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Finiteness of homotopy groups related to the non-abelian tensor product

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Abstract
Using finiteness-related results for non-abelian tensor products, we prove finiteness conditions for the homotopy groups $\pi_n(X)$ in terms of the number of tensors. In particular, we establish a quantitative version of the classical Blakers–Massey triad connectivity theorem. Moreover, we study other finiteness conditions and equivalence properties that arise from the non-abelian tensor square. Finally, we give applications to homotopy pushouts, especially in the case of Eilenberg–MacLane spaces.

Keywords Finiteness conditions · Non-abelian tensor product of groups · Eilenberg–MacLane spaces · Homotopy groups

Mathematics Subject Classification 20E34 · 20F50 · 20J06 · 55P20 · 55Q05

1 Introduction

In [8] Brown and Loday presented a topological significance for the non-abelian tensor product of groups. The non-abelian tensor product is used to describe the third relative homotopy group of a triad as a *non-abelian tensor product* of the second homotopy groups of appropriate subspaces. More specifically, in [8, Corollary 3.2], the third triad homotopy group is

$$\pi_3(X, A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C),$$

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where X is a pointed space and $\{A, B\}$ is an open cover of X such that A, B and $C = A \cap B$ are connected and (A, C), (B, C) are 1-connected.

In [12], Ellis gives a finiteness criterion for the triad homotopy group in terms of the finiteness of the involved groups (see also [2,3]). More generally, finiteness conditions on $\pi_n(X)$ when the excision theorem holds are given by the finiteness of $\pi_n(A)$, $\pi_n(B)$ and $\pi_{n-1}(C)$. However, when the excision property does not hold, the failure is measured by the triad homotopy groups $\pi_n(X, A, B)$ with $n \geq 3$. Therefore, the finiteness of $\pi_n(X)$ also depends on the triad homotopy groups $\pi_n(X, A, B)$. Thus, in order to give a bound for $\pi_n(X)$, it is needed to study finiteness conditions on $\pi_n(X, A, B)$. In [9], another application of the non-abelian tensor product is given by Brown and Loday, where they extended the classical Blakers–Massey triad connectivity theorem, which states that if A, B and $A \cap B$ are connected, $\{A, B\}$ is an open cover of X, $(A, A \cap B)$ is p-connected, and $(B, A \cap B)$ is q-connected, then $\pi_{p+q+1}(X, A, B)$ is isomorphic to the non-abelian tensor product

$$\pi_{p+1}(A, A \cap B) \otimes \pi_{q+1}(B, A \cap B).$$

The hypothesis $p, q \geq 2$ is broadened to $p, q \geq 1$, and the hypothesis $\pi_1(A \cap B) = 0$ is removed.

For the convenience of the reader we repeat the relevant definitions (cf. [2,3,16]). Let G and H be groups each of which acts upon the other (on the right),

$$G \times H \to G,\ (g, h) \mapsto g^h; \quad H \times G \to H,\ (h, g) \mapsto h^g$$

and on itself by conjugation, in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = \left(\left(g^{g_1^{-1}}\right)^h\right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left(\left(h^{h_1^{-1}}\right)^g\right)^{h_1}.$$

In this situation we say that G and H act *compatibly* on each other. Let H^φ be a copy of H, isomorphic via $\varphi : H \to H^\varphi,\ h \mapsto h^\varphi$, for all $h \in H$. Consider the group $\eta(G, H)$ defined in [16] as

$$\begin{aligned}\eta(G, H) = \langle G \cup H^\varphi \mid & [g, h^\varphi]^{g_1} = [g^{g_1}, (h^{g_1})^\varphi],\ [g, h^\varphi]^{h_1^\varphi} = [g^{h_1}, (h^{h_1})^\varphi], \\ & \forall g, g_1 \in G,\ h, h_1 \in H\rangle.\end{aligned}$$

