastic semigroups techniques, used in this paper, allow us to consider quite general singular growth and death rates as opposed to [24] but, on the other hand, places some restrictions on the birth process. Thus our results are complementary to those of [23, 24]. We note that instead one also can use known results on compact semigroups but this would require analyzing the whole spectrum of the generator which is rather difficult. In our method no compactness properties of the solution semigroup are required.

The most difficult part of the reduction of the problem to a Markov problem, which is the cornerstone of our method, is to check that the Perron eigenvector of the adjoint equation is bounded and separated from zero. It is then interesting to note that even a nonlinear model of cell population (cf. [17]) can be reduced to a Markov semigroup.

2 Formulation of the Results

We collect all assumptions concerning our model. We assume that $g : [a, 1] \to (0, \infty)$ is a continuously differentiable function (hence, in particular, g is bounded away from zero), $\mu : [a, 1) \to [0, \infty)$ and $b_k : [a, 1) \to [0, \infty)$ are continuous functions. Further we assume that for every $\overline{m} \in (a, 1)$

$$\int_{\overline{m}}^{1} \mathcal{P}(m, [a, \overline{m})) \, dm > 0. \tag{12}$$

This assumption has the following biological interpretation: for every $\overline{m} \in (a, 1)$ there is a mother with maturity greater than $\overline{m}$ which has daughters with maturity less than $\overline{m}$. It means that if (12) was not satisfied, then for some $\overline{m}$ all daughters of any mother with $m \geq \overline{m}$ also would have maturity greater than $\overline{m}$ and thus we could restrict the set of parameters to the interval $[\overline{m}, 1]$.

We also assume that a mother of maturity m cannot have daughters with maturity exceeding $m - h$, that is,

$$\mathcal{P}(m, [a, m - h]) = 1 \quad \text{for all } m \in [a, 1]. \tag{13}$$

Condition (13) can be rewritten in an equivalent form

$$\mathcal{P}(m, (m - h, 1]) = 0 \quad \text{for all } m \in [a, 1]. \tag{14}$$

From (14) and the definition of b it follows in particular that $b(m) = 0$ for $m \in [a, a + h]$. We assume that $\inf\{b(m) : m \in [1 - h, 1)\} > 0$, which means that any cell with maturity $\geq 1 - h$ can divide. Another immediate consequence of (13) is that $(Pf)(m)$ depends only on values of f in $[m + h, 1]$ and, in particular, $Pf = 0$ on $[1 - h, 1]$. Since the maximal value of the parameter m is 1, we have to assume that $\int_{a}^{1} \mu(m) \, dm = \infty$, that is, all cells die or divide before or at reaching maturity 1. Moreover, there is a constant $\bar{C} > 0$ such that

$$\bar{C}^{-1} \mu(m) \leq b(m) \leq \bar{C} \mu(m) \quad \text{for } m \geq 1 - h. \tag{15}$$

Remark 1 The assumption (15) is natural because $\mu(m)$ includes the rate of loss by division. For example if we consider only the division into two cells then $\mu(m) = \mu_0(m) + b_2(m)$ and $b(m) = 2b_2(m)$, where $\mu_0(m)$ is the mortality rate and $b_2(m)$ is the division rate. In this case we can assume that $\mu_0(m)$ is a bounded function and $b_2(m)$ is a strictly positive function for $m \geq 1 - h$, which implies (15).

Let P be defined by (8) and denote by A_{b^a} the differential operator $-(gN)' - \mu N$ restricted to the domain

$$D(A_{b^a}) = \{N \in L^1[a, 1] \cap AC(a, 1) : \mu N, (gN)' \in L^1[a, 1], N \text{ satisfies } (10)\},$$

where $AC(a, 1)$ denotes the space functions which are absolutely continuous on each compact subinterval of $[a, 1]$.

Theorem 1 *The operator* $(A_{b^a} + P, D(A_{b^a}))$ *generates a strongly continuous positive semigroup, denoted* $\{T(t)\}_{t\geq 0}$, *on* $L^1[a, 1]$.

The proof of Theorem 1 is given in Sect. 3. The main result of the paper is the following theorem on asynchronous exponential growth of the population.

Theorem 2 *Assume that one of the following conditions holds*:

(I) *there exists a measurable function* $q : [a, 1] \times [a, 1] \to [0, \infty)$ *such that*

$$\int_a^1 \int_a^1 q(y, m)\, dm\, dy > 0 \quad and \quad \mathcal{P}(m, A) \geq \int_A q(y, m)\, dy \qquad (16)$$

for any $m \in [a, 1]$ *and* $A \in \Sigma$,

(II) *there exist* $m_0 \in (a, 1)$, $\varepsilon > 0$ *such that* $b^a(m) > 0$ *for* $m \in (m_0 - \varepsilon, m_0 + \varepsilon)$,

(III) *there exist* $m_0 \in (a, 1)$, $\varepsilon > 0$ *and a* C^1-*function* $r : (m_0 - \varepsilon, m_0 + \varepsilon) \to [a, 1]$ *such that* $g(r(m_0)) \neq r'(m_0)g(m_0)$ *and*

$$\mathcal{P}(m, \{r(m)\}) \geq \varepsilon \quad for \ m \in (m_0 - \varepsilon, m_0 + \varepsilon). \qquad (17)$$

Then there exist a continuous function $f_* : [a, 1] \to [0, \infty)$, $f_*(m) > 0$ *for* $m \in (a, 1)$ *and a continuous function* $w : [a, 1] \to (0, \infty)$ *such that* $e^{-\lambda t} N(t, \cdot)$ *converges in* $L^1[a, 1]$ *to* $f_* \int_a^1 N(0, m)w(m)\, dm$.

Condition (I) amounts to $m \mapsto \mathcal{P}^r(m, \cdot)$ being not zero almost everywhere. The proof of Theorem 2 is given in Sects. 5 and 6. It should be noted that in some special cases none of conditions I, II, III holds. For example, if m is the weight of a cell, $g(m) = m$ and we have equal division, that is, $\mathcal{P}(m, A) = 2$ if $m/2 \in A$ and $\mathcal{P}(m, A) = 0$ if $m/2 \notin A$. Then the weight of daughter cells at any given time does not depend on the moment of division. In this case the distribution of masses of daughters strongly depends on the initial distribution. Of course in this case the semigroup $\{T(t)\}_{t \geq 0}$ has no asynchronous exponential growth. An interesting example of a linear population model which has no asynchronous exponential growth property is a model with telomere loss [3].

