The Dirichlet problem for the prescribed Ricci curvature equation on cohomogeneity one manifolds

Artem Pulemotov^{1,2}

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Abstract Let M be a domain enclosed between two principal orbits on a cohomogeneity one manifold M_1 . Suppose that T and R are symmetric invariant $(0, 2)$ -tensor fields on M and ∂M , respectively. The paper studies the prescribed Ricci curvature equation $\text{Ric}(G) = T$ for a Riemannian metric G on M subject to the boundary condition $G_{\partial M} = R$ (the notation $G_{\partial M}$ here stands for the metric induced by G on ∂M). Imposing a standard assumption on M_1 , we describe a set of requirements on T and R that guarantee global and local solvability.

Keywords Ricci curvature · Dirichlet problem · Cohomogeneity one

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1 Introduction

Suppose that M is a smooth manifold of dimension 3 or higher (possibly with boundary) and T is a symmetric $(0, 2)$ -tensor field on M . The present paper investigates the prescribed Ricci curvature equation

$$\text{Ric}(G) = T, \quad (1.1)$$

where the unknown G is a Riemannian metric on M . Mathematicians have been studying (1.1) since at least the early 1980s. We invite the reader to see [4, 7] for the history of the subject. The list of recent references not mentioned in [4, 7] includes but is not limited to [15, 16, 22, 26–28].

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✉ Artem Pulemotov
a.pulemotov@uq.edu.au

¹ School of Mathematics and Physics, The University of Queensland, St Lucia, QLD 4072, Australia

² Department of Mathematics, The University of Chicago, 5734 South University Ave, Chicago, IL 60637-1514, USA

The solvability of boundary-value problems for Eq. (1.1) is, by and large, an unexplored topic. The author of the present paper made progress on this topic in [24]. The main theorems of [24] concern the local solvability of Dirichlet- and Neumann-type problems for (1.1) (i.e. solvability in a neighbourhood of a boundary point on M).

It is worth noting that D. DeTurck's study of (1.1) underlay his discovery of the DeTurck trick. In a similar fashion, new knowledge about boundary-value problems for (1.1) may help answer questions about the existence and uniqueness of solutions to boundary-value problems for the Ricci flow and the Einstein equation. Such questions were investigated in [1, 2, 5, 10, 17, 23, 25, 29] and other works. A large number still remain open.

Let M_1 be a smooth connected manifold of dimension 3 or higher with $\partial M_1 = \emptyset$. Consider a compact Lie group $\mathcal{G}$ acting on M_1 . Suppose that the orbit space $M_1/\mathcal{G}$ is one dimensional. It is then customary to call M_1 a cohomogeneity one manifold. Such manifolds enjoy numerous applications in geometry and mathematical physics; see, e.g. [6, 8, 11, 13] and the references of [21]. In what follows, we suppose that M is the closure of a domain on M_1 contained between two principal $\mathcal{G}$ -orbits. The boundary of M is then equal to the union of these orbits. It will be convenient for us to assume that the tensor field T introduced above is defined on all of M_1 , not just M . The purpose of the present paper is to study the global and local existence of solutions to a Dirichlet-type problem for Eq. (1.1) on M . Our main results are stated as Theorems 2.2, 2.5, and 2.6.

2 Main results

Because the space $M_1/\mathcal{G}$ is one dimensional, it must be homeomorphic to the real line, the closed interval, the half-line, or the circle. In the first and the fourth case, there are no singular orbits on M_1 . For the sake of convenience, we will assume that $M_1/\mathcal{G}$ is homeomorphic to $\mathbb{R}$. It is easy to state analogues of our theorems in the situations where this assumption does not hold. Pick a point in M_1 and denote by $\mathcal{H}$ the isotropy group of this point. We will use the symbol $\mathfrak{g}$ for the Lie algebra of $\mathcal{G}$. Choose an $\text{Ad}(\mathcal{G})$ -invariant scalar product Q on $\mathfrak{g}$. Suppose that $\mathfrak{p}$ is the orthogonal complement of the Lie algebra of $\mathcal{H}$ in $\mathfrak{g}$ with respect to Q . We standardly identify $\mathfrak{p}$ with the tangent space of $\mathcal{G}/\mathcal{H}$ at $\mathcal{H}$. The isotropy representation of $\mathcal{G}/\mathcal{H}$ then yields the structure of an $\mathcal{H}$ -module on $\mathfrak{p}$. The following requirement will be imposed throughout the rest of the paper.

Hypothesis 2.1 *The $\mathcal{H}$ -module $\mathfrak{p}$ appears as an orthogonal sum*

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n \quad (2.1)$$

of pairwise nonisomorphic irreducible $\mathcal{H}$ -modules $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

Roughly speaking, Hypothesis 2.1 ensures that $\mathcal{G}$ -invariant $(0, 2)$ -tensor fields on $\mathcal{G}/\mathcal{H}$ are diagonal; cf. (2.4) and (2.6) below. This hypothesis is rather standard. It has come up in several papers including [12, 13].

Let the tensor field T be $\mathcal{G}$ -invariant. Assume that it is possible to construct a diffeomorphism

$$\Psi : \mathbb{R} \times (\mathcal{G}/\mathcal{H}) \rightarrow M_1 \quad (2.2)$$

such that the map $\Psi(r, \cdot)$ is $\mathcal{G}$ -equivariant for every $r \in \mathbb{R}$ and the equality

$$\Psi^*T = dr \otimes dr + T_r, \quad r \in \mathbb{R}, \quad (2.3)$$

holds true. Here, T_r is a $(0, 2)$ -tensor field on $\mathcal{G}/\mathcal{H}$ defined for each $r \in \mathbb{R}$. It is fully determined by how it acts on $\mathfrak{p}$. In view of Hypothesis 2.1, there exist smooth functions $\phi_1, \dots, \phi_n$ from $\mathbb{R}$ to $\mathbb{R}$ such that

$$T_r(X, Y) = \phi_1(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \phi_n(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \quad (2.4)$$

The notation $\text{pr}_{\mathfrak{p}_k} X$ and $\text{pr}_{\mathfrak{p}_k} Y$ refers to the projections of X and Y onto $\mathfrak{p}_k$ for $k = 1, \dots, n$.

If the tensor field T is positive definite, it is always possible to construct the diffeomorphism Ψ . Indeed, in this case, we can interpret T as a Riemannian metric on M_1 and consider a unit speed geodesic η with respect to this metric. Assuming η is orthogonal to all the principal orbits, we define $\Psi(r, g\mathcal{H}) = g\eta(r)$ for all $r \in \mathbb{R}$ and $g \in \mathcal{G}$. This construction is quite standard; for example, it was used in [13, 18, 21].

In what follows, we suppose that Ψ is the identity map and

$$M = [0, \sigma] \times \mathcal{G}/\mathcal{H} \quad (2.5)$$

for some $\sigma > 0$. This does not cause any loss of generality. Let R be a symmetric positive definite $\mathcal{G}$ -invariant $(0, 2)$ -tensor field on ∂M . We will use R to impose a boundary condition on (1.1). Denote by R^0 and R^σ the restrictions of R to $\{0\} \times \mathcal{G}/\mathcal{H}$ and $\{\sigma\} \times \mathcal{G}/\mathcal{H}$, respectively. Given $\tau \in [0, \sigma]$, it will be convenient for us to identify the tangent spaces to $\{\tau\} \times \mathcal{G}/\mathcal{H}$ at the point $\{\tau\} \times \mathcal{H}$ with $\mathfrak{p}$ in the natural way. We observe that R is fully determined by how R^0 and R^σ act on $\mathfrak{p}$. Thanks to Hypothesis 2.1, there exist positive numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying the equalities

$$\begin{aligned} R^0(X, Y) &= a_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + a_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \\ R^\sigma(X, Y) &= b_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + b_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \end{aligned} \quad (2.6)$$

Fix a number $\alpha > 0$ such that

$$|\phi_i(r)| \leq \alpha, \quad i = 1, \dots, n, \quad r \in [0, \sigma], \quad (2.7)$$

along with a pair of numbers $\omega_1, \omega_2 > 0$ such that

$$\omega_1 \leq a_i, b_i \leq \omega_2, \quad i = 1, \dots, n. \quad (2.8)$$

Denote by d_i the dimension of $\mathfrak{p}_i$ for $i = 1, \dots, n$. Given a Riemannian metric G defined on a neighbourhood of ∂M , we write $G_{\partial M}$ for the metric induced by G on ∂M . Our first result is a sufficient condition for the global solvability of a Dirichlet-type problem for (1.1).