It is a well-known fact (see [16, Proposition 2.2]) that the subgroup $[G, H^\varphi]$ of $\eta(G, H)$ is canonically isomorphic with the *non-abelian tensor product* $G \otimes H$, as defined by Brown and Loday in their seminal paper [8], the isomorphism being induced by $g \otimes h \mapsto [g, h^\varphi]$ (see also Ellis and Leonard [13]). It is clear that the subgroup $[G, H^\varphi]$ is normal in $\eta(G, H)$ and one has the decomposition

$$\eta(G, H) = \big([G, H^\varphi] \cdot G\big) \cdot H^\varphi,$$

where the dots mean (internal) semidirect products. We observe that when $G = H$ and all actions are conjugations, $\eta(G, H)$ becomes the group $\nu(G)$ introduced in [20]. Recall that an element $\alpha \in \eta(G, H)$ is called a *tensor* if $\alpha = [a, b^\varphi]$ for suitable $a \in G$ and $b \in H$. We write $T_\otimes(G, H)$ to denote the set of all tensors (in $\eta(G, H)$). When $G = H$ and all actions are by conjugation, we simply write $T_\otimes(G)$ instead of $T_\otimes(G, G)$. A number of structural results for the non-abelian tensor product of groups (and related constructions) in terms of the set of tensors were presented in [2–5,21].

Our contribution is to give finiteness conditions and bounds on the triad homotopy groups $\pi_n(X, A, B)$. Furthermore, a finiteness condition and bound is given to $\pi_n(X)$. We establish the following related results.

Theorem A *Let X be a union of open subspaces A, B such that A, B and $C = A \cap B$ are path-connected, and the pairs (A, C) and (B, C) are, respectively, p-connected and q-connected. Suppose that $\pi_n(A)$, $\pi_n(B)$, $\pi_{n-1}(C)$ and the set of tensors $\mathrm{T}_{\otimes}(\pi_{p+1}(A, C), \pi_{q+1}(B, C))$ are finite, where $n = p+q+1$. Then $\pi_n(X)$ is a finite group with $\{a, b, c, m\}$-bounded order, where $|\pi_n(A)| = a$, $|\pi_n(B)| = b$, $|\pi_{n-1}(C)| = c$ and $m = |\mathrm{T}_{\otimes}(\pi_{p+1}(A, C), \pi_{q+1}(B, C))|$.*

An application of Theorem A is that $\pi_3(K(C_{r^\infty}, 2) \vee K(C_{s^\infty}, 2))$ is trivial, where "$\vee$" is the wedge sum, r and s are primes, and $K(G, n)$ is an *Eilenberg–MacLane space* (i.e., a topological space having just one non-trivial homotopy group $\pi_n(K(G, n)) \cong G$). See also Remark 2.1 and Corollary 2.2.

In [8], Brown and Loday show that the third homotopy group of the suspension of an Eilenberg–MacLane space $K(G, 1)$ satisfies

$$\pi_3(SK(G, 1)) \cong J_2(G),$$

where $J_2(G)$ denotes the kernel of the derived map $\kappa : [G, G^\varphi] \to G'$, given by $[g, h^\varphi] \mapsto [g, h]$ (cf. [21, Chapter 2 and 3]). Many authors have studied bounds on the order of $\pi_3(SK(G, 1))$ (cf. [1,6,7,17]). We deduce a finiteness criterion for $\pi_3(SX)$ in terms of $\pi_2(X)$ and the number of tensors $\mathrm{T}_{\otimes}(G)$, where $\pi_1(X) \cong G$ and SX is the suspension of the space X (see Remark 2.4).

Theorem B *Let X be a connected space and $\pi_1(X) = G$. Suppose that the set of tensors $\mathrm{T}_{\otimes}(G)$ has exactly m tensors in $\nu(G)$ and $\pi_2(X)$ is finite with $|\pi_2(X)| = a$. Then $\pi_3(SX)$ is a finite group with $\{a, m\}$-bounded order.*

It is well known that the finiteness of the non-abelian tensor square $[G, G^\varphi]$ does not imply the finiteness of the group G (see Remark 2.7(b)). In [18], Parvizi and Niroomand prove that if G is a finitely generated group and the non-abelian tensor square $[G, G^\varphi]$ is finite, then G is finite. We obtain equivalent conditions (see the following theorem) and a related topological result (see Corollary 3.4).