3 Proof of Theorem 1

Consider the 'formal' equation for the resolvent of A_{b^a},

$$\lambda N(m) + (g(m)N(m))' + \mu(m)N(m) = f(m),$$

whose solution is given by

$$N_\lambda(m) = \frac{e^{-\lambda G(m) - Q(m)}}{g(m)} \int_a^m e^{\lambda G(s) + Q(s)} f(s)\, ds + C\frac{e^{-\lambda G(m) - Q(m)}}{g(m)}, \qquad (18)$$

where $G(m) = \int_a^m (1/g(s))\, ds$ and $Q(m) = \int_a^m (\mu(s)/g(s))\, ds$. To shorten notation we denote

$$e_{-\lambda}(m) := e^{-\lambda G(m) - Q(m)}, \qquad e_\lambda(m) := e^{\lambda G(m) + Q(m)}$$

and note two useful formulae. For $s \in [a, 1)$

$$\int_s^1 \frac{e^{-\lambda G(m)}}{g(m)}\, dm = -\frac{1}{\lambda} \int_s^1 \frac{d}{dm} e^{-\lambda G(m)}\, dm \leq \frac{e^{-\lambda G(s)}}{\lambda} \qquad (19)$$

and, adding and subtracting λ in the numerator below and noting that $e_{-\lambda}(1) = 0$, similarly we obtain

$$\int_s^1 \frac{\mu(m)e_{-\lambda}(m)}{g(m)}\,dm = e_{-\lambda}(s) - \lambda \int_s^1 \frac{e_{-\lambda}(m)}{g(m)}\,dm. \tag{20}$$

Returning to (18) and arguing as in [5] with help of (19) and (20) we see that the first term in (18)

$$N_{\lambda,0}(m) = \frac{e_{-\lambda}(m)}{g(m)} \int_a^m e_\lambda(s) f(s)\,ds$$

is the resolvent $R(\lambda, A_0)$ of the operator A_0 corresponding to the homogeneous Dirichlet boundary condition at $m = a$ and it satisfies

$$\int_a^1 N_{\lambda,0}(m)\,dm \leq \frac{1}{\lambda}\|f\|, \quad \lambda > 0, \tag{21}$$

$$\int_a^1 \mu(m)N_{\lambda,0}(m)\,dm \leq \|f\|. \tag{22}$$

We introduce the notation $\langle f, g \rangle := \int_a^1 f(m)g(m)\,dm$. To deal with the boundary term, we let $m \to a$ in (18) and, using the boundary condition, we obtain

$$N_\lambda(m) = N_{\lambda,0}(m) + \frac{e_{-\lambda}(m)\langle b^a, N_\lambda \rangle}{g(m)}$$

which can be solved for $\langle b^a, N_\lambda \rangle$, yielding

$$\langle b^a, N_\lambda \rangle = \frac{\langle b^a, N_{\lambda,0} \rangle}{1 - \langle b^a, g^{-1}e_{-\lambda} \rangle},$$

provided $\Psi(\lambda) := \langle b^a, g^{-1}e_{-\lambda} \rangle < 1$. Let us consider this condition. Using (15), we see that $\Psi(\lambda) \leq \bar{C} \int_a^1 \mu(s)\frac{e_{-\lambda}(s)}{g(s)}\,ds$. Since $\mu e_{-\lambda}/g \leq \mu e^{-Q}/g$, with the latter function integrable by (20), the Lebesgue dominated convergence theorem yields $\Psi(\lambda) \to 0$ as $\lambda \to \infty$. Thus, for some $\bar{\lambda}$ and $b_\infty < 1$, $\langle b^a, g^{-1}e_{-\lambda} \rangle \leq b_\infty$ for all $\lambda > \bar{\lambda}$ and for such λ, using (19) with $s = a$ and (22), we have the estimate

$$\|N_\lambda\| \leq \|N_{\lambda,0}\| + \frac{1}{\lambda}\left|\frac{\langle b^a, N_{\lambda,0} \rangle}{1 - \langle b^a, g^{-1}e_{-\lambda} \rangle}\right| \leq \|N_{\lambda,0}\| + \frac{\bar{C}}{\lambda}\frac{\|f\|}{1 - b_\infty} \leq \frac{1 + \bar{C} - b_\infty}{(1 - b_\infty)\lambda}\|f\|.$$

By the previous calculations $\mu e_{-\lambda}/g$ is integrable and thus $R(\lambda, A_{b^a})$ for $\lambda > \bar{\lambda}$ is defined by

$$R(\lambda, A_{b^a})f = R(\lambda, T_0)f + \frac{e_{-\lambda}}{g}\frac{\langle b^a, R(\lambda, T_0)f \rangle}{1 - \langle b^a, g^{-1}e_{-\lambda} \rangle}. \tag{23}$$

Next we prove that for any $\lambda > 0$

$$\lim_{m \to 1^-} e_\lambda(m) \int_m^1 \frac{e_{-\lambda}(m')}{g(m')}\,dm' = 0. \tag{24}$$

Indeed, consider the family of functionals $\{\xi_m\}_{m \in (1-\epsilon, 1]}$ for some $\epsilon > 0$ defined by

$$\xi_m f = e_\lambda(m) \int_m^1 \frac{e_{-\lambda}(m')}{g(m')} f(m')\,dm'$$

for $f \in L^1[a, 1]$. We have

$$|\xi_m f| \leq e_\lambda(m) \int_m^1 \frac{e_{-\lambda}(m')}{g(m')}|f(m')|\,dm' \leq c_g \int_a^1 |f(m')|\,dm'$$

on account of monotonicity of e_λ with $c_g = 1/\inf_{m\in[a,1]} g(m) < +\infty$ by assumption. More-over, for f with support in $[a, 1 - \delta]$ with any $\delta > 0$ we have $\lim_{m\to 1^-} \xi_m f = 0$ and, by a corollary to the Banach-Steinhaus theorem, the above limit extends by density for any $f \in L^1[a, 1]$ and, in particular, for $f \equiv 1$ which yields (24).

Now, we observe that $\bar\lambda$, defined before (23), is universal for all b^a satisfying $b^a \leq \bar C\mu$. Let us consider $b^a_\mu = \bar C\mu$. Then, for $f \geq 0$ and fixed $\lambda > \bar\lambda$, we have

$$\|R(\lambda, A_{b^a_\mu})f\| = \|R(\lambda, A_0)f\| + \frac{\langle b^a_\mu, R(\lambda, A_0)f\rangle}{1 - \langle b^a_\mu, g^{-1}e_{-\lambda}\rangle} \int_a^1 \frac{e_{-\lambda}(m)}{g(m)} \, dm = \int_a^1 f(s)\Phi(s) \, ds$$

where, by (20),

$$\Phi(s) = e_\lambda(s) \int_s^1 \frac{(1 + C'\mu(m))e_{-\lambda}(m)}{g(m)} \, dm = C' + (1 - C'\lambda)e_\lambda(s) \int_s^1 \frac{e_{-\lambda}(m)}{g(m)} \, dm \quad (25)$$

with

$$C' = \frac{\bar C}{1 - \langle b^a, g^{-1}e_\lambda\rangle} \int_a^1 \frac{e_{-\lambda}(m)}{g(m)} \, dm.$$

Now, Φ is a continuous function on $[a, 1]$ by (24) and it is strictly positive on $[a, 1)$ as both the integrand and e_λ are positive on this interval. Moreover, if $s \to 1^-$ then, again by (24), the rightmost term of (25) converges to zero and $\Phi(s) \to C' > 0$. Thus $\Phi(s) \geq C_0 > 0$ on $[a, 1]$, where C_0 is a constant and therefore $\|R(\lambda, A_{b^a_\mu})f\| \geq C_0\|f\|$ for any $\lambda > \bar\lambda$. It is a classical result, see e.g. [14], that $A_{b^a_\mu}$ is densely defined. Thus, by the Arendt-Batty-Robinson theo-rem, [1, Theorem 2.5] or [4, Theorem 3.39], the operator $(A_{b^a_\mu}, D(A_{b^a_\mu}))$ generates a positive semigroup. Since we are in an L^p space, the type of the semigroup equals the spectral bound of $A_{b^a_\mu}$.