Theorem 2.2 *There exist functions $\rho_0 : (0, \infty)^2 \rightarrow (0, \infty)$ and $\sigma_0 : (0, \infty)^5 \rightarrow (0, \infty)$, both independent of the tensor fields T and R , such that the following statement is satisfied: if the formulas*

$$\left| \frac{d}{dr} \phi_i(r) \right| \leq c_1 \sigma, \quad |a_i - b_i| \leq c_2 \sigma^2, \quad i = 1, \dots, n, \quad r \in [0, \sigma], \quad (2.9)$$

and the formulas

$$\begin{aligned} \sum_{i=1}^n d_i \left(\max\{\phi_i(r), 0\} + \frac{\omega_2^2}{\omega_1^2} \min\{\phi_i(r), 0\} \right) &> \rho_0(\omega_1, \omega_2), \\ \sigma &< \sigma_0(\alpha, \omega_1, \omega_2, c_1, c_2), \quad r \in [0, \sigma], \end{aligned} \quad (2.10)$$

hold for some $c_1, c_2 > 0$, then the manifold M supports a $\mathcal{G}$ -invariant Riemannian metric G solving the equation $\text{Ric}(G) = T$ on M under the boundary condition $G_{\partial M} = R$.

Remark 2.3 When proving Theorem 2.2, we will obtain explicit expressions for ρ_0 and σ_0 . These expressions (at least the one for σ_0) will be rather unsightly.

Remark 2.4 The first formula in (2.9) essentially forbids the part of T tangent to the $\mathcal{G}$ -orbits to change dramatically from one orbit to another. The second one says that R^0 should not be very different from R^σ . Note that formulas (2.9) are automatically satisfied when R^0 coincides with R^σ on $\mathfrak{p}$ and $\phi_1, \dots, \phi_n$ are constant. The meaning of (2.10) is that the tensor field T has to be large in the directions tangent to the $\mathcal{G}$ -orbits and small in the direction transverse to the $\mathcal{G}$ -orbits.

Our second result establishes the local solvability of (1.1) in the interior of M . Given $\tau \in [0, \sigma]$, we denote by Γ^τ the $\mathcal{G}$ -orbit $\{\tau\} \times \mathcal{G}/\mathcal{H}$ on M .

Theorem 2.5 *For each $\tau \in (0, \sigma)$, there exists a $\mathcal{G}$ -invariant Riemannian metric G^τ on M such that the equality $\text{Ric}(G^\tau) = T$ holds on some neighbourhood of Γ^τ .*

Next, we establish the local solvability of (1.1) near ∂M .

Theorem 2.6 *There exists a $\mathcal{G}$ -invariant Riemannian metric G^{bdy} on M such that $\text{Ric}(G^{bdy}) = T$ on some neighbourhood of ∂M and $G_{\partial M}^{bdy} = R$.*

The proofs of Theorems 2.5 and 2.6 will rely on Proposition 3.9 appearing below. This proposition will also demonstrate that a $\mathcal{G}$ -invariant metric solving the prescribed Ricci curvature equation near a $\mathcal{G}$ -orbit on M is uniquely determined by the metric it induces on this $\mathcal{G}$ -orbit and by the orbit's second fundamental form.

Remark 2.7 Assume that $\mathcal{G}$ is the special orthogonal group $\text{SO}(d)$ and M_1 coincides with $\mathbb{R}^d$ less a closed ball around the origin. One may then be able to study boundary-value problems for (1.1) with the methods of [9]; see also [19]. These methods consist in reducing the prescribed Ricci curvature equation to a first-order ordinary differential equation for a single real-value function. The authors of [9] were able to achieve such a reduction by exploiting the fact that $\text{SO}(d)$ -invariant metrics on $\mathbb{R}^d$ are globally conformally flat. In essence, their arguments relied on a clever change of variable in the prescribed Ricci curvature equation.

Remark 2.8 Instead of requiring that Hypothesis 2.1 hold for $\mathcal{G}/\mathcal{H}$, we may assume that $\mathcal{G}/\mathcal{H}$ is an abelian Lie group. The $\mathcal{H}$ -module $\mathfrak{p}$ can then be written in the form (2.1) with the $\mathcal{H}$ -modules $\mathfrak{p}_k$ being one dimensional for all $k = 1, \dots, n$. As before, we suppose that there exists a diffeomorphism Ψ satisfying formula (2.3). In our current situation, however, it is not necessarily the case that there are smooth functions $\phi_1, \dots, \phi_n$ from $[0, \sigma]$ to $\mathbb{R}$ obeying equality (2.4). Assume that such functions do exist. Suppose also that one can find positive numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that (2.6) holds. Thus, we demand that T and R be diagonal with respect to (2.1). It is then possible to prove the assertions of Theorems 2.5, 2.6, and 2.2 using the reasoning of Sect. 3.

Remark 2.9 Instead of assuming the existence of Ψ above, one may assume that there is a diffeomorphism such that (2.3) holds with this diffeomorphism substituted for Ψ and a minus sign in front of $dr \otimes dr$. The techniques in the present paper seem to be effective for treating this case. We will not dwell on any further details.

Example 2.10 Given an interval $I \subset \mathbb{R}$ and a number $\epsilon \geq 0$, denote

$$D_I = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \in I\}, \quad \mathcal{S}_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \epsilon^2\}.$$

Assume that $\mathcal{G}$ is the product $\mathrm{SO}(2) \times \mathrm{SO}(2)$. Define M_1 to equal $\mathcal{D}_{(\frac{\chi}{2}, 2)} \times \mathcal{S}_1$ with $\chi > 0$. The standard action of $\mathrm{SO}(2)$ on $\mathbb{R}^2$ gives rise to an action of $\mathcal{G}$ on M_1 . The orbits of this action are the tori $\mathcal{S}_\epsilon \times \mathcal{S}_1$ with $\epsilon \in (\frac{\chi}{2}, 2)$. The isotropy group of an arbitrarily chosen point in M_1 consists of nothing but the identity element in $\mathcal{G}$. Consider a $(0, 2)$ -tensor field T on M_1 . It is convenient for us to assume that T is positive definite, although this assumption can be relaxed. Suppose that T is rotationally symmetric in the sense of [14, 24]. This means T is $\mathcal{G}$ -invariant and diagonal with respect to the cylindrical coordinates on $\mathcal{D}_{[0, 2]} \times \mathcal{S}_1$. We define M to equal $\mathcal{D}_{[\chi, 1]} \times \mathcal{S}_1$. Thus, M is a solid torus less a neighbourhood of the core circle. Consider a symmetric positive definite $(0, 2)$ -tensor field R on ∂M . We suppose that R is $\mathcal{G}$ -invariant and diagonal in the coordinates induced on ∂M by the cylindrical coordinates on $\mathcal{D}_{[0, 2]} \times \mathcal{S}_1$. In the current setting, Theorem 2.2 (along with Remark 2.8) yields a sufficient condition for the solvability of the equation $\mathrm{Ric}(G) = T$ on all of M , subject to $G_{\partial M} = R$. No such condition previously appeared in the literature. Theorems 2.5 and 2.6 imply local solvability; cf. [24].

3 The proofs

In what follows, we assume that T is positive definite and $c_1 = c_2 = 1$. Thus, the function σ_0 , whose existence Theorem 2.2 asserts, becomes a function of three variables, not five. These assumptions will make our arguments easier to follow. Removing them is straightforward.

3.1 Preparatory material

We begin by stating a formula for the Ricci curvature of a $\mathcal{G}$ -invariant metric on M . This formula will involve two arrays of numbers, $(\beta_k)_{k=1}^n$ and $(\gamma_{k,l}^m)_{k,l,m=1}^n$. In order to introduce them, denote by $[\cdot, \cdot]$ and P the Lie bracket and the Killing form of the Lie algebra $\mathfrak{g}$. The irreducibility of the summands in decomposition (2.1) implies the existence of nonnegative numbers $\beta_1, \dots, \beta_n$ such that

$$P(X, Y) = -\beta_k Q(X, Y), \quad k = 1, \dots, n, \quad X, Y \in \mathfrak{p}_k.$$

Because the group $\mathcal{G}$ is compact and Hypothesis 2.1 holds, at least one of these numbers must be strictly positive. Suppose that d is the dimension of M . We choose a Q -orthonormal basis $(\tilde{e}_i)_{i=1}^{d-1}$ of the space $\mathfrak{p}$ adapted to (2.1). In addition to $\beta_1, \dots, \beta_n$, let us define

$$\gamma_{k,l}^m = \frac{1}{d_k} \sum Q([\tilde{e}_{\iota_k}, \tilde{e}_{\iota_l}], \tilde{e}_{\iota_m})^2$$

for $m, k, l = 1, \dots, n$. The sum here is taken over all ι_k, ι_l , and ι_m such that $\tilde{e}_{\iota_k} \in \mathfrak{p}_k, \tilde{e}_{\iota_l} \in \mathfrak{p}_l$, and $\tilde{e}_{\iota_m} \in \mathfrak{p}_m$. Note that $\gamma_{k,l}^m$ is independent of the choice of $(\tilde{e}_i)_{i=1}^{d-1}$.