Theorem C *Let G be a finitely generated group. The following properties are equivalent.*

(a) *The group G is finite;*
(b) *The set of tensors $\mathrm{T}_{\otimes}(G)$ is finite;*
(c) *The non-abelian tensor square $[G, G^\varphi]$ is finite;*
(d) *The derived subgroup G' is locally finite, and the kernel $J_2(G) \cong \pi_3(SK(G, 1))$ is periodic;*
(e) *The derived subgroup G' is locally finite, and the diagonal subgroup $\Delta(G)$ is periodic;*
(f) *The derived subgroup G' is locally finite, and the subgroup $\tilde{\Delta}(G) = \langle [g, h^\varphi][h, g^\varphi] \mid g, h \in G\rangle$ is periodic;*
(g) *The non-abelian tensor square $[G, G^\varphi]$ is locally finite.*

In algebraic topology, the non-abelian tensor product arises from a homotopy pushout, see [8,10]. The homotopy pushout (or homotopy amalgamated sum) is well known for its application in the classical Seifert–van Kampen theorem as well as Higher Homotopy Seifert–van Kampen theorem in the case of a covering by two open sets. Our contribution is to apply the ideas of Theorem A and of constructions related to tensor products in order to obtain finiteness results related to the homotopy pushout. For instance, see Proposition 4.2 in the fourth section for an application to homotopy pushout of Eilenberg–Maclane spaces.

The paper is organized as follows. In the next section we describe finiteness criteria for the group $\pi_n(X)$ in terms of the number of tensors. In particular, we establish a quantitative version of the classical Blakers–Massey triad connectivity theorem. In the third section we examine some necessary conditions on finiteness for the group G in terms of certain torsion elements of the non-abelian tensor square $[G, G^{\varphi}]$. In the final section, as an application we obtain finiteness criteria for the homotopy pushout that depends on the number of tensors in the non-abelian tensor product of groups.

2 Finiteness conditions

Let (X, A, B) be a triad, that is, A and B are subspaces of X, containing the base point in $C = A \cap B$, such that the triad homotopy group $\pi_n(X, A, B)$ for $n \geq 3$ fits into a long exact sequence

$$\cdots \to \pi_n(B, C) \to \pi_n(X, A) \to \pi_n(X, A, B) \to \pi_{n-1}(B, C) \to \cdots .$$

Let X be a pointed space and $\{A, B\}$ an open cover of X such that A, B and $C = A \cap B$ are connected and (A, C), (B, C) are 1-connected, see [22].

Using the relative homotopy long exact sequences and the third triad homotopy group, Ellis and McDermott obtain an interesting bound on the order of $\pi_3(X)$ (cf. [14, Proposition 5]). We have obtained (as a consequence of Theorem A) a more general version. In contrast to the bound in [14], we do not require that $\pi_2(A, C)$ and $\pi_2(B, C)$ be finite groups and we do not need to estimate $\pi_2(X, C)$.

Following the same setting, we apply the extended Blakers–Massey triad connectivity [9, Theorem 4.2]

$$\pi_{p+q+1}(X, A, B) \cong \pi_{p+1}(A, A \cap B) \otimes \pi_{q+1}(B, A \cap B),$$

in order to present a finiteness condition and a bound to $\pi_n(X)$ in terms of the set of tensors $\mathrm{T}_{\otimes}(\pi_{p+1}(A, C), \pi_{q+1}(B, C))$, where $n = p + q + 1$.

Proof of Theorem A Consider the relative homotopy long exact sequences, as in [22],

$$\pi_n(B) \to \pi_n(B, C) \to \pi_{n-1}(C)$$
$$\pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$$
$$\pi_n(B, C) \to \pi_n(X, A) \to \pi_n(X, A, B).$$

By the first exact sequence, we deduce that $\pi_n(B, C)$ is a finite group with $\{b, c\}$-bounded order. According to Brown–Loday's result [9, Theorem 4.2], the group $\pi_n(X, A, B)$ is isomorphic to the non-abelian tensor product $M \otimes N$, where $M = \pi_{p+1}(A, C)$ and $N = \pi_{q+1}(B, C)$. As $|T_{\otimes}(\pi_{p+1}(A, C), \pi_{q+1}(B, C))| = m$, we have $\pi_n(X, A, B)$ finite with m-bounded order ([3, Theorem B]). From this we conclude that the group $\pi_n(X, A)$ is finite with $\{b, m\}$-bounded order. In the same manner we can see that $\pi_n(X)$ is finite with $\{a, b, c, m\}$-bounded order. The proof is complete. □

Remark 2.1 A direct application of Theorem A is that $\pi_3(K(G, 2) \vee K(H, 2))$ is a finite group with m-bounded order, where $m = |\mathrm{T}_{\otimes}(G, H)|$. In particular, $\pi_3(K(C_{r^\infty}, 2) \vee K(C_{s^\infty}, 2))$ is trivial, where s and r are primes and $K(G, n)$ is an Eilenberg–MacLane space (a topological space having just one non-trivial homotopy group $\pi_n(K(G, n)) \cong G$).