Now let us consider an arbitrary b^a satisfying $b^a \leq \bar C\mu$. Then, by (23), for all $\lambda \geq \bar\lambda$, $0 \leq R(\lambda, A_{b^a}) \leq R(\lambda, A_{b^a_\mu})$ and hence $0 \leq R^n(\lambda, A_{b^a}) \leq R^n(\lambda, A_{b^a_\mu})$. Since $A_{b^a_\mu}$ generates a semigroup, its resolvent satisfies the Hille-Yosida estimates and hence also $\|R^n(\lambda, A_{b^a})\| \leq M(\lambda - \bar\lambda)^{-n}$, for some $M > 0$ and all $n \in \mathbb{N}$. Thus, $(A_{b^a}, D(A_{b^a}))$ generates a positive semigroup.

In the last step we analyze the operator $PR(\lambda, A_{b^a})$ and, more precisely, its spectral radius $r(PR(\lambda, A_{b^a}))$. The key observation is that, by (14), the support of Pf is contained in $[a, 1 - h]$ for any $f \in L^1[a, 1]$. Thus, by writing

$$(PR(\lambda, A_{b^a}))^n f = (PR(\lambda, A_{b^a}))^{n-1} PR(\lambda, A_{b^a})f,$$

we can consider the powers $(PR(\lambda, A_{b^a}))^n f$ with $n \geq 2$ as acting on $L^1[a, 1 - h]$. Also, since we deal with positive operators, we can restrict ourselves to nonnegative functions. Hence, let $0 \leq v \in L^1[a, 1 - h]$. Then we can write

$$(R(\lambda, A_0)v)(m) = \begin{cases} (R(\lambda, A'_0)v)(m) & \text{for } m \in [a, 1 - h), \\ \frac{e_{-\lambda}(m)}{g(m)} L_\lambda(v) & \text{for } m \in [1 - h, 1], \end{cases}$$

where A'_0 is the operator A_0 restricted to $[a, 1 - h)$ and

$$L_\lambda(v) = \int_a^{1-h} e_\lambda(s)v(s) \, ds \leq e_\lambda(1 - h)\|v\|.$$

Hence

$$\int_a^1 (PR(\lambda, A_0)v)(m)\,dm$$

$$= \int_a^1 b^r(m)(R(\lambda, A_0)v)(m)\,dm$$

$$= \int_a^{1-h} b^r(m)(R(\lambda, A_0')v)(m)\,dm + L_\lambda(v) \int_{1-h}^1 \frac{b^r(m)e_{-\lambda}(m)}{g(m)}\,dm$$

$$\leq \frac{\sup_{m \in [a, 1-h]} b^r(m)\|v\|}{\lambda} + \|v\| \int_{1-h}^1 \frac{b^r(m)e^{-\lambda(G(m)-G(1-h))-Q(m)+Q(1-h)}}{g(m)}\,dm,$$

where we used the standard result that the transport problem with homogeneous Dirichlet boundary conditions and bounded coefficients generates a contractive semigroup. As for the second term, we note that $G(m) > G(1-h)$ for $m > 1-h$ as R is strictly increasing. Also, $b^r e_{-\lambda}/g \leq b^r e^{-Q}/g$ and the latter is integrable by (15) and (20). Thus, from the Lebesgue dominated convergence theorem, the integral converges to 0 as $\lambda \to \infty$ and we can write $\|PR(\lambda, A_0)v\| \leq c_1(\lambda)\|v\|$ for any $0 \leq v \in L^1[a, 1]$ with support in $[a, 1-h]$, where $c_1(\lambda) \to 0$ as $\lambda \to \infty$.

Next, for such functions v, again using (15) and argument as above, we find that

$$\int_a^1 P\left(\frac{e_{-\lambda}}{g} \frac{\langle b^a, R(\lambda, A_0)v\rangle}{1 - \langle b^a, g^{-1}e_{-\lambda}\rangle}\right)(m)\,dm = \frac{\langle b^a, R(\lambda, A_0)v\rangle}{1 - \langle b^a, g^{-1}e_{-\lambda}\rangle} \int_a^1 \frac{b^r(m)e_{-\lambda}(m)}{g(m)}\,dm$$

$$\leq \frac{c_2(\lambda)}{1 - c_3(\lambda)} c_3(\lambda)\|v\| = c_4(\lambda)\|v\|,$$

where $c_3(\lambda)$, and hence $c_4(\lambda)$, tend to 0 as $\lambda \to \infty$. Now, for arbitrary $f \in L^1[a, 1]$, for $r(PR(\lambda, A_{b^a}))$ we take $\lambda > \bar{\lambda}$ and such that $c_1(\lambda) + c_4(\lambda) < 1/2$. Then

$$\|(PR(\lambda, A_{b^a}))^n f\| \leq \|(PR(\lambda, A_{b^a}))^{n-1}\| \|PR(\lambda, A_{b^a})\| \|f\| \leq \frac{1}{2^{n-1}} \|PR(\lambda, A_{b^a})\| \|f\|$$

and

$$r(PR(\lambda, A_{b^a})) = \lim_{n \to \infty} \sqrt[n]{\|(PR(\lambda, A_{b^a}))^n\|} \leq \frac{1}{2}.$$

Hence, for sufficiently large λ the resolvent $R(\lambda, A_{b^a} + P)$ is given by a norm converging series of positive operators and thus it is a positive operator. Therefore we can apply Desch's theorem, see e.g. [7] or [4, Theorem 5.13], to claim that $(A_{b^a}, D(A_{b^a}))$ generates a positive semigroup. $\square$

Remark 2 Arguing as in (24) for $R(\lambda, A_0)$ and using $e_{-\lambda}(m)/g(m) \to 0$ as $m \to 1^-$ we see from (23) that if $u \in D(A_{b^a})$ then

$$\lim_{m \to 1^-} u(m) = 0. \tag{26}$$

4 Reduction of $\{T(t)\}_{t \geq 0}$ to a Stochastic Semigroup

Equation (7) can be rewritten as an evolution equation

$$u'(t) = Bu(t), \tag{27}$$

where

$$Bf(m) = g(m)f'(m) - \mu(m)f(m) + \int_a^1 f(y)\mathcal{P}(m, dy). \qquad (28)$$

We recall that a positive semigroup is called stochastic if it is conservative on nonnegative initial data. The semigroup $\{T(t)\}_{t\geq 0}$ of Theorem 1 in general is not stochastic. However, using a suitable eigenvectors of the operator B, given by (28), we are able to construct a semigroup $\{U(t)\}_{t\geq 0}$ which is similar to $\{T(t)\}_{t\geq 0}$ and which is stochastic. This will facilitate analysis of stability of $\{T(t)\}_{t\geq 0}$.