Consider a Riemannian metric G on M . Suppose that $h, f_1, \dots, f_n$ are smooth functions from $[0, \sigma]$ to $(0, \infty)$. Let G be defined by the equality

$$G = h^2(r) dr \otimes dr + G_r, \quad r \in [0, \sigma]. \quad (3.1)$$

The tensor field G_r in the right-hand side is the $\mathcal{G}$ -invariant Riemannian metric on $\mathcal{G}/\mathcal{H}$ such that

$$G_r(X, Y) = f_1^2(r) Q(\mathrm{pr}_{\mathfrak{p}_1} X, \mathrm{pr}_{\mathfrak{p}_1} Y) + \dots + f_n^2(r) Q(\mathrm{pr}_{\mathfrak{p}_n} X, \mathrm{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \quad (3.2)$$

In the sequel, the prime next to a real-valued function on $[0, \sigma]$ will denote the derivative of this function.

Lemma 3.1 *The Ricci curvature of the Riemannian metric G given by (3.1) and (3.2) obeys the equality*

$$\text{Ric}(G) = \text{Ric}^{\text{lin}} + \text{Ric}_r^{\text{orb}}, \quad r \in [0, \sigma],$$

where Ric^{lin} is the $(0, 2)$ -tensor field on $[0, \sigma]$ satisfying

$$\text{Ric}^{\text{lin}} = - \sum_{k=1}^n d_k \left(\frac{f_k''}{f_k} - \frac{h' f_k'}{h f_k} \right) dr \otimes dr$$

and $\text{Ric}_r^{\text{orb}}$ is the $\mathcal{G}$ -invariant $(0, 2)$ -tensor field on $\mathcal{G}/\mathcal{H}$ satisfying

$$\begin{aligned} \text{Ric}_r^{\text{orb}}(X, Y) = & \sum_{i=1}^n \left(\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} \right. \\ & \left. - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} + \frac{f_i h' f_i'}{h^3} \right) Q(\text{pr}_{\mathfrak{p}_i} X, \text{pr}_{\mathfrak{p}_i} Y) \end{aligned}$$

for $X, Y \in \mathfrak{p}$.

Proof The terms involving dr are computed and listed under Proposition 1.14 in [18]. Let us find $\text{Ric}_r^{\text{orb}}$. Hypothesis 2.1 implies

$$\text{Ric}_r^{\text{orb}}(X, Y) = 0$$

when $X \in \mathfrak{p}_i$ and $Y \in \mathfrak{p}_j$ for $i, j = 1, \dots, n$ such that $i \neq j$. Remark 1.16 in [18] states that

$$\begin{aligned} \text{Ric}_r^{\text{orb}}(X, X) = & \left(\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} \right. \\ & \left. - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} + \frac{f_i h' f_i'}{h^3} \right) Q(X, X) \end{aligned}$$

when $X \in \mathfrak{p}_i$ and $i = 1, \dots, n$. In view of Hypothesis 2.1, the desired expression for $\text{Ric}_r^{\text{orb}}$ immediately follows. $\square$

If the Ricci curvature of G coincides with T , then Lemma 3.1 yields the equalities

$$\begin{aligned} - \sum_{k=1}^n d_k \left(\frac{f_k''}{f_k} - \frac{h' f_k'}{h f_k} \right) &= 1, \\ \frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} + \frac{f_i h' f_i'}{h^3} &= \phi_i, \quad i = 1, \dots, n. \end{aligned} \tag{3.3}$$

The following result is essentially a restatement of the contracted second Bianchi identity.

Lemma 3.2 *Assume that the Ricci curvature of the metric G given by (3.1) and (3.2) obeys the equality*

$$\text{Ric}(G) = \bar{\sigma}(r) dr \otimes dr + T_r, \quad r \in [0, \sigma],$$

with $\bar{\sigma}$ being a smooth function on $[0, \sigma]$ and T_r satisfying (2.4). Then,

$$\frac{\bar{\sigma}'}{2h^2} - \frac{\bar{\sigma}h'}{h^3} = \sum_{k=1}^n d_k \left(\frac{\phi'_k}{2f_k^2} - \frac{\bar{\sigma}f'_k}{h^2 f_k} \right). \quad (3.4)$$

Proof Fix a Q -orthonormal basis $(\tilde{e}_i)_{i=1}^{d-1}$ of the space $\mathfrak{p}$ adapted to the decomposition (2.1). Recall that we identify $\mathfrak{p}$ with the tangent space of $\mathcal{G}/\mathcal{H}$ at $\mathcal{H}$. Given $r_0 \in [0, \sigma]$, let us construct a $\mathcal{G}$ -invariant G -orthonormal frame field $(e_i)_{i=1}^d$ on a neighbourhood U of $(r_0, \mathcal{H})$ in M so that the following requirements are met:

1. The equality $e_i = (0, \frac{1}{f_i(r)} \tilde{e}_i)$ holds at $(r, \mathcal{H})$ for every $i = 1, \dots, d-1$ as long as $(r, \mathcal{H}) \in U$.
2. The vector field e_d coincides with $(\frac{1}{h(r)} \frac{\partial}{\partial r}, 0)$ on U .

The contracted second Bianchi identity then implies

$$\sum_{i=1}^d \nabla_{e_i} \text{Ric}(G)(e_i, e_d) = \frac{1}{2} e_d \left(\sum_{i=1}^d \text{Ric}(G)(e_i, e_i) \right).$$

The symbol ∇ in the left-hand side denotes the covariant derivative in the tensor bundle over M given by the Levi-Civita connection of G . We calculate and see that the equalities

$$\begin{aligned} \sum_{i=1}^d \nabla_{e_i} \text{Ric}(G)(e_i, e_d) &= \sum_{i=1}^d e_i (\text{Ric}(G)(e_i, e_d)) - \sum_{i=1}^d \text{Ric}(G)(\nabla_{e_i} e_i, e_d) \\ &\quad - \sum_{i=1}^d \text{Ric}(G)(e_i, \nabla_{e_i} e_d) \\ &= e_d (\text{Ric}(G)(e_d, e_d)) - \sum_{i=1}^d G(\nabla_{e_i} e_i, e_d) \text{Ric}(G)(e_d, e_d) \\ &\quad - \sum_{i=1}^d G(\nabla_{e_i} e_d, e_i) \text{Ric}(G)(e_i, e_i) \\ &= \frac{\bar{\sigma}'}{h^3} - \frac{2\bar{\sigma}h'}{h^4} + \sum_{k=1}^n d_k \frac{\bar{\sigma}f'_k}{h^3 f_k} - \sum_{k=1}^n d_k \frac{f'_k}{h f_k^3} \phi_k, \end{aligned}$$

as well as the equality

$$\frac{1}{2} e_d \left(\sum_{i=1}^d \text{Ric}(G)(e_i, e_i) \right) = \sum_{k=1}^n d_k \left(\frac{\phi'_k}{2h f_k^2} - \frac{f'_k}{h f_k^3} \phi_k \right) + \frac{\bar{\sigma}'}{2h^3} - \frac{\bar{\sigma}h'}{h^4},$$

hold at $(r_0, \mathcal{H})$. The assertion of the lemma follows immediately. $\square$

Denote by f and ϕ the functions $(f_1, \dots, f_n)$ and $(\phi_1, \dots, \phi_n)$ acting from $[0, \sigma]$ to $(0, \infty)^n$. We can rewrite the second equality in (3.3) as

$$f''(r) = F(h(r), h'(r), f(r), f'(r), \phi(r)), \quad r \in [0, \sigma], \quad (3.5)$$

with $F : (0, \infty) \times \mathbb{R} \times (0, \infty)^n \times \mathbb{R}^{n+n} \rightarrow \mathbb{R}^n$ given by the formulas

$$\begin{aligned} F(p, q, x, y, z) &= (F_1(p, q, x, y, z), \dots, F_n(p, q, x, y, z)), \\ F_i(p, q, x, y, z) &= \frac{\beta_i p^2}{2x_i} + p^2 \sum_{k,l=1}^n \gamma_{i,k}^l \frac{x_i^4 - 2x_k^4}{4x_i x_k^2 x_l^2} \\ &\quad - \sum_{k=1}^n d_k \frac{y_i y_k}{x_k} + \frac{y_i^2}{x_i} + \frac{q y_i}{p} - \frac{p^2}{x_i} z_i, \quad i = 1, \dots, n, \\ p &\in (0, \infty), \quad q \in \mathbb{R}, \quad x = (x_1, \dots, x_n) \in (0, \infty)^n, \\ y &= (y_1, \dots, y_n) \in \mathbb{R}^n, \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n. \end{aligned}$$