Using the same idea as in the previous remark, we have the following corollary.

Corollary 2.2 *Let A and B be p-connected and q-connected locally contractible spaces, respectively. Suppose that $|\pi_n(A)| = a$, $|\pi_n(B)| = b$ and $|\,\mathrm{T}_{\otimes}(\pi_{p+1}(A), \pi_{q+1}(B))| = m$, where $n = p + q + 1$. Then $\pi_n(A \vee B)$ is finite with $\{a, b, m\}$-bounded order.*

The previous corollary is an example where we can use the non-abelian tensor product to overcome the failure of the excision property, which reflects the fact that $\pi_n(A \vee B)$ is different from $\pi_n(A) \oplus \pi_n(B)$ in general, for $n \geq 2$.

The following result provides a finiteness criterion for the group G in terms of the number of tensors in the non-abelian tensor square $[G, G^{\varphi}]$.

Corollary 2.3 *Let X be a connected space and $\pi_1(X) = G$. Suppose that the first homology group of X, $H_1(X, \mathbb{Z})$, is finitely generated and the set of tensors $\mathrm{T}_{\otimes}(G) \subseteq \nu(G)$ has exactly m tensors. Then both $H_1(X, \mathbb{Z})$ and $\pi_1(X) = G$ are finite with m-bounded orders.*

Proof By Theorem A, the non-abelian tensor square $[G, G^{\varphi}]$ is finite with m-bounded order. Consequently, the derived subgroup G' is finite with m-bounded order. Since G^{ab} is finitely generated, we deduce that the abelianization G^{ab} is isomorphic to a subgroup of the non-abelian tensor square $[G^{ab}, (G^{ab})^{\varphi}]$. On the other hand, $[G^{ab}, (G^{ab})^{\varphi}]$ is finite with m-bounded order because $[G^{ab}, (G^{ab})^{\varphi}]$ is an homomorphic image of $[G, G^{\varphi}]$. Therefore the abelianization G^{ab} is finite with m-bounded order. Consequently, G is finite with m-bounded order. The proof is complete. □

In the case of the suspension triad $(SX; C_+X, C_-X)$, see [8,22], we can obtain finiteness criteria and bounds to the order of $\pi_3(SX)$, where C_-X and C_+X are the two cones of X in SX.

Proof of Theorem B Consider the long exact sequence

$$\cdots \to \pi_2(X) \to \pi_3(SX) \to \pi_2(\Omega SX, X) \to \cdots$$

where ΩSX is the space of loops in SX, maps from the circle S^1 to SX, equipped with the compact–open topology. By Brown–Loday's result [8, Proposition 3.3], $\pi_2(\Omega SX, X)$ is isomorphic to the non-abelian tensor square $[G, G^{\varphi}]$ and by hypothesis $|\,\mathrm{T}_{\otimes}(G)| = m$; hence $[G, G^{\varphi}]$ is finite with m-bounded order ([3, Theorem B]). Since $|\pi_2(X)| = a$, it follows that $\pi_3(SX)$ is finite with $\{a, m\}$-bounded order. □

Remark 2.4 In the above result, it is worth noting that $\pi_2(X)$ does not need to be trivial; therefore, the result is more general compared to the bounds for $\pi_3(SK(G, 1))$ when $X = K(G, 1)$. However, in [8] it is proved that if $\pi_1(X) = G$ and $\pi_2(X)$ is trivial, then $\pi_3(SX) \cong J_2(G) = \ker(\kappa)$, where $\kappa : [G, G^{\varphi}] \to G'$ is given by $[g, h^{\varphi}] \mapsto [g, h]$.