We begin with the following proposition.

Proposition 1 *The operator B has an eigenvector w which satisfies the following inequalities*

$$\alpha \leq w(m) \leq \beta \quad \text{for all } m \in [a, 1),$$

where $0 < \alpha < \beta$.

We divide the proof of Proposition 1 into a sequence of lemmata.

First observe that $Bw_\lambda = \lambda w_\lambda$ if and only if w_λ is a bounded function such that w_λ is absolutely continuous in each interval $[a, 1 - \varepsilon]$, $\varepsilon > 0$, and w_λ satisfies the following equation

$$w'_\lambda(m) = d_\lambda(m)w_\lambda(m) - \frac{1}{g(m)}Fw_\lambda(m), \qquad (29)$$

where

$$d_\lambda(m) = \frac{\mu(m) + \lambda}{g(m)}, \quad \text{and} \quad Ff(m) = \int_a^1 f(y)\mathcal{P}(m, dy). \qquad (30)$$

We note that due to (8) and the domain of definition of P, the function Ff is locally integrable on $[a, 1)$ for any $f \in L^\infty[a, 1]$. The following lemma gathers some properties of the operator F.

Lemma 1 *Let $z \in [a + h, 1]$. If $w_1, w_2 \in L^\infty[a, 1]$ and $w_1(m) = w_2(m)$ for $m \in [a, z - h]$, then $Fw_1(m) = Fw_2(m)$ for $m \in [a, z]$.*

Proof Let $w \in L^\infty[a, 1]$. As $\mathcal{P}(m, (z - h, 1]) = 0$ for $m \in [a, z]$ we have

$$Fw(m) = \int_a^{z-h} w(y)\,\mathcal{P}(m, dy), \quad \text{for } m \leq z. \qquad (31)$$

Applying formula (31) to the function $w = w_1 - w_2$ we get $Fw_1(m) = Fw_2(m)$ for $m \in [a, z]$. $\qquad \square$

By Lemma 1, the restriction of the function Fw_λ to the interval $[a, z]$, $z \in [a + h, 1]$, depends only on the restriction of w_λ to the interval $[a, z - h]$. In particular $Fw_\lambda(m) = 0$ for $m \in [a, a + h]$. This implies that equation (29) takes the form

$$w'_\lambda(m) = d_\lambda(m)w_\lambda(m) \quad \text{for } m \in [a, a + h]. \qquad (32)$$

If we assume that $w_\lambda(a) = 1$, then equation (32) has a unique continuous solution. Since the function Fw_λ, restricted to $[a + h, a + 2h]$, depends only on the function w_λ restricted

to $[a, a + h]$ and it is integrable on $[a, a + 2h]$, equation (29) has a unique solution in the interval $[a, a + 2h]$ such that $w_\lambda(a) = 1$. This solution is an absolutely continuous function and depends continuously on λ. Using this method we can solve (29) successively for $m \in [a + 2h, a + 3h], \ldots$. We summarize the above observation.

Lemma 2 *For every $\lambda \in \mathbb{R}$ equation (29) has the unique solution $w_\lambda : [a, 1) \to \mathbb{R}$ such that $w_\lambda(a) = 1$. The solution w_λ is an absolutely continuous function in each interval $[a, 1 - \varepsilon]$, $\varepsilon > 0$. This solution depends continuously on λ, i.e. for every $\varepsilon > 0$ the map $\varphi : \mathbb{R} \to C[a, 1 - \varepsilon]$ given by $\varphi(\lambda) = w_\lambda$ is continuous.*

Lemma 3 *There exists $\lambda_1 > 0$ such that for every $\lambda \geq \lambda_1$ the function w_λ is increasing.*

Proof Let λ_1 be a constant such that

$$\lambda_1 > b(m) + \frac{\bar{C}}{h} g(m) \quad \text{for } m \leq 1 - \frac{h}{2}. \tag{33}$$

Since $b(m) = 0$ for $m \leq a + h$, the function w_λ satisfies the equation $w_\lambda' = d_\lambda w_\lambda$ in the interval $[a, a + h]$. Further, $d_\lambda > 0$ implies that w_λ is increasing in $[a, a + h]$. Now, we assume that w_λ is increasing in $[a, m_0]$, where $m_0 \leq 1 - \frac{3}{2}h$. We claim that w_λ is increasing in the interval $[a, m_0 + h]$. Indeed, from (31) we have

$$F w_\lambda(m) \leq b(m) \max\{w_\lambda(x) : x \in [a, m - h]\} \leq b(m) w_\lambda(m_0)$$

for $m \in [m_0, m_0 + h]$. Thus

$$w_\lambda'(m) = d_\lambda(m) w_\lambda(m) - \frac{1}{g(m)} F w_\lambda(m) \geq \frac{1}{g(m)} (\lambda w_\lambda(m) - b(m) w_\lambda(m_0)) \tag{34}$$

for $m \in [m_0, m_0 + h]$. Since $\lambda > b(m)$ for $m \leq 1 - \frac{h}{2}$ and since the function w_λ is locally absolutely continuous inequality, (34) implies that w_λ is increasing in the interval $[m_0, m_0 + h]$. Hence, the function w_λ is increasing in the interval $[a, 1 - \frac{h}{2}]$. Moreover, (33) and the inequality $F w_\lambda(m) \leq b(m) w_\lambda(m)$ for $m \leq 1 - \frac{h}{2}$ imply that $w_\lambda'(m) \geq \frac{\bar{C}}{h} w_\lambda(m)$ and, consequently,

$$w_\lambda\left(1 - \frac{h}{2}\right) \geq e^{\frac{\bar{C}}{h} \frac{h}{2}} w_\lambda(1 - h) > \bar{C} w_\lambda(1 - h). \tag{35}$$

From (35) we obtain

$$F w_\lambda(m) \leq b(m) w_\lambda(1 - h) < \bar{C}^{-1} b(m) w_\lambda\left(1 - \frac{h}{2}\right) \leq \mu(m) w_\lambda\left(1 - \frac{h}{2}\right) \tag{36}$$

for $m \in [1 - h, 1]$. Since $d_\lambda(m) > \frac{\mu(m)}{g(m)}$, inequality (36) implies that

$$w_\lambda'(m) = d_\lambda(m) w_\lambda(m) - \frac{1}{g(m)} F w_\lambda(m) > d_\lambda(m) \left[w_\lambda(m) - w_\lambda\left(1 - \frac{h}{2}\right) \right].$$

Writing this inequality as $[w_\lambda(m) - w_\lambda(1 - h/2)]' > d_\lambda(m)[w_\lambda(m) - w_\lambda(1 - h/2)]$, we find $w_\lambda(m) > w_\lambda(1 - h/2)$ for $m \geq h/2$ and consequently the function w_λ is increasing in $[1 - \frac{h}{2}, 1)$. □