The prime next to a vector-valued function means component-wise differentiation. Combining the two equalities in (3.3), we find

$$H_1(f(r), f'(r)) = h^2(r) H_2(f(r), \phi(r)), \quad r \in [0, \sigma], \quad (3.6)$$

with the mappings $H_1 : (0, \infty)^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $H_2 : (0, \infty)^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the formulas

$$\begin{aligned} H_1(x, y) &= 1 - \sum_{k=1}^n d_k \left(\sum_{l=1}^n d_l \frac{y_k y_l}{x_k x_l} - \frac{y_k^2}{x_k^2} \right), \\ H_2(x, z) &= \sum_{k=1}^n d_k \left(\frac{z_k}{x_k^2} - \frac{\beta_k}{2x_k^2} - \sum_{l,m=1}^n \gamma_{k,l}^m \frac{x_k^4 - 2x_l^4}{4x_k^2 x_l^2 x_m^2} \right), \\ x &= (x_1, \dots, x_n) \in (0, \infty)^n, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n, \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n. \end{aligned}$$

It will be convenient for us to denote

$$H(x, y, z) = \sqrt{H_1(x, y) H_2^{-1}(x, z)}$$

for $x \in (0, \infty)^n$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}^n$ such that $H_2(x, z) \neq 0$ and $H_1(x, y) H_2^{-1}(x, z) \geq 0$.

Solving (3.4) for $h'(r)$ and substituting 1 for $\bar{\sigma}(r)$, we arrive at the following conclusion: if $\text{Ric}(G)$ coincides with T , then

$$h'(r) = K(h(r), f(r), f'(r), \phi'(r)), \quad r \in [0, \sigma]. \quad (3.7)$$

Here, $K : (0, \infty)^{1+n} \times \mathbb{R}^{n+n} \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} K(p, x, y, w) &= \sum_{i=1}^n d_i \left(\frac{p y_i}{x_i} - \frac{p^3 w_i}{2x_i^2} \right), \\ p &\in (0, \infty), \quad x = (x_1, \dots, x_n) \in (0, \infty)^n, \\ y &= (y_1, \dots, y_n) \in \mathbb{R}^n, \quad w = (w_1, \dots, w_n) \in \mathbb{R}^n. \end{aligned}$$

Let a and b denote the vectors $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ with the numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ coming from (2.6). If the metric $G_{\partial M}$ induced by G on ∂M equals R , then

$$f(0) = a, \quad f(\sigma) = b. \quad (3.8)$$

We also point out that whenever (3.6) holds, we must have

$$H_1(f(0), f'(0)) = h^2(0) H_2(f(0), \phi(0)). \quad (3.9)$$

3.2 Proof of Theorem 2.2 (less the key lemma)

Intuitively, our plan for proving Theorem 2.2 is to find a metric G satisfying two requirements. The first one is that $\text{Ric}(G)$ equal T in the directions tangent to the $\mathcal{G}$ -orbits. The other is that G and T obey the contracted second Bianchi identity. When both of these requirements are met, it must be the case that $\text{Ric}(G) = T$. We define ρ_0 by the formula

$$\rho_0(p, q) = 2 \sum_{k=1}^n d_k \left(\frac{\beta_k q^2}{2p^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{q^6}{4p^6} \right), \quad p, q \in (0, \infty).$$

Lemma 3.3 *Assume that inequalities (2.9) and the first inequality in (2.10) are satisfied. There exists a function $\sigma_0 : (0, \infty)^3 \rightarrow (0, \infty)$ such that the following statement holds: if σ is less than $\sigma_0(\alpha, \omega_1, \omega_2)$, then we can find smooth $f : [0, \sigma] \rightarrow (0, \infty)^n$ and $h : [0, \sigma] \rightarrow (0, \infty)$ solving Eqs. (3.5) and (3.7) under the boundary conditions (3.8) and (3.9).*

We will present the proof of Lemma 3.3 in Sect. 3.3. Meanwhile, fix a function σ_0 satisfying the assertion of this lemma. Suppose that σ is less than $\sigma_0(\alpha, \omega_1, \omega_2)$. Let $f : [0, \sigma] \rightarrow (0, \infty)^n$ and $h : [0, \sigma] \rightarrow (0, \infty)$ be smooth functions obeying (3.5), (3.7), (3.8) and (3.9). We define the metric G on M through (3.1)–(3.2). It is easy to see that the Ricci curvature of G must equal

$$\bar{\sigma}(r) dr \otimes dr + T_r, \quad r \in [0, \sigma],$$

for some $\bar{\sigma} : [0, \sigma] \rightarrow \mathbb{R}$. The induced metric $G_{\partial M}$ coincides with R . The proof of Theorem 2.2 will be complete if we demonstrate that $\bar{\sigma}(r) = 1$ for all $r \in [0, \sigma]$.

Lemma 3.2 implies

$$\bar{\sigma}' = \frac{2\bar{\sigma}h'}{h} + \sum_{i=1}^n d_i \left(\frac{h^2 \phi_i'}{f_i^2} - \frac{2\bar{\sigma} f_i'}{f_i} \right). \quad (3.10)$$

Thanks to (3.7), this formula will still hold if we replace $\bar{\sigma}$ in it by the function identically equal to 1 on $[0, \sigma]$. Furthermore, invoking Lemma 3.1 and the boundary conditions (3.8)–(3.9), we find

$$\begin{aligned} \bar{\sigma}(0) &= - \sum_{k=1}^n d_k \left(\frac{f_k''(0)}{a_k} - \frac{h'(0)f_k'(0)}{h(0)a_k} \right) \\ &= - \sum_{k=1}^n d_k \left(\frac{\beta_k h^2(0)}{2a_k^2} + h^2(0) \sum_{l,m=1}^n \gamma_{k,l}^m \frac{a_k^4 - 2a_l^4}{4a_k^2 a_l^2 a_m^2} \right. \\ &\quad \left. - \sum_{l=1}^n d_l \frac{f_k'(0)f_l'(0)}{a_k a_l} + \frac{f_k'^2(0)}{a_k^2} - \frac{h^2(0)}{a_k^2} \phi_k(0) \right) \\ &= h^2(0)H_2(a, \phi(0)) + (1 - H_1(a, f'(0))) = 1. \end{aligned}$$

The standard theorems on the uniqueness of solutions to ordinary differential equations now yield $\bar{\sigma}(r) = 1$ for $r \in [0, \sigma]$.

3.3 Proof of Lemma 3.3

From now on and until the end of Sect. 3.3, we assume that inequalities (2.9) and the first inequality in (2.10) are satisfied. Let $\tilde{f} : [0, \sigma] \rightarrow \mathbb{R}^n$ be defined by

$$\tilde{f}(r) = a \frac{\sigma - r}{\sigma} + b \frac{r}{\sigma}, \quad r \in [0, \sigma].$$

We seek the function f , whose existence Lemma 3.3 asserts, in a neighbourhood of $\tilde{f}$. This, in particular, will help us ensure the positivity of the components of f . Similarly, we look for the function h in a neighbourhood of the function $\tilde{h}$ to be introduced in Lemma 3.4. Our arguments will involve the constants

$$H_0 = H(\tilde{f}(0), \tilde{f}'(0), \phi(0)),$$

$$K_0 = \sup_{p \in [\frac{H_0}{2}, \frac{3H_0}{2}]} \sup_{r \in [0, \sigma]} |K(p, \tilde{f}(r), \tilde{f}'(r), \phi'(r))|.$$

The second inequality in (2.9) and the first inequality in (2.10) imply that H_0 is well defined under the assumptions of Lemma 3.4. Recall that the letter d stands for the dimension of M . It is evident that $\sum_{i=1}^n d_i = d - 1$.

Lemma 3.4 *Let $\rho_1, \sigma_1 > 0$ be given by the formulas*

$$\rho_1 = \max \left\{ 4 \left(\sum_{k=1}^n d_k \left(\frac{\alpha}{\omega_1^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{\omega_2^4}{2\omega_1^6} \right) \right)^{\frac{1}{2}}, \frac{9}{4} \left(\frac{\rho_0(\omega_1, \omega_2)}{2\omega_2^2} \right)^{-\frac{1}{2}} \right\},$$

$$\sigma_1 = \min \left\{ 1, \frac{\omega_1}{4d}, \frac{2\omega_1^2}{(2\rho_1^2\omega_1 + \rho_1^4)(d-1)} \right\}.$$

If $\sigma \leq \sigma_1$, then the problem

$$\begin{aligned} \tilde{h}'(r) &= K(\tilde{h}(r), \tilde{f}(r), \tilde{f}'(r), \phi'(r)), \quad r \in [0, \sigma], \\ \tilde{h}(0) &= H_0, \end{aligned} \tag{3.11}$$

has a unique smooth solution $\tilde{h} : [0, \sigma] \rightarrow (\frac{1}{\rho_1}, \rho_1)$.