Combining the above bounds to the order of the non-abelian tensor square and [8, Proposition 4.10] we obtain, under appropriate conditions on the set of tensors $\mathrm{T}_{\otimes}(G)$, some finiteness criteria and bounds to the orders of $\pi_3(SK(G, 1))$ and $\pi_2^S(K(G, 1))$, the second stable homotopy group of an Eilenberg–MacLane space.

Corollary 2.5 *Let G be a group. Suppose that the set $\mathrm{T}_{\otimes}(G)$ has exactly m tensors in $\nu(G)$. Then*

(a) *The second stable homotopy group $\pi_2^S(K(G, 1))$ is finite with m-bounded order;*
(b) *$\pi_3(SK(G, 1))$ is finite with m-bounded order.*

In particular, when G is a Prüfer group C_{p^∞} in the previous corollary, then $\pi_2^S(K(G,1))$ and $\pi_3(SK(G,1))$ are trivial.

Next we give a necessary and sufficient condition for the finiteness of the third homotopy group of the suspension of an Eilenberg–Maclane space.

Proposition 2.6 *Let G be a BFC group such that G^{ab} is finitely generated. Then, $\pi_3(SK(G,1))$ is finite if and only if $\pi_1(K(G,1))$ is finite.*

Proof Assume that $J_2(G) \cong \pi_3(SK(G,1))$ is finite and consider the following short exact sequence

$$0 \to J_2(G) \to [G, G^\varphi] \to G' \to 1.$$

Since G' is finite by Neumann's theorem [19, 14.5.11], we have that the non-abelian tensor square $[G, G^\varphi]$ is finite. Since G^{ab} is finitely generated, we deduce that the abelianization G^{ab} is isomorphic to a subgroup of the non-abelian tensor square $[G^{ab}, (G^{ab})^\varphi]$. On the other hand, $[G^{ab}, (G^{ab})^\varphi]$ is finite because $[G^{ab}, (G^{ab})^\varphi]$ is a homomorphic image of $[G, G^\varphi]$. From this we deduce that the abelianization G^{ab} is finite and so G is finite as well.

Conversely, suppose that $\pi_1(K(G,1))$ is finite. Consequently, the non-abelian tensor square $[G, G^\varphi]$ is finite and so, $\pi_3(SK(G,1))$ is finite. The proof is complete. □

Remark 2.7 (a) Assume that G is a BFC group and the abelianization G^{ab} is finitely generated. Consider the following short exact sequence (cf. [21, Section 2]),

$$1 \to \Delta(G) \to J_2(G) \to H_2(G) \to 1,$$

where $J_2(G)$ is isomorphic to $\pi_3(SK(G,1))$ and $H_2(G)$ is the second homology group. Recall that the diagonal subgroup $\Delta(G)$ is given by $\Delta(G) = \langle [g, g^\varphi] \mid g \in G \rangle$. The subgroup $\Delta(G)$ is finite if and only if $\pi_1(K(G,1))$ is finite (if and only if the group $\pi_3(SK(G,1))$ is finite). See [8, Section 2] for more details.
(b) Note that the equivalences above are no longer guaranteed without the hypothesis that G^{ab} is a finitely generated group. For instance, the Prüfer group $G = C_{p^\infty}$ is an infinite group such that the non-abelian tensor square $[G, G^\varphi]$ is trivial and so, finite. In particular, $J_2(G) \cong \pi_3(SK(G,1))$ is also trivial.

A direct application of Corollaries 2.3 and 2.5 to the suspension of an Eilenberg–Maclane space $K(G,1)$ and the second stable homotopy group of $K(G,1)$ is the following finiteness condition.

Corollary 2.8 *Let G be a group. Then, $\pi_3(SK(G,1))$ and G' are finite if and only if $\mathrm{T}_\otimes(G)$ is finite. Moreover, if G is perfect then $\pi_2^S(K(G,1))$ and G are finite if and only if $\mathrm{T}_\otimes(G)$ is finite.*

3 Torsion elements in the non-abelian tensor square

This section will be devoted to obtain some finiteness conditions for the group G in terms of the torsion elements in the non-abelian tensor square. Specifically, our proofs involve looking at the description of the diagonal subgroup $\Delta(G) \leqslant [G, G^\varphi]$. Such a description has previously been used by the authors [2,21].