Lemma 4 *There exists λ_2 such that for each $\lambda \leq \lambda_2$ the function w_λ has a zero in the interval $(a, 1)$.*

Proof Let $\widetilde{m} \in (a + h, 1)$ be a point such that $b(\widetilde{m}) > 0$. Since b is a continuous function, there exist positive constants ε, δ such that $\delta < h, \widetilde{m} + \delta < 1$ and

$$b(m) \geq \varepsilon \quad \text{for } m \in [\widetilde{m} - \delta, \widetilde{m} + \delta]. \tag{37}$$

Let us denote $\alpha_\lambda = \max\{d_\lambda(m) : m \leq \widetilde{m} + \delta\}$ and choose λ_2 such that for every $\lambda \leq \lambda_2$ we have $\alpha_\lambda < 0$ and

$$\varepsilon \delta e^{-\alpha_\lambda(h-\delta)} > g(m) \quad \text{for } m \leq \widetilde{m} + \delta. \tag{38}$$

We prove that the function w_λ has a zero in the interval $[a, \widetilde{m} + \delta]$. Conversely, suppose that $w_\lambda(m) > 0$ in the interval $[a, \widetilde{m} + \delta]$. Since $w'_\lambda(m) \leq \alpha_\lambda w_\lambda(m)$ for $m \leq \widetilde{m} + \delta$, the function w_λ is decreasing in the interval $[a, \widetilde{m} + \delta]$ and

$$w_\lambda(m) \geq e^{-\alpha_\lambda(\widetilde{m}-m)} w_\lambda(\widetilde{m}) \quad \text{for } m \leq \widetilde{m}. \tag{39}$$

From inequality (37) it follows that

$$Fw_\lambda(m) \geq \varepsilon \inf\{w_\lambda(x) : x \leq m - h\} = \varepsilon w_\lambda(m - h) \tag{40}$$

for $m \in [\widetilde{m} - \delta, \widetilde{m} + \delta]$. From (39) and (40) it follows that

$$Fw_\lambda(m) \geq \varepsilon e^{-\alpha_\lambda(\widetilde{m}-m+h)} w_\lambda(\widetilde{m}) \geq \varepsilon e^{-\alpha_\lambda(h-\delta)} w_\lambda(\widetilde{m}). \tag{41}$$

Inequalities (38), (41) and $w'_\lambda(m) \leq -\frac{1}{g(m)} Fw_\lambda(m)$ imply that

$$w'_\lambda(m) \leq -\varepsilon e^{-\alpha_\lambda(h-\delta)} \frac{1}{g(m)} w_\lambda(\widetilde{m}) < -\frac{1}{\delta} w_\lambda(\widetilde{m}). \tag{42}$$

Thus

$$w_\lambda(\widetilde{m} + \delta) < w_\lambda(\widetilde{m}) - \delta\left(\frac{1}{\delta} w_\lambda(\widetilde{m})\right) = 0 \tag{43}$$

and therefore the function w_λ has a zero in the interval $[a, \widetilde{m} + \delta]$. $\qquad\square$

Now we denote by Λ the subset of $\mathbb{R}$ which contains all constants $\lambda \in \mathbb{R}$ such that $w_\lambda(m_\lambda) = 0$ for some $m_\lambda \in (a, 1)$. According to Lemma 4 the set Λ is nonempty and, by Lemma 3, it is bounded from above. Let $\lambda_s = \sup \Lambda$.

Lemma 5 *There exists $\alpha > 0$ such that $w_{\lambda_s}(m) \geq \alpha$ for $m \in [a, 1)$.*

Proof The solution w_λ depends continuously on λ. From this and from the definition of λ_s, it follows that $w_{\lambda_s}(m) \geq 0$ for $m \in [a, 1)$. We claim that $w_{\lambda_s}(m) > 0$ for $m \in [a, 1)$. Suppose, contrary to our claim, that there exists $m_s \in (a, 1)$ such that $w_{\lambda_s}(m_s) = 0$ and $w_{\lambda_s}(m) > 0$ for $m \in [a, m_s)$. Since $w_{\lambda_s} \geq 0$, we have $Fw_{\lambda_s} \geq 0$. Thus the function w_{λ_s} satisfies the inequality $w'_{\lambda_s} \leq d_{\lambda_s} w_{\lambda_s}$ and the condition $w_{\lambda_s}(m_s) = 0$. By standard arguments from the theory of differential inequalities we obtain $w_{\lambda_s}(m) \leq 0$ for $m \geq m_s$. On the other hand, the function w_{λ_s} is nonnegative. Thus $w_{\lambda_s}(m) = 0$ for $m \geq m_s$ and from (29) we have $Fw_{\lambda_s}(m) = 0$ for $m \geq m_s$. But since $w_{\lambda_s}(m) > 0$ for $m \in [a, m_s)$, inequality (12) implies that $\int_{m_s}^1 Fw_{\lambda_s}(m) \, dm > 0$, which contradicts the fact that $Fw_{\lambda_s}(m) = 0$ for $m \in [m_s, 1)$. Thus $w_{\lambda_s}(m) > 0$ for $m \in [a, 1)$. Let $\varepsilon = \min_{m \leq 1-h} w_{\lambda_s}(m)$. Since $\mathcal{P}_k(m, [a, 1-h]) = 1$ and $w_{\lambda_s} \geq \varepsilon \mathbf{1}_{[a,1-h]}$ we have, using also (31),

$$Fw_{\lambda_s}(m) \geq \varepsilon F\mathbf{1}_{[a,1-h]}(m) = \varepsilon \sum_{k=1}^{\infty} kb_k(m)\mathcal{P}_k(m, [a, 1-h]) = \varepsilon b(m), \quad m \in [a, 1). \tag{44}$$

This implies that for all $m \in (a, 1)$

$$w'_{\lambda_s}(m) \leq d_{\lambda_s}(m)w_{\lambda_s}(m) - \varepsilon j(m), \tag{45}$$

where $j(m) = b(m)/g(m)$. From (15) it follows that there exists a constant $\gamma > 1$ such that $d_{\lambda_s}(m) \leq \gamma j(m)$ for $m \geq 1 - h$. We check that $w_{\lambda_s} \geq \alpha$, where $\alpha = \varepsilon/2\gamma$. Conversely, suppose that for some $m_0 \in (1 - h, 1)$ we have $w_{\lambda_s}(m_0) < \alpha$. Then $w_{\lambda_s}(m) < \alpha$ for $m \in [m_0, 1)$. Indeed, if there exists $m_1 > m_0$ such that $w_{\lambda_s}(m_1) = \alpha$ and $w_{\lambda_s}(m) \leq \alpha$ for $m \in [m_0, m_1]$, then from

$$w'_{\lambda_s}(m) \leq d_{\lambda_s}(m)w_{\lambda_s}(m) - \varepsilon j(m) < -\frac{\varepsilon}{2}j(m) < 0, \quad m \in [m_0, m_1], \tag{46}$$

it follows that $w_{\lambda_s}(m) < \alpha$ for $m \in [m_0, m_1]$, which contradicts the definition of m_1. Thus $w_{\lambda_s}(m) < \alpha$ for $m \in [m_0, 1)$. From (46) it follows that

$$w_{\lambda_s}(m) - w_{\lambda_s}(m_0) = \int_{m_0}^m w'_{\lambda_s}(x)\,dx \leq -\frac{\varepsilon}{2}\int_{m_0}^m j(x)\,dx. \tag{47}$$