Proof Assume that $\sigma < \sigma_1$. Employing the standard theory of ordinary differential equations, it is easy to show that the problem (3.11) has a unique smooth solution on the interval $\left[0, \min \left\{ \sigma, \frac{H_0}{2\kappa} \right\} \right]$ as long as $K_0 \leq \kappa$. The values of this solution must lie in $\left[\frac{H_0}{2}, \frac{3H_0}{2} \right]$.

Our assumptions imply

$$\frac{1}{\rho_1} \leq \frac{H_0}{2} < \frac{3H_0}{2} \leq \rho_1.$$

In view of (2.9), the estimate

$$K_0 \leq \frac{(2\rho_1\omega_1\sigma + \rho_1^3\sigma)(d-1)}{2\omega_1^2}$$

holds true. Keeping these facts in mind, we conclude that problem (3.11) has a unique smooth solution

$$\tilde{h} : \left[0, \min \left\{ \sigma, \frac{2\omega_1^2}{(2\rho_1^2\omega_1\sigma + \rho_1^4\sigma)(d-1)} \right\} \right] \rightarrow \left(\frac{1}{\rho_1}, \rho_1 \right).$$

At the same time, whenever $\sigma \leq \sigma_1$, the equality

$$\sigma = \min \left\{ \sigma, \frac{2\omega_1^2}{(2\rho_1^2\omega_1\sigma + \rho_1^4\sigma)(d-1)} \right\}$$

is satisfied. This means $\bar{h}$ is actually defined on $[0, \sigma]$. $\square$

From this moment on and until the end of Sect. 3.3, let us assume that $\sigma \leq \sigma_1$. It then makes sense to talk about $\bar{h}$. Our plan is to prove, for small σ , the existence of smooth $u : [0, \sigma] \rightarrow \mathbb{R}^n$ and $v : [0, \sigma] \rightarrow \mathbb{R}$ solving the equations

$$\begin{aligned} u''(r) &= F(\bar{h}(r) + v(r), \bar{h}'(r) + v'(r), \bar{f}(r) + u(r), \bar{f}'(r) + u'(r), \phi(r)), \\ v'(r) &= -\bar{h}'(r) + K(\bar{h}(r) + v(r), \bar{f}(r) + u(r), \bar{f}'(r) + u'(r), \phi'(r)), \quad r \in [0, \sigma], \end{aligned} \quad (3.12)$$

under the boundary conditions

$$\begin{aligned} u(0) &= u(\sigma) = 0, \\ v(0) &= -\bar{h}(0) + H(\bar{f}(0) + u(0), \bar{f}'(0) + u'(0), \phi(0)). \end{aligned} \quad (3.13)$$

We will then set $f = \bar{f} + u$ and $h = \bar{h} + v$. It is obvious that these functions will obey (3.5), (3.7), (3.8) and (3.9).

Our proof of the existence of u and v will rely on the Schauder fixed point theorem. Let us introduce the space $\mathcal{B}$ of all the pairs (v_1, v_2) such that $v_1 : [0, \sigma] \rightarrow \mathbb{R}^n$ is C^1 -differentiable and $v_2 : [0, \sigma] \rightarrow \mathbb{R}$ is continuous. We endow $\mathcal{B}$ with the norm

$$|(v_1, v_2)|_{\mathcal{B}} = \sup_{r \in [0, \sigma]} |v_1(r)|_{\mathbb{R}^n} + \sigma \sup_{r \in [0, \sigma]} |v_1'(r)|_{\mathbb{R}^n} + \sup_{r \in [0, \sigma]} |v_2(r)|,$$

where $|\cdot|_{\mathbb{R}^n}$ is the Euclidean norm in $\mathbb{R}^n$. Denote by $B(L)$ the closed ball in $\mathcal{B}$ of radius $L > 0$ centred at 0. We will now define a map $\mathcal{C} : B(L) \rightarrow \mathcal{B}$ and show that $\mathcal{C}$ has a fixed point (u, v) under appropriate conditions. The functions u and v will satisfy (3.12) and (3.13).

Assume that the radius L is less than or equal to $\frac{\sigma}{2} \min \left\{ \omega_1, \frac{1}{\rho_1} \right\}$. Given $(\mu, v) \in B(L)$, let $u_{\mu, v}$ be the unique solution of the problem

$$\begin{aligned} u''_{\mu, v}(r) &= \bar{F}(\bar{h}(r) + v(r), \bar{f}(r) + \mu(r), \bar{f}'(r) + \mu'(r), \phi(r), \phi'(r)), \quad r \in [0, \sigma], \\ u_{\mu, v}(0) &= u_{\mu, v}(\sigma) = 0, \end{aligned} \quad (3.14)$$

where

$$\bar{F}(p, x, y, z, w) = F(p, K(p, x, y, w), x, y, z), \quad p \in (0, \infty), \quad x \in (0, \infty)^n, \quad y, z, w \in \mathbb{R}^n.$$

It is obvious that such a solution exists. Moreover, it is easy to write down an explicit formula for it (the formula is quite lengthy, and we will not present it here; the reader may find it in, e.g. [20, Section XII.4]). We will set $\mathcal{C}(\mu, v) = (u_{\mu, v}, v_{\mu, v})$ for a properly chosen $v_{\mu, v} : [0, \sigma] \rightarrow \mathbb{R}$. Before we can describe $v_{\mu, v}$, however, we need to state the following auxiliary result.

Lemma 3.5 *Let Θ be given by the formula*

$$\Theta = \sqrt{d} \max_{i=1, \dots, n} \left| \frac{4\beta_i \rho_1^2}{\omega_1} + 1536\rho_1^2 \sum_{k, l=1}^n \gamma_{i, k}^l \frac{\omega_2^4}{\omega_1^5} + 2\omega_1 + (2\omega_1 + 2\omega_1^2 + 8\rho_1^2)(d-1) + \frac{8\alpha\rho_1^2}{\omega_1} \right|.$$

If (μ, ν) lie in $B(L)$, then the estimate

$$\sup_{r \in [0, \sigma]} |\bar{F}(\bar{h}(r) + \nu(r), \bar{f}(r) + \mu(r), \bar{f}'(r) + \mu'(r), \phi(r), \phi'(r))|_{\mathbb{R}^n} \leq \Theta \quad (3.15)$$

holds true. Moreover, in this case, we have

$$|u_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\sigma^2}{8} \Theta, \quad |u'_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\sigma}{2} \Theta, \quad r \in [0, \sigma]. \quad (3.16)$$

Proof Inequality (3.15) is a straightforward consequence of the definition of $\bar{F}$. To obtain (3.16), it suffices to write down an explicit formula for $u_{\mu, \nu}$ and perform elementary estimation of its terms [for the second part of (3.16), one needs to differentiate before estimating]. We refer to [20, Section XII.4] for the details of this argument. $\square$

From now on and until the end of Sect. 3.3, we assume

$$\sigma \leq \min \left\{ \sigma_1, \sqrt{\frac{\omega_1}{\Theta}}, \frac{\omega_1}{2d\Theta} \right\}. \quad (3.17)$$

Given $(\mu, \nu) \in B(L)$, let us introduce $v_{\mu, \nu} : [0, \sigma] \rightarrow \mathbb{R}$ through the formula

$$\begin{aligned} v_{\mu, \nu}(r) = & -\bar{h}(0) + H(\bar{f}(0) + u_{\mu, \nu}(0), \bar{f}'(0) + u'_{\mu, \nu}(0), \phi(0)) \\ & + \int_0^r (-\bar{h}'(s) + K(\bar{h}(s) + \nu(s), \bar{f}(s) + u_{\mu, \nu}(s), \bar{f}'(s) \\ & + u'_{\mu, \nu}(s), \phi'(s))) \, ds, \quad r \in [0, \sigma]. \end{aligned} \quad (3.18)$$

Lemma 3.5 and inequality (3.17) imply the estimates

$$\sup_{r \in [0, \sigma]} |u_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\omega_1}{2}, \quad \sup_{r \in [0, \sigma]} |u'_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\omega_1}{4d},$$

which ensure that the right-hand side of (3.18) is well defined (indeed, the expression $H(\bar{f}(0), y, \phi(0))$ is well defined and positive whenever $|y|_{\mathbb{R}^n} \leq \frac{\omega_1}{2d}$). We now set $\mathcal{C}(\mu, \nu) = (u_{\mu, \nu}, v_{\mu, \nu})$. Our next goal is to show that when σ is sufficiently small and the radius L is appropriately chosen, the map $\mathcal{C}$ satisfies the conditions of the Schauder theorem.