It is well known that the finiteness of the non-abelian tensor square $[G, G^\varphi]$ does not imply the finiteness of the group G (see Remark 2.7, above). In [18], Parvizi and Niroomand prove that if G is a finitely generated group and the non-abelian tensor square $[G, G^{\varphi}]$ is finite, then G is finite. Later, in [2], the authors prove that if G is a finitely generated locally graded group and the exponent of the non-abelian tensor square $\exp([G, G^{\varphi}])$ is finite, then G is finite. The next result can be viewed as a generalization of the above results.

Theorem C *Let G be a finitely generated group. The following properties are equivalents.*

(a) *The group G is finite;*
(b) *The set of tensors* $\mathrm{T}_{\otimes}(G)$ *is finite;*
(c) *The non-abelian tensor square* $[G, G^{\varphi}]$ *is finite;*
(d) *The derived subgroup* G' *is locally finite, and the kernel* $J_2(G) \cong \pi_3(SK(G, 1))$ *is periodic;*
(e) *The derived subgroup* G' *is locally finite, and the diagonal subgroup* $\Delta(G)$ *is periodic;*
(f) *The derived subgroup* G' *is locally finite, and the subgroup* $\tilde{\Delta}(G)$ *is periodic;*
(g) *The non-abelian tensor square* $[G, G^{\varphi}]$ *is locally finite.*

Proof (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) are direct.
(b) $\Rightarrow$ (c). Suppose that $\mathrm{T}_{\otimes}(G)$ is finite. By [2, Theorem A], the non-abelian tensor square $[G, G^{\varphi}]$ is finite.
(f) $\Rightarrow$ (g). By Schmidt's theorem [19, 14.3.1], it suffices to prove that the abelianization G^{ab} is finite.

Since G^{ab} is finitely generated, we deduce that

$$G^{ab} = T \times F,$$

where T is the torsion part and F the free part of G^{ab} (cf. [19, 4.2.10]). If the abelianization G^{ab} is not periodic, then there exists an element of infinite order $x \in G$ such that $xG' \in F$. In particular, $[x, x^{\varphi}][x, x^{\varphi}] = [x, x^{\varphi}]^2$ is an infinite element in $\tilde{\Delta}(G)$. Consequently, F is trivial and $G^{ab} = T$ is finite.

(g) $\Rightarrow$ (a). First we prove that the abelianization G^{ab} is finite. Arguing as in the above paragraph, we deduce that $G^{ab} = T \times F$, where T is the torsion part and F the free part of G^{ab}. From [21, Remark 5] we conclude that $\Delta(G^{ab})$ is isomorphic to

$$\Delta(T) \times \Delta(F) \times (T \otimes_{\mathbb{Z}} F),$$

where $T \otimes_{\mathbb{Z}} F$ is the usual tensor product of $\mathbb{Z}$-modules. In particular, the free part of $\Delta(G^{ab})$ is precisely $\Delta(F)$. Now, the canonical projection $G \twoheadrightarrow G^{ab}$ induces an epimorphism $q : \Delta(G) \twoheadrightarrow \Delta(G^{ab})$. Since $\Delta(G)$ is locally finite, it follows that $\Delta(G^{ab})$ is also locally finite. Consequently, F is trivial and thus G^{ab} is periodic and, consequently, finite.

It remains to prove that the derived subgroup G' is finite. Since G is finitely generated and G^{ab} is finite, it follows that the derived subgroup G' is finitely generated ([19, 1.6.11]). As G' is an homomorphic image of the non-abelian tensor square $[G, G^{\varphi}]$, we have G' is finite. From this we deduce that G is finite. The proof is complete. □

Remark 3.1 Note that in the above result the finitely generated hypothesis is essential. For instance, if p is an odd prime, consider the Prüfer group $A = C_{p^{\infty}}$. Define the semidirect product $\mathcal{D}_{p^{\infty}} = A \cdot C_2$, where $C_2 = \langle c \rangle$ and

$$a^c = a^{-1},$$

for every $a \in A$. Hence, the group $\mathcal{D}_{p^{\infty}}$ is a locally finite group in which the abelianization is finite. By Moravec's result [15], the non-abelian tensor square $[\mathcal{D}_{p^{\infty}}, \mathcal{D}^{\varphi}_{p^{\infty}}]$ is locally finite, but $\mathcal{D}_{p^{\infty}}$ is infinite.