However, since $\int_{m_0}^1 j(x)\,dx = \infty$, (47) yields $\lim_{m \to 1} w_{\lambda_s}(m) = -\infty$, which contradicts the fact that $w_{\lambda_s} \geq 0$ and completes the proof that $w_{\lambda_s} \geq \alpha$. □

Lemma 6 *There exists $\beta > 0$ such that $w_{\lambda_s}(m) \leq \beta$ for $m \in [a, 1)$.*

Proof Now we show that there exists $\beta > 0$ such that $w_{\lambda_s} \leq \beta$. Let $M > 0$ be such that $w_{\lambda_s}(m) < M$ for $m \in [a, 1 - h]$. Let

$$\gamma = \max_{m \leq 1} \frac{|\lambda_s| + 1}{g(m)} \quad \text{and} \quad \beta = \bar{C}e^\gamma M. \tag{48}$$

We claim that $w_{\lambda_s} \leq \beta$. Suppose to the contrary that there exists $\bar{m} \in (1 - h, 1)$ such that $w_{\lambda_s}(\bar{m}) > \beta$. From the continuous dependence of the functions w_λ on λ it follows that there exists $\lambda \in \Lambda$ such that $|\lambda - \lambda_s| < 1$, $w_\lambda(\bar{m}) > \beta$, $w_\lambda(m) > 0$ for $m \in [a, \bar{m}]$, and $w_\lambda(m) < M$ for $m \in [a, 1 - h]$. According to (31) we have

$$Fw_\lambda(m) \leq Mb(m) \quad \text{for } m \in [a, 1]. \tag{49}$$

Since

$$d_\lambda(m) = \frac{\lambda + \mu(m)}{g(m)} \geq -\gamma + \frac{\mu(m)}{g(m)} \geq -\gamma + \bar{C}^{-1}j(m), \tag{50}$$

the function w_λ satisfies the inequality

$$w'_\lambda(m) \geq (-\gamma + \bar{C}^{-1}j(m))w_\lambda(m) - Mj(m) \geq -\gamma w_\lambda(m) \tag{51}$$

provided that $w_\lambda(m) \geq \bar{C}M$. However, $w_\lambda(\bar{m}) > \beta = \bar{C}e^\gamma M > \bar{C}M$ hence $w_\lambda \geq \bar{C}M$ in some neighborhood of $\bar{m}$. Then, in this neighborhood, (51) yields

$$w_\lambda(m) \geq e^{-\gamma(m-\bar{m})}w_\lambda(\bar{m}) > e^{-\gamma}\beta = \bar{C}M \tag{52}$$

and, due to the strict inequality, we can extend (51) to a larger domain. Consequently, $w_\lambda(m) > \bar{C}M$ for $m \in [\bar{m}, 1)$ and hence $w_\lambda(m) > 0$ for all $m \in [a, 1)$, which is impossible because $\lambda \in \Lambda$ and the function w_λ has a zero in the interval $[a, 1)$. This implies that $w_{\lambda_s} \leq \beta$. □

Now, denote by M_w the operator of multiplication by the solution w of (29) for an appropriate $\lambda \in \mathbb{R}$. Since w is an absolutely continuous function $w: [a, 1) \to (0, \infty)$ satisfying

$\alpha \leq w(m) \leq \beta$ for all $m \in [a, 1)$, where $0 < \alpha < \beta$, M_w is an isomorphism of $L^1[a, 1]$. Defining

$$U(t) = e^{-\lambda t} M_w T(t) M_{w^{-1}}, \tag{53}$$

we see that $\{U(t)\}_{t\geq 0}$ is a strongly continuous positive semigroup on $L^1[a, 1]$ with the generator

$$\mathcal{A}u = M_w(A_{b^a} + P)M_{w^{-1}}u - \lambda u = \frac{w'gu}{w} - (gu)' - \mu u + wP\left(\frac{u}{w}\right) - \lambda u$$

$$= -(gu)' - \frac{u}{w}\int_a^1 w(y)\mathcal{P}(\cdot, dy) + wP\left(\frac{u}{w}\right) \tag{54}$$

on the domain $D(\mathcal{A}) = M_{w^{-1}} D(A_{b^a})$. In particular, for $u \in D(\mathcal{A})$

$$g(a)u(a) = \int_a^1 \frac{b^a(m)w(a)}{w(m)} u(m)\, dm. \tag{55}$$

Since for $u \in D(\mathcal{A})$ the last two terms in the definition of $\mathcal{A}$ are in $L^1[a, 1]$ (as $u/w \in L_b^1[a, 1]$), also $(gu)' \in L^1[a, 1]$ and we can integrate termwise to get

$$\int_a^1 (\mathcal{A}u)(m)\, dm$$

$$= g(a)u(a) - \lim_{m\to 1^-} g(m)u(m)$$

$$\quad - \int_a^1 \frac{u(m)}{w(m)}\left(w(a)b^a(m) + \int_a^1 w(y)\mathcal{P}^r(m, dy)\right) dm$$

$$\quad + \int_a^1 w(m)\left(P\left(\frac{u}{w}\right)\right)(m)\, dm$$

$$= -\int_a^1 \frac{u(m)}{w(m)} \int_a^1 w(y)\mathcal{P}^r(m, dy)\, dm + \int_a^1 w(m)\left(P\left(\frac{u}{w}\right)\right)(m)\, dm = 0,$$

where we used in the following order, (26), definition of b^a and $\mathcal{P}^r$, (55) and the definition of the operator P. Thus, we have proved

Proposition 2 $\{U(t)\}_{t\geq 0}$ *is a stochastic semigroup on* $L^1[a, 1]$.

5 Invariant Density

We need some auxiliary definitions. Denote by D the set of all densities i.e., the set defined by

$$D = \left\{ f \in L^1[a, 1] : f \geq 0,\ \int_a^1 f(m)\, dm = 1 \right\}.$$

A density f_* is called *invariant* under a stochastic semigroup $\{U(t)\}_{t\geq 0}$ if $U(t)f_* = f_*$ for every $t \geq 0$.

Now we will investigate the existence of an invariant density of the semigroup $\{U(t)\}_{t\geq 0}$. Before, however, we introduce some simplifying notation. Let $\tilde{b}(m) = b^a(m)w(a)/w(m)$ and $Rf = wP(\frac{f}{w})$. Then, using (8), we can write $\frac{1}{w}P^*w = R^*\mathbf{1}$

Since $Rf = wP(\frac{f}{w})$ and $0 < \alpha \le w(m) \le \beta$ for $m \in [a, 1]$, we have

$$\frac{\alpha}{\beta} Pf \le Rf \le \frac{\beta}{\alpha} Pf \tag{56}$$

for $f \ge 0$, $f \in L_b^1[a, 1]$. We also have

$$\frac{\alpha}{\beta} b^r(m) \le R^* \mathbf{1}(m) \le \frac{\beta}{\alpha} b^r(m) \quad \text{and} \quad \frac{\alpha}{\beta} b^a(m) \le \tilde{b}(m) \le \frac{\beta}{\alpha} b^a(m) \tag{57}$$

for $m \in [a, 1]$.