Suppose that $\theta > 0$ is a constant obeying the inequalities

$$\begin{aligned} |H(x, y, z) - H(x, \hat{y}, z)| & \leq \theta |y - \hat{y}|_{\mathbb{R}^n}, \\ |H(x, y, z) - H(x, \hat{y}, z)| & \leq \theta \sum_{k, l=1}^n |y_k y_l - \hat{y}_k \hat{y}_l|, \\ x & \in [\omega_1, \omega_2]^n, \\ y = (y_1, \dots, y_n) & \in \left[-\frac{\omega_1}{2d}, \frac{\omega_1}{2d} \right]^n, \\ \hat{y} = (\hat{y}_1, \dots, \hat{y}_n) & \in \left[-\frac{\omega_1}{2d}, \frac{\omega_1}{2d} \right]^n, \\ z & \in \left\{ (z_1, \dots, z_n) \in [0, \alpha]^n \mid \sum_{k=1}^n d_k z_k \geq \rho_0(\omega_1, \omega_2) \right\}, \end{aligned} \quad (3.19)$$

and the inequality

$$|K(p, x, y, w) - K(\hat{p}, \hat{x}, \hat{y}, w)| \leq \theta(|p - \hat{p}| + |x - \hat{x}|_{\mathbb{R}^n} + |y - \hat{y}|_{\mathbb{R}^n}),$$

$$p, \hat{p} \in \left[\frac{1}{2\rho_1}, 2\rho_1 \right], \quad x, \hat{x} \in \left[\frac{\omega_1}{2}, 2\omega_2 \right]^n, \quad y, \hat{y} \in \left[-\frac{\omega_1}{2d}, \frac{\omega_1}{2d} \right]^n, \quad w \in [-1, 1]^n.$$
(3.20)

It is obvious that such a θ exists. We define

$$\Sigma = \Theta + \theta n^2(2\Theta + \Theta^2 + \omega_1),$$

$$\sigma_0(\alpha, \omega_1, \omega_2) = \min \left\{ \sigma_1, \sqrt{\frac{\omega_1}{\Theta}}, \frac{\omega_1}{2d\Theta}, \frac{\omega_1}{2\Sigma}, \frac{1}{2\rho_1\Sigma} \right\}.$$

Note that σ_0 is the function whose existence (along with ρ_0) Lemma 3.3 asserts. Let us also set $L_0 = \sigma^2 \Sigma$. From now on, we will assume that the second inequality in (2.10) holds, i.e. $\sigma < \sigma_0(\alpha, \omega_1, \omega_2)$. This implies, in particular, that L_0 cannot exceed $\frac{\sigma}{2} \min \left\{ \omega_1, \frac{1}{\rho_1} \right\}$.

Lemma 3.6 *The image $\mathcal{CB}(L_0)$ is contained in $B(L_0)$.*

Proof Take a pair (μ, ν) from $B(L_0)$. Our goal is to show that $\mathcal{C}(\mu, \nu)$ lies in $B(L_0)$. Lemma 3.5 yields the estimate

$$|(u_{\mu, \nu}, v_{\mu, \nu})|_{\mathcal{B}} \leq \sigma^2 \Theta + \sup_{r \in [0, \sigma]} |v_{\mu, \nu}(r)|.$$

Remembering the second formula in (2.9), we also find

$$\begin{aligned} |v_{\mu, \nu}(r)| &\leq \left| -\bar{h}(0) + H(\bar{f}(0) + u_{\mu, \nu}(0), \bar{f}'(0) + u'_{\mu, \nu}(0), \phi(0)) \right| \\ &\quad + \sigma \sup_{s \in [0, r]} \left| -\bar{h}(s) + K(\bar{h}(s) + \nu(s), \bar{f}(s) + u_{\mu, \nu}(s), \bar{f}'(s) + u'_{\mu, \nu}(s), \phi'(s)) \right| \\ &= \left| H(\bar{f}(0), \bar{f}'(0) + u'_{\mu, \nu}(0), \phi(0)) - H(\bar{f}(0), \bar{f}'(0), \phi(0)) \right| \\ &\quad + \sigma \sup_{s \in [0, r]} \left| K(\bar{h}(s) + \nu(s), \bar{f}(s) + u_{\mu, \nu}(s), \bar{f}'(s) + u'_{\mu, \nu}(s), \phi'(s)) \right. \\ &\quad \left. - K(\bar{h}(s), \bar{f}(s), \bar{f}'(s), \phi'(s)) \right| \\ &\leq \theta \sum_{k, l=1}^n (\sigma |(u_{\mu, \nu})'_k(0)| + \sigma |(u_{\mu, \nu})'_l(0)| + |(u_{\mu, \nu})'_k(0)(u_{\mu, \nu})'_l(0)|) \\ &\quad + \sigma \theta \sup_{s \in [0, r]} (|\nu(s)| + |u_{\mu, \nu}(s)|_{\mathbb{R}^n} + |u'_{\mu, \nu}(s)|_{\mathbb{R}^n}) \\ &\leq \sigma^2 \theta n^2 (2\Theta + \Theta^2 + \omega_1), \quad r \in [0, \sigma], \end{aligned}$$

where $(u_{\mu, \nu})_k$ and $(u_{\mu, \nu})_l$ are the k th and the l th components of $u_{\mu, \nu}$. Consequently, it must be the case that

$$|(u_{\mu, \nu}, v_{\mu, \nu})|_{\mathcal{B}} \leq \sigma^2 (\Theta + \theta n^2 (2\Theta + \Theta^2 + \omega_1)) = \sigma^2 \Sigma = L_0.$$

□

Lemma 3.7 *The map $\mathcal{C} : B(L_0) \rightarrow \mathcal{B}$ is continuous.*

Proof Without loss of generality, assume that the constant θ fixed above satisfies

$$\begin{aligned} |\bar{F}(p, x, y, z, w) - \bar{F}(\hat{p}, \hat{x}, \hat{y}, \hat{z}, \hat{w})|_{\mathbb{R}^n} &\leq \theta(|p - \hat{p}| + |x - \hat{x}|_{\mathbb{R}^n} + |y - \hat{y}|_{\mathbb{R}^n}), \\ p, \hat{p} &\in \left[\frac{1}{2\rho_1}, 2\rho_1 \right], \quad x, \hat{x} \in \left[\frac{\omega_1}{2}, 2\omega_2 \right]^n, \quad y, \hat{y} \in \left[-\frac{\omega_1}{2}, \frac{\omega_1}{2} \right]^n, \\ z &\in [0, \alpha]^n, \quad w \in [-1, 1]^n. \end{aligned} \quad (3.21)$$

Suppose that the pairs (μ_1, v_1) and (μ_2, v_2) lie in $B(L_0)$. The first formula in (3.14), along with inequalities (3.16) and (3.21), imply

$$\begin{aligned} \sup_{r \in [0, \sigma]} |u_{\mu_1, v_1}(r) - u_{\mu_2, v_2}(r)|_{\mathbb{R}^n} &\leq \frac{\sigma\theta}{8} |(\mu_1, v_1) - (\mu_2, v_2)|_{\mathcal{B}}, \\ \sup_{r \in [0, \sigma]} |u'_{\mu_1, v_1}(r) - u'_{\mu_2, v_2}(r)|_{\mathbb{R}^n} &\leq \frac{\theta}{2} |(\mu_1, v_1) - (\mu_2, v_2)|_{\mathcal{B}}. \end{aligned}$$

Using (3.18)–(3.20), we also find

$$\begin{aligned} \sup_{r \in [0, \sigma]} |v_{\mu_1, v_1}(r) - v_{\mu_2, v_2}(r)| &\leq \theta |u'_{\mu_1, v_1}(0) - u'_{\mu_2, v_2}(0)|_{\mathbb{R}^n} + \theta \int_0^\sigma |v_1(s) - v_2(s)| \, ds \\ &\quad + \theta \int_0^\sigma (|u_{\mu_1, v_1}(s) - u_{\mu_2, v_2}(s)|_{\mathbb{R}^n} \\ &\quad + |u'_{\mu_1, v_1}(s) - u'_{\mu_2, v_2}(s)|_{\mathbb{R}^n}) \, ds \\ &\leq \left(\frac{3\theta^2}{2} + \theta \right) |(\mu_1, v_1) - (\mu_2, v_2)|_{\mathcal{B}}. \end{aligned}$$