Proposition 3.2 *Let G be a polycyclic-by-finite group. Suppose that the non-abelian tensor square $[G, G^{\varphi}]$ is periodic. Then G is finite.*

Proof Since the derived subgroup G' is an epimorphic image of the non-abelian tensor square $[G, G^{\varphi}]$, it follows that G' is also periodic. In particular, we can deduce that G' is finite. Now, arguing as in the proof of Theorem C, we deduce that G^{ab} is finite. The proof is complete. □

It is well known that if G is a group with exponent $\exp(G) \in \{2, 3, 4, 6\}$, then G is locally finite (Burnside, Levi–van der Waerden, Sanov, M. Hall see [19, Section 14.2] for more details). We also examine the finiteness of the group G when the non-abelian tensor square $[G, G^{\varphi}]$ has small exponent.

Corollary 3.3 *Let $n \in \{2, 3, 4, 6\}$ and let G be a finitely generated group. Assume that the exponent of the non-abelian tensor square $\exp([G, G^{\varphi}])$ is exactly n. Then G is finite.*

Proof It suffices to see that the non-abelian tensor square $[G, G^{\varphi}]$ is locally finite (see [19, Section 14.2] for more details). By Theorem C, the group G is finite. The proof is complete. □

The following corollary is a topological version of Theorem C.

Corollary 3.4 *Let G be a finitely generated group, and let X be a topological space such that $\pi_1(X) = G$ and $\pi_2(X)$ is trivial. Assume that $\pi_3(SX)$ is finitely generated. Then the following properties are equivalent.*

(a) *The group $\pi_1(X) = G$ is finite;*
(b) *The derived subgroup G' is locally finite, and $\pi_3(SX)$ is periodic;*
(c) *The derived subgroup G' is locally finite, and $F(\pi_3(SX)) \cong F(H_2(G))$, where $F(H)$ is the free part of a group H;*
(d) *The derived subgroup G' is locally finite, and $F(\pi_3(SX)) \cong F(\pi_4(S^2X))$, where $F(H)$ is the free part of a group H.*

Proof Since $\pi_2(X)$ is trivial, it follows that $J_2(G) \cong \pi_3(SX)$ (cf. [8, Proposition 3.3]). As $J_2(G)$ is finitely generated, we have that $H_2(G)$ and $\pi_4(S^2X)$ are finitely generated groups. Consider the short exact sequences, as in [8],

$$1 \to \Delta(G) \to J_2(G) \to H_2(G) \to 1,$$
$$1 \to \tilde{\Delta}(G) \to J_2(G) \to \pi_4(S^2X) \to 1,$$
$$1 \to J_2(G) \to [G, G^{\varphi}] \to G' \to 1.$$

Therefore, the result follows by applying Theorem C. □

4 Application to homotopy pushout

We end this paper by proving some finiteness criteria for the homotopy pushout. Consider the following commutative square of spaces

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{b} & X \end{array}$$

and denote by $F(f)$, $F(g)$ and $F(a)$ the homotopy fibers of f, g and a, respectively, and let $F(X)$ be the homotopy fiber of $F(g) \to F(a)$. The previous square is called a *homotopy pushout* when the canonical map of squares from the double mapping cylinder $\mathrm{M}(f, g)$ to X is a weak equivalence of spaces at the four corners, for more details see [8]. For the homotopy pushout, we have an analogous of Theorem A.

Proposition 4.1 *Let the following square of spaces be a homotopy pushout*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{b} & X \end{array}$$

Suppose that $\pi_3(A)$, $\pi_2(F(g))$ and the set of tensors $\mathrm{T}_\otimes(G, H)$ are finite, where $G = \pi_1(F(f))$ and $H = \pi_1(F(g))$. Then $\pi_3(X)$ is a finite group with $\{n_a, n_b, m\}$-bounded order, where $|\pi_3(A)| = n_a$, $|\pi_2 F(g)| = n_b$ and $|\mathrm{T}_\otimes(G, H)| = m$.