Proposition 3 *The semigroup $\{U(t)\}_{t \ge 0}$ has a unique invariant density $v_* \in D(\mathcal{A})$. Additionally, $v_* > 0$ on $(a, 1)$ and $v_*(1) = 0$.*

Proof An invariant density v_* of the semigroup $\{U(t)\}_{t \ge 0}$ is a solution of the following stationary system

$$(g(m)v(m))' = -\tilde{b}(m)v(m) - R^* \mathbf{1}(m)v(m) + Rv(m), \tag{58}$$

$$g(a)v(a) = \int_a^1 \tilde{b}(m)v(m)\,dm. \tag{59}$$

We check that the system (58)–(59) has a unique solution such that $v(1 - h) = 1$. First, since $Rv(m) = 0$ for $m \ge 1 - h$, in the interval $[1 - h, 1]$ the function v satisfies the differential equation $v'(m) = -\gamma(m)v(m)$, where $\gamma(m) = (g'(m) + \tilde{b}(m) + R^* \mathbf{1}(m))/g(m)$. Hence

$$v(m) = \exp\left\{ -\int_{1-h}^m \gamma(y)\,dy \right\}$$

for $m \ge 1 - h$. Arguing as in the proof of Theorem 1, by (57), $\lim_{m \to 1^-} v(m) = 0$ and $\int_{1-h}^1 b(m)v(m)\,dm < \infty$. Since, as the operator P, $Rv(m)$ only depends on values of v on $[m + h, 1]$, the last inequality implies that $Rv(m)$ is a well defined and integrable function on $[1 - 2h, 1 - h]$. Thus we can solve equation (58) on the interval $[1 - 2h, 1 - h]$ matching the value $v(1 - h)$ and step by step we construct an absolutely continuous solution of this equation on the interval $[a, 1)$. By (57) and the integrability of bv, all terms on the right hand side of (58) are integrable over the interval $[a, 1]$. Hence, integrating (58) termwise, we obtain

$$g(1)v(1) - g(a)v(a) = -\int_a^1 \tilde{b}(m)v(m)\,dm - \int_a^1 R^* \mathbf{1}(m)v(m)\,dm + \int_a^1 Rv(m)\,dm.$$

Since $v(1) = 0$ and $\int_a^1 R^* \mathbf{1}(m)v(m)\,dm = \int_a^1 Rv(m)\,dm$, we get

$$g(a)v(a) = \int_a^1 \tilde{b}(m)v(m)\,dm.$$

From (58) we see that $(gv)' \in L^1[a, 1]$. Denoting $u = wv$, where w was used to define $\{U(t)\}_{t \ge 0}$, we have

$$(gu)' = (gvw)' = w(gv)' + gvw' = w(gv)' + gv\mu(w'/\mu).$$

By construction and (15), $v\mu$ is integrable and w/μ is bounded, by (29) and again by (15). The fact that u satisfies other requirements for belonging to $D(A_{b^a})$ is clear. Therefore $v \in D(\mathcal{A})$. Now we check that $v(m) > 0$ for $m \in (a, 1)$. Suppose to the contrary that for

some $m \in (a, 1)$ $v(m) = 0$ and $v(y) > 0$ for $y \in (m, 1)$. Then, integrating both sides of (58) over the interval $[a, m]$, we obtain

$$g(m)v(m) - g(a)v(a) = -\int_a^m \tilde{b}(y)v(y)\,dy - \int_a^m R^* \mathbf{1}(y)v(y)\,dy + \int_a^m Rv(y)\,dy.$$

From the boundary condition (59), it follows that

$$\int_m^1 \tilde{b}(y)v(y)\,dy - \int_a^m R^* \mathbf{1}(y)v(y)\,dy + \int_a^1 R^* \mathbf{1}_{[a,m]}(y)v(y)\,dy = 0.$$

Let $\bar{\mathcal{P}}(m, dy) = \frac{w(y)}{w(m)}\mathcal{P}^r(m, dy)$. Then $R^* f(m) = \int_a^1 f(y)\,\bar{\mathcal{P}}(m, dy)$ and we have

$$\int_m^1 \tilde{b}(y)v(y)\,dy - \int_a^m \bar{\mathcal{P}}(y, [a, 1])v(y)\,dy + \int_a^1 \bar{\mathcal{P}}(y, [a, m])v(y)\,dy = 0,$$

but, since $\bar{\mathcal{P}}(y, [m, 1]) = 0$ for $y \leq m$, we obtain

$$\int_m^1 \tilde{b}(y)v(y)\,dy + \int_m^1 \bar{\mathcal{P}}(y, [a, m])v(y)\,dy = 0.$$

As $v(y) > 0$ for $y > m$, we have $\tilde{b}(y) = 0$ and $\bar{\mathcal{P}}(y, [a, m]) = 0$ for almost every $y \geq m$. It means that $\mathcal{P}(y, \{a\}) = b^a(y) = 0$ and $\mathcal{P}^r(y, [a, m]) = 0$ for almost every $y \geq m$, which contradicts (12). Finally, we set $v_* = v/\|v\|$ and v^* is a unique invariant density of the semigroup $\{U(t)\}_{t \geq 0}$. □

6 Proof of Theorem 2

We precede the proof of Theorem 2 with some remarks. Let $\{U(t)\}_{t \geq 0}$ be the stochastic semigroup defined in Proposition 2. The semigroup $\{U(t)\}_{t \geq 0}$ is called *asymptotically stable* if there is an invariant density v_* such that

$$\lim_{t \to \infty} \|U(t)v - v_*\| = 0 \quad \text{for } v \in D. \tag{60}$$

Observe that if $\{U(t)\}_{t \geq 0}$ is asymptotically stable, then for every function $v \in L^1[a, 1]$ we have

$$\lim_{t \to \infty} U(t)v = v_* \int_a^1 v(m)\,dm \quad \text{in } L^1[a, 1].$$

Since

$$e^{-\lambda t} N(t, m) = \frac{u(t, m)}{w(m)} \quad \text{and} \quad \int_a^1 u(0, m)\,dm = \int_a^1 N(0, m)w(m)\,dm,$$

we have

$$e^{-\lambda t} N(t, \cdot) \to \frac{v_*}{w} \int_a^1 N(0, m)w(m)\,dm \quad \text{in } L^1[a, 1]. \tag{61}$$

Thus it remains to check that under the assumptions of Theorem 2, the semigroup $\{U(t)\}_{t \geq 0}$ is asymptotically stable. It means that it is enough to show asymptotic stability of the semigroup $\{U(t)\}_{t \geq 0}$ if one of the conditions (I), (II) or (III) of Theorem 2 holds.