Consequently, it must be the case that

$$|C(\mu_1, v_1) - C(\mu_2, v_2)|_{\mathcal{B}} \leq \left(\frac{3\theta^2}{2} + 2\theta \right) |(\mu_1, v_1) - (\mu_2, v_2)|_{\mathcal{B}},$$

which tells us $\mathcal{C}$ is continuous. $\square$

Lemma 3.8 *The closure of the set $\mathcal{CB}(L_0)$ in $\mathcal{B}$ is a compact subset of $\mathcal{B}$.*

Proof Suppose that $((\mu_j, v_j))_{j=1}^\infty$ are pairs from $B(L_0)$. It suffices to prove that the sequence $((u_{\mu_j, v_j}, v_{\mu_j, v_j}))_{j=1}^\infty$ has a convergent subsequence. The mean value theorem and Lemma 3.5 yield the estimates

$$\begin{aligned} |u_{\mu_j, v_j}(r_1) - u_{\mu_j, v_j}(r_2)|_{\mathbb{R}^n} &\leq \sup_{r \in [0, \sigma]} |u'_{\mu_j, v_j}(r)|_{\mathbb{R}^n} |r_1 - r_2| \leq \frac{\sigma}{2} \Theta |r_1 - r_2|, \\ |u'_{\mu_j, v_j}(r_1) - u'_{\mu_j, v_j}(r_2)|_{\mathbb{R}^n} &\leq \sup_{r \in [0, \sigma]} |u''_{\mu_j, v_j}(r)|_{\mathbb{R}^n} |r_1 - r_2| \leq \Theta |r_1 - r_2|, \quad j = 1, 2, \dots, \end{aligned}$$

for $r_1, r_2 \in [0, \sigma]$. Recalling formulas (3.11) and (3.20), we also obtain

$$\begin{aligned} |v_{\mu_j, v_j}(r_1) - v_{\mu_j, v_j}(r_2)| &\leq \sup_{r \in [0, \sigma]} |v'_{\mu_j, v_j}(r)| |r_1 - r_2| \\ &\leq \sup_{r \in [0, \sigma]} |-\bar{h}'(r) + K(\bar{h}(r) + v_j(r), \bar{f}(r) + u_{\mu_j, v_j}(r), \bar{f}'(r) + u'_{\mu_j, v_j}(r), \phi'(r))| |r_1 - r_2| \\ &\leq \theta \sup_{r \in [0, \sigma]} (|v_j(r)| + |u_{\mu_j, v_j}(r)|_{\mathbb{R}^n} + |u'_{\mu_j, v_j}(r)|_{\mathbb{R}^n}) |r_1 - r_2| \\ &\leq \theta(\sigma^2 \Sigma + \sigma \Theta) |r_1 - r_2|, \quad j = 1, 2, \dots, \quad r_1, r_2 \in [0, \sigma]. \end{aligned}$$

It follows that the sequences $(u_{\mu_j, v_j})_{j=1}^\infty$, $(u'_{\mu_j, v_j})_{j=1}^\infty$, and $(v_{\mu_j, v_j})_{j=1}^\infty$ are equicontinuous. Furthermore, because $CB(L_0)$ is a subset of $B(L_0)$, they are uniformly bounded. These facts, along with the Arzelà–Ascoli theorem, imply that $((u_{\mu_j, v_j}, v_{\mu_j, v_j}))_{j=1}^\infty$ must have a convergent subsequence. $\square$

According to the lemmas above, the map $\mathcal{C} : B(L_0) \rightarrow \mathcal{B}$ is continuous, and its image is a precompact subset of $B(L_0)$. Consequently, the Schauder theorem (see, e.g. [20, Chapter XII, Corollary 0.1]) allows us to conclude that there exists a pair $(u, v) \in B(L_0)$ satisfying the equality $\mathcal{C}(u, v) = (u, v)$. It is easy to understand that u and v obey (3.12) and (3.13). A simple bootstrapping argument demonstrates that u and v are smooth. We define $f = \bar{f} + u$ and $h = \bar{h} + v$. Clearly, these functions take values in $(0, \infty)^n$ and $(0, \infty)$, respectively, and solve (3.5) and (3.7) under the conditions (3.8) and (3.9). Thus, Lemma 3.3 is established.

3.4 Proof of Theorems 2.5 and 2.6

Given $\tau \in [0, \sigma]$ and $\kappa > 0$, set

$$J_\kappa^\tau = (\tau - \kappa, \tau + \kappa) \cap [0, \sigma], \quad \mathcal{X}_\kappa^\tau = J_\kappa^\tau \times \mathcal{G}/\mathcal{H}.$$

Obviously, $\mathcal{X}_\kappa^\tau$ is a neighbourhood of Γ^τ in M . Assume that G^τ is a Riemannian metric on M and $G_{\Gamma^\tau}^\tau$ is the metric on Γ^τ induced by G^τ . Let $\Pi(G^\tau)$ be the second fundamental form of Γ^τ in M with respect to G^τ and to the unit normal whose scalar product with $(\frac{\partial}{\partial r}, 0)$ is less than 0.

For each $\tau \in [0, \sigma]$, consider a symmetric positive definite $\mathcal{G}$ -invariant $(0, 2)$ -tensor field R^τ on Γ^τ . In order to keep our notation consistent, we assume that R^0 and R^σ are the restrictions of R to $\{0\} \times \mathcal{G}/\mathcal{H}$ and $\{\sigma\} \times \mathcal{G}/\mathcal{H}$. It is evident that R^τ is fully determined by how it acts on $\mathfrak{p}$. There exist numbers $a_{\tau,1}, \dots, a_{\tau,n} > 0$ satisfying

$$R^\tau(X, Y) = a_{\tau,1}^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + a_{\tau,n}^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

Let us also fix, for every $\tau \in [0, \sigma]$, a symmetric $\mathcal{G}$ -invariant tensor field S^τ on Γ^τ . There are $\delta_{\tau,1}, \dots, \delta_{\tau,n} \in \mathbb{R}$ such that

$$S^\tau(X, Y) = \delta_{\tau,1} Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \delta_{\tau,n} Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

Proposition 3.9, which we are about to state, underlies Theorems 2.5 and 2.6. The author's paper [24] contains similar results, though established in a different setting. Another closely related theorem was obtained in [3]. The methods we use in the present paper are different from those of [24], as we explain in Remark 3.10 below.

Proposition 3.9 *Suppose that $\tau \in [0, \sigma]$. The following two statements are equivalent:*

1. *For some $\kappa > 0$, there exists a $\mathcal{G}$ -invariant Riemannian metric G^τ on M such that $\text{Ric}(G^\tau) = T$ on $\mathcal{X}_\kappa^\tau$, $G_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi(G^\tau) = S^\tau$.*
2. *The inequality*

$$\sum_{k=1}^n d_k \left(\frac{\beta_k}{2a_{\tau,k}^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{a_{\tau,k}^4 - 2a_{\tau,l}^4}{4a_{\tau,k}^2 a_{\tau,l}^2 a_{\tau,m}^2} - \sum_{l=1}^n d_l \frac{\delta_{\tau,k} \delta_{\tau,l}}{a_{\tau,k}^2 a_{\tau,l}^2} + \frac{\delta_{\tau,k}^2}{a_{\tau,k}^4} - \frac{1}{a_{\tau,k}^2} \phi_k(\tau) \right) < 0 \quad (3.22)$$

is satisfied.

If these statements hold and $\check{G}^\tau$ is a $\mathcal{G}$ -invariant metric on M such that $\text{Ric}(\check{G}^\tau) = T$ on $\mathcal{X}_\kappa^\tau$, $\check{G}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi(\check{G}^\tau) = S^\tau$, then $\check{G}^\tau$ must coincide with G^τ on $\mathcal{X}_\kappa^\tau$.