Proof By [8, Theorem 3.1] we have that $\pi_1(F(X)) \simeq \pi_1(F(f)) \otimes \pi_1(F(g))$. Since $|T_\otimes(\pi_1(F(f)), \pi_1(F(g)))| = m$ then $\pi_1(F(X))$ is finite with m-bounded order ([3, Theorem B]). Using $|\pi_3(A)| = n_a$, $|\pi_2 F(g)| = n_b$ and the long exact sequences of the fibrations

$$F(X) \to F(g) \to F(a)$$
$$F(a) \to A \to X,$$

it follows that $\pi_3(X)$ is a finite group with $\{n_a, n_b, m\}$-bounded. □

Many authors have studied some finiteness conditions for the non-abelian tensor product of groups (cf. [2,4,11,12,15,18]). For instance, in [15], Moravec proved that if G, H are locally finite groups acting compatibly on each other, then so is $G \otimes H$. In [11], Donadze, Ladra and Thomas proved interesting finiteness criteria for the non-abelian tensor product in terms of the involved groups. In [2,3], the authors prove a finiteness criterion for the non-abelian tensor product of groups in terms of the number of tensors. In the case of homotopy pushout for Eilenberg–Maclane spaces it is possible to obtain results direct from the study of the non-abelian tensor product of groups. We obtain the following related result.

Proposition 4.2 *Let M, N be normal subgroups of a group G and form the homotopy pushout*

$$\begin{array}{ccc} K(G,1) & \longrightarrow & K(G/N,1) \\ \downarrow & & \downarrow \\ K(G/M,1) & \longrightarrow & X \end{array}$$

(a) *Suppose that the subgroups M and N are locally finite. Then $\pi_2(X)$ and $\pi_3(X)$ are locally finite.*
(b) *Suppose that M is a non-abelian free group of finite rank and N is a finite group. Then $\pi_2(X)$ and $\pi_3(X)$ are finite.*
(c) *Suppose that the set of tensors $\mathrm{T}_\otimes(M, N) \subseteq \eta(M, N)$ is finite. Then $\pi_3(X)$ is a finite group with m-bounded order, where $|\mathrm{T}_\otimes(M, N)| = m$.*

Proof According to Brown and Loday's result [8, Corollary 3.4], the group $\pi_3(X)$ is isomorphic to a subgroup of the non-abelian tensor product $[M, N^\varphi]$ and $\pi_2(X)$ is isomorphic to $(M \cap N)/[M, N]$. In particular, the group $\pi_2(X)$ is a homomorphic image of $M \cap N$.

(a) As $M \cap N$ is locally finite, we have that $\pi_2(X)$ is locally finite. Now, since M and N are locally finite, it follows that the non-abelian tensor product $[M, N^{\varphi}]$ is locally finite [15]. Consequently, the group $\pi_3(X)$ is locally finite.
(b) Since M is finite and N is a non-abelian free group, we deduce that $\pi_2(X)$ is finite. According to Donadze, Ladra and Thomas' result [11, Corollary 4.7], we conclude that the non-abelian tensor product $[M, N^{\varphi}]$ is finite and so $\pi_3(X)$ is finite.
(c) According to Theorem A, the non-abelian tensor product $[M, N^{\varphi}]$ is finite with m-bounded order. In particular, the group $\pi_3(X)$ is finite with m-bounded order. The proof is complete.□

Remark 4.3 Note that Proposition 4.2 (c) in a certain sense cannot be improved. For instance, if $M = N = C_{p^\infty}$ then the group $\pi_2(X) \cong C_{p^\infty}$ is infinite, whereas $\pi_3(X)$ is trivial.

As an interesting consequence of Proposition 4.2 we obtain that, if $G = MN$ such that $M \cap N$ and $[M, N^{\varphi}]$ are trivial, then X is 3-connected, i.e., $\pi_n(X)$ is trivial for $n = 1, 2$ and 3; see the following example.

Example 4.4 When $G = C_{r^\infty} \times C_{s^\infty}$ in Proposition 4.2 with r and s primes (not necessarily distinct), we have that X is 3-connected. In fact, $\pi_1(X)$ is trivial by van Kampen theorem and $\pi_2(X) = 0$ as a consequence of the van Kampen theorem for maps. Finally, the triviality of $\pi_3(X)$ follows from the proposition above.

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