Proposition 4 *Assume that one of the conditions* (I), (I) *or* (III) *holds. Then the semigroup* $\{U(t)\}_{t \geq 0}$, *defined in Proposition* 2, *is asymptotically stable.*

We precede the proof of Proposition 4 by an auxiliary definition. Let $\{U(t)\}_{t\geq 0}$ be a semigroup on the space $L^p(X, \Sigma, \nu)$, $1 \leq p \leq \infty$. We say that this semigroup $\{U(t)\}_{t\geq 0}$ is *partially integral* if there is $t_0 > 0$ such that the operator $U(t_0)$ can be written in the form

$$U(t_0) f(m) = \int k(m, y) f(y) \nu(dy) + Sf(m), \tag{62}$$

where S is a positive operator on L^p and the kernel k is a measurable nonnegative function such that

$$\int \int k(m, y) \nu(dy) \nu(dm) > 0. \tag{63}$$

In order to prove Proposition 4 we need the following criterion for asymptotic stability of stochastic semigroups.

Proposition 5 ([25]) *Let $\{U(t)\}_{t\geq 0}$ be a partially integral stochastic semigroup on the space $L^1(X, \Sigma, \nu)$. Assume that the semigroup $\{U(t)\}_{t\geq 0}$ has a unique invariant density f_*. If $f_* > 0$ then the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable.*

It remains to prove the following lemma.

Lemma 7 *Assume that one of the conditions* (I), (II) *or* (III) *holds. Then the semigroup $\{U(t)\}_{t\geq 0}$ defined in Proposition* 2 *is partially integral.*

Proof Let $\{S(t)\}_{t\geq 0}$ be the strongly continuous semigroup on $L^1[a, 1]$ generated by $(A_{b^a}, D(A_{b^a}))$. Since the semigroup $\{T(t)\}_{t\geq 0}$ generated by $A_{b^a} + P$ can be considered as the Miyadera-Voigt perturbation of $\{S(t)\}_{t\geq 0}$, see [4, Lemma 5.14],

$$T(t)f = S(t)f + \int_0^t T(t - \tau) P S(\tau) f \, d\tau, \quad f \in L^1[a, 1], \tag{64}$$

which implies, in particular,

$$T(t)f \geq S(t)f + \int_0^t S(t - \tau) P S(\tau) f \, d\tau, \quad 0 \leq f \in L^1[a, 1]. \tag{65}$$

According to [9, Chap. V, Lemma B (i)], if R is a positive operator and Q is a positive and integral operator, then the operators RQ and QR are integral operators. Therefore, if the semigroup $\{S(t)\}_{t\geq 0}$ is partially integral or the operator P is partially integral, then from (65) it follows that the semigroup $\{T(t)\}_{t\geq 0}$ is partially integral.

If condition (I) holds, then

$$Pf(m) \geq \int_a^1 q(m, y) f(y) \, dy \tag{66}$$

for $f \geq 0$. Thus, the operator P is partially integral which implies that the semigroup $\{T(t)\}_{t\geq 0}$ is partially integral.

If condition (II) holds, then $b^a(m) > 0$ on some nontrivial interval. We check that the semigroup $\{S(t)\}_{t\geq 0}$ is partially integral, and consequently, the semigroup $\{T(t)\}_{t\geq 0}$ is partially integral. In this case there exist $m_0 \in (a, 1)$, $\varepsilon > 0$ such that $b^a(m) > 0$ for $m \in (m_0 - \varepsilon, m_0 + \varepsilon)$. Without loss of generality we can assume that $m_0 + \varepsilon < 1$. Let us denote by $\pi_t m_0$ the solution $m(t)$ of the equation $m' = g(m)$ with the initial condition $m(0) = m_0$. Fix $t > 0$ such that $\pi_{-t} m_0 \geq m_0 - \varepsilon$. Let u be the solution of the equation $u'(t) = A_{b^a} u$ with the initial condition $u(0, m) = v(m)$. Then there exists $\gamma_1 > 0$ such that

$u(s, y) \geq \gamma_1 v(\pi_{-s} y)$ for $y \in (m_0, m_0 + \varepsilon)$ and $s \leq t$. Let $m \leq \pi_t a$. Then there exists $\gamma_2 > 0$ such that $u(t, m) > \gamma_2 u(t - \tau(m), a)$, where $\tau(m)$ is given by the formula $\pi_{\tau(m)} a = m$. From the boundary condition (10) we obtain

$$u(t, m) \geq \gamma_2 u(t - \tau(m), a) \geq \frac{\gamma_2}{g(a)} \int_a^1 b^a(y) u(t - \tau(m), y) \, dy$$

$$\geq \frac{\gamma_1 \gamma_2}{g(a)} \int_{m_0}^{m_0 + \varepsilon} b^a(y) v(\pi_{\tau(m) - t} y) \, dy$$

$$\geq \frac{\gamma_1 \gamma_2}{g(a)} \int_{\pi_{\tau(m) - t} m_0}^{\pi_{\tau(m) - t}(m_0 + \varepsilon)} \frac{b^a(\pi_{t - \tau(m)} x) g(\pi_{t - \tau(m)} x)}{g(x)} v(x) \, dx$$

for $m \leq \pi_t a$. Thus $S(t) v(m) \geq \int k(x, m) v(x) \, dx$, where k is a positive function for $m \in (a, \pi_t a)$ and $x \in (\pi_{\tau(m) - t} m_0, \pi_{\tau(m) - t}(m_0 + \varepsilon))$ and the operator $S(t)$ is partially integral.

Now assume that condition (III) holds. Let $f \in L^1[a, 1]$ be a nonnegative function. We check that for small t the operator $T(t)$ is partially integral, which is equivalent to the fact that the conjugated operator $T^*(t)$ has a nontrivial integral part. We have

$$T^*(t) f(m) \geq \int_0^t [S^*(\tau) P^* S^*(t - \tau) f](m) \, dt \geq \varepsilon \int_0^t f(\pi_{t-\tau} r(\pi_\tau m)) \, d\tau, \qquad (67)$$

where ε is a positive constant. Let $y(\tau, m) = \pi_{t-\tau} r(\pi_\tau m)$. Since

$$\frac{\partial y}{\partial \tau}(0, m_0) = -g(r(m_0)) + r'(m_0) g(m_0) \neq 0$$

we have also $\frac{\partial y}{\partial \tau}(\tau, m) \neq 0$ for small t, $\tau \in [0, t]$ and $m \in (m_0 - \delta, m_0 + \delta)$. From (67) it follows that there are constants $\eta > 0$, $\delta' > 0$, $s_1 \in [a, 1]$, $s_2 \in [a, 1]$, $s_2 > s_1$ and $t_0 > 0$ such that

$$T^*(t_0) f(m) \geq \eta \int_{s_1}^{s_2} f(y) \, dy \qquad (68)$$

for $m \in (m_0 - \delta', m_0 + \delta')$. From (68) it follows that the semigroup $\{T(t)\}_{t \geq 0}$ is partially integral.

Since $\{U(t)\}_{t \geq 0}$ is given by (53), it is also partially integral. □

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