Proof Suppose that there exist $\kappa > 0$ and a $\mathcal{G}$ -invariant Riemannian metric G^τ on M such that $\text{Ric}(G^\tau) = T$ on $\mathcal{X}_\kappa^\tau$, $G_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi(G^\tau) = S^\tau$. Employing Lemma 3.1 and the fact that T is positive definite, one can show that G^τ satisfies the formula

$$G^\tau = h_\tau^2(r) dr \otimes dr + G_r^\tau, \quad r \in J_\kappa^\tau. \quad (3.23)$$

Here, h_τ is a smooth function acting from J_κ^τ to $(0, \infty)$. The tensor field G_r^τ is a $\mathcal{G}$ -invariant Riemannian metric on $\mathcal{G}/\mathcal{H}$. It is clear that

$$G_r^\tau(X, Y) = f_{\tau,1}^2(r) Q(\text{pr}_{p_1} X, \text{pr}_{p_1} Y) + \cdots + f_{\tau,n}^2(r) Q(\text{pr}_{p_n} X, \text{pr}_{p_n} Y), \quad X, Y \in \mathfrak{p}, \quad (3.24)$$

for some smooth functions $f_{\tau,1}, \dots, f_{\tau,n}$ from J_κ^τ to $(0, \infty)$. The equality $\text{Ric}(G^\tau) = T$ and Lemma 3.1 imply

$$H_1(f_\tau(r), f'_\tau(r)) = h_\tau^2(r) H_2(f_\tau(r), \phi(r)), \quad r \in J_\kappa^\tau. \quad (3.25)$$

The notation f_τ here stands for $(f_{\tau,1}, \dots, f_{\tau,n})$. Because $G_{\Gamma^\tau}^\tau = R^\tau$ and $\Pi(G^\tau) = S^\tau$, we also have

$$f_\tau(\tau) = a_\tau, \quad f'_\tau(\tau) = -h_\tau(\tau) \delta_\tau^a,$$

where $a_\tau = (a_{\tau,1}, \dots, a_{\tau,n})$ and $\delta_\tau^a = (\frac{\delta_{\tau,1}}{a_{\tau,1}}, \dots, \frac{\delta_{\tau,n}}{a_{\tau,n}})$. Keeping these two formulas in mind and using (3.25), we easily calculate that the quantity in the left-hand side of (3.22) is equal to $-\frac{1}{h_\tau^2(\tau)}$. This quantity must, therefore, be negative.

Assume now that (3.22) holds. Let us prove the existence of $\kappa > 0$ and a metric G^τ on M such that $\text{Ric}(G^\tau) = T$ on $\mathcal{X}_\kappa^\tau$, $G_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi(G^\tau) = S^\tau$. Consider the system of ordinary differential equations

$$\begin{aligned} f''_\tau(r) &= \bar{F}(h_\tau(r), f_\tau(r), f'_\tau(r), \phi(r), \phi'(r)), \\ h'_\tau(r) &= K(h_\tau(r), f_\tau(r), f'_\tau(r), \phi'(r)), \end{aligned} \quad (3.26)$$

for the unknown functions f_τ and h_τ . We supplement this system with the conditions

$$\begin{aligned} f_\tau(\tau) &= a_\tau, \\ f'_\tau(\tau) &= -(H_2(a_\tau, \phi(\tau)) + (1 - H_1(a_\tau, \delta_\tau^a)))^{-\frac{1}{2}} \delta_\tau^a, \\ h_\tau(\tau) &= (H_2(a_\tau, \phi(\tau)) + (1 - H_1(a_\tau, \delta_\tau^a)))^{-\frac{1}{2}}. \end{aligned} \quad (3.27)$$

Note that, thanks to (3.22), the right-hand sides of the last two formulas are well defined. The standard theory of ordinary differential equations tells us that problem (3.26)–(3.27) has a solution. To be more precise, for some number $\kappa > 0$, there exist smooth functions $f_\tau : J_\kappa^\tau \rightarrow (0, \infty)^n$ and $h_\tau : J_\kappa^\tau \rightarrow (0, \infty)$ solving (3.26) on J_κ^τ and satisfying (3.27). With these functions at hand, we define a $\mathcal{G}$ -invariant Riemannian metric G^τ on $\mathcal{X}_\kappa^\tau$ by formulas (3.23) and (3.24). We extend it to all of M arbitrarily. It follows from (3.26) that

$$\text{Ric}(G^\tau) = \hat{\sigma}(r) dr \otimes dr + T_r, \quad r \in J_\kappa^\tau,$$

for some $\hat{\sigma} : J_\kappa^\tau \rightarrow \mathbb{R}$. Employing Lemma 3.2 and arguing as in Sect. 3.2, one demonstrates that $\hat{\sigma}$ must be identically equal to 1 on J_κ^τ . This means $\text{Ric}(G^\tau) = T$ on $\mathcal{X}_\kappa^\tau$. Conditions (3.27) imply that $G_{\Gamma^\tau}^\tau = R^\tau$ and $\Pi(G^\tau) = S^\tau$.

Suppose now that statements 1 and 2 in Proposition 3.9 hold true. We may assume that the metric G^τ satisfies (3.23) and (3.24). Then, the functions f_τ and h_τ solve (3.26)–(3.27) on J_k^τ . Consider a $\mathcal{G}$ -invariant Riemannian metric $\check{G}^\tau$ on M such that $\text{Ric}(\check{G}^\tau) = T$ on $\mathcal{X}_k^\tau$, $\check{G}_{\Gamma^\tau}^\tau = R^\tau$, and $\text{II}(\check{G}^\tau) = S^\tau$. Our objective is to show that $\check{G}^\tau$ coincides with G^τ on $\mathcal{X}_k^\tau$. By analogy with (3.23), we write

$$\check{G}^\tau = \check{h}_\tau^2(r) dr \otimes dr + \check{G}_r^\tau, \quad r \in J_k^\tau.$$

In the right-hand side, $\check{h}_\tau : J_k^\tau \rightarrow (0, \infty)$ is a smooth function. The tensor field $\check{G}_r^\tau$ is a $\mathcal{G}$ -invariant Riemannian metric on $\mathcal{G}/\mathcal{H}$. There are smooth functions $\check{f}_{\tau,1}, \dots, \check{f}_{\tau,n}$ from J_k^τ to $(0, \infty)$ such that

$$\check{G}_r^\tau(X, Y) = \check{f}_{\tau,1}^2(r) Q(\text{pr}_{p_1} X, \text{pr}_{p_1} Y) + \dots + \check{f}_{\tau,n}^2(r) Q(\text{pr}_{p_n} X, \text{pr}_{p_n} Y), \quad X, Y \in \mathfrak{p}.$$

It will be convenient for us to denote $\check{f}_\tau = (\check{f}_{\tau,1}, \dots, \check{f}_{\tau,n})$. Because $\text{Ric}(\check{G}^\tau) = T$, $\check{G}_{\Gamma^\tau}^\tau = R^\tau$, and $\text{II}(\check{G}^\tau) = S^\tau$, formulas (3.26)–(3.27) would still hold on J_k^τ if we substituted $\check{f}_\tau, \check{f}_\tau', \check{f}_\tau'', \check{h}_\tau$, and $\check{h}_\tau'$ in them for $f_\tau, f_\tau', f_\tau'', h_\tau$, and h_τ' . The standard theory of ordinary differential equations then implies that $\check{f}_\tau = f_\tau$ and $\check{h}_\tau = h_\tau$ on J_k^τ . Consequently, $\check{G}^\tau$ coincides with G^τ on $\mathcal{X}_k^\tau$. Thus, the proof is complete. $\square$

Remark 3.10 One may establish Proposition 3.9 by adapting the methods employed in the paper [24]. The main idea behind those methods is to modify the right-hand side of (1.1) by a G -dependent diffeomorphism making the equation more easily solvable. Such an approach relies on the work of DeTurck (see [7, Chapter 5] for an overview) and is similar in spirit to the DeTurck trick for the Ricci flow. Conversely, it seems possible to prove the existence and uniqueness results in [24] with the techniques employed above.

Remark 3.11 Suppose that we are in the situation described in Remark 2.8. Thus, $\mathcal{G}/\mathcal{H}$ is an abelian Lie group, and Hypothesis 2.1 fails to hold. In this case, statement 2 of Proposition 3.9 implies statement 1, but establishing the converse implication may be problematic. Roughly speaking, this is because, when Hypothesis 2.1 does not hold, the metric G^τ need not be diagonal with respect to (2.1). For the same reason, proving the assertion about $\check{G}^\tau$ may be troublesome with our methods.

Remark 3.12 Statement 1 of Proposition 3.9 is equivalent to statement 2 even if T is not positive definite. Yet, our methods do not yield the assertion about $\check{G}^\tau$ in this case. Recall that if T is not positive definite, we need to assume the existence of a diffeomorphism Ψ satisfying (2.2), (2.3) and (2.4).

Theorems 2.5 and 2.6 follow from Proposition 3.9 by choosing R^τ and S^τ in such a way that

$$\begin{aligned} R^\tau(X, Y) &= \left(\frac{\sigma - \tau}{\sigma} a_1 + \frac{\tau}{\sigma} b_1 \right) Q(\text{pr}_{p_1} X, \text{pr}_{p_1} Y) \\ &\quad + \dots + \left(\frac{1 - \tau}{\sigma} a_n + \frac{\tau}{\sigma} b_n \right) Q(\text{pr}_{p_n} X, \text{pr}_{p_n} Y), \\ S^\tau(X, Y) &= \beta Q(X, Y), \quad X, Y \in \mathfrak{p}, \end{aligned}$$

for a sufficiently large $\beta > 0$.

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