Erratum to: Geometric models for the spectra of certain Gelfand pairs associated with Heisenberg groups

Chal Benson · Gail Ratcliff

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The paper contains an error which necessitates some revisions to the proof of our main result, Theorem 1.2. In fact, the polynomial functions $\widehat{D}_\alpha \in \mathbb{C}[\mathfrak{a}^*]$ ($\alpha \in \Lambda$), discussed in Section 2.2, need not, in general, be invariant under the little Weyl group W_α as stated. One needs to introduce a “ ϱ -shift” (half the sum of the positive roots) to achieve W_α -invariance. This means that although $\text{top}(\widehat{D}_\alpha) \in \mathbb{C}[\mathfrak{a}^*]^{W_\alpha}$, the polynomial $\widehat{D}_\alpha$ itself need not lie in the image of the mapping $\bar{\rho}$ given in Eqs. 2.1 and 2.3. Thus, we cannot define polynomials $E_\alpha \in \mathbb{C}[V_\mathbb{R}]^K$ as in Definition 2.1 or obtain the related functions $\varepsilon_L \in C(\mathfrak{h}_V^*)^K$ as claimed in Proposition 4.2.

Lemma A.2 below provides a technical tool needed to revise the proof for Theorem 1.2. First, we require the following substitute for Lemma 2.7 from the paper.

Lemma A.1 *For a well-behaved multiplicity free action $K:V$ and $\alpha, \beta \in \Lambda$ one has*

$$(\text{top}(\widehat{D}_\alpha))(\beta) = p_\alpha(v_\beta).$$

Proof Equation 2.3 yields

$$\bar{\rho}(p_\alpha)(\beta) = \tilde{p}_\alpha(z, \eta(z, \beta))$$

for any $z \in V$ satisfying $h_\beta(z) \neq 0$. Condition (i) in Definition 2.4 allows us to take $z = v_\beta$ and condition (ii) gives $\eta(v_\beta, \beta) = \langle \cdot, v_\beta \rangle = v_\beta^*$. So then

$$(\text{top}(\widehat{D}_\alpha))(\beta) = \bar{\rho}(p_\alpha)(\beta) = \tilde{p}_\alpha(v_\beta, v_\beta^*) = p_\alpha(v_\beta).$$

□

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C. Benson (✉) · G. Ratcliff
Department of Mathematics, East Carolina University Greenville,
Greenville, NC 27858, USA
e-mail: bensonf@ecu.edu

G. Ratcliff
e-mail: ratcliffg@ecu.edu

Lemma A.2 Let $K:V$ be a well-behaved multiplicity free action, (β_n) a sequence in Λ and (λ_n) a sequences in $\mathbb{R}^\times$ with $\lim \lambda_n = 0$ and $(|\lambda_n|\beta_n)$ converging in $\mathfrak{a}^*$. Then $\lim \widehat{L}_\alpha(\phi_{\beta_n, \lambda_n}) = \lim p_\alpha(\sqrt{2|\lambda_n|} v_{\beta_n})$ for all $\alpha \in \Lambda$.

Proof Recall that $\widehat{L}_\alpha(\phi_{\beta_n, \lambda_n}) = (2|\lambda_n|)^{|\alpha|} \widehat{D}_\alpha(\beta_n)$ by Eq. 3.1. So now

$$\lim \widehat{L}_\alpha(\phi_{\beta_n, \lambda_n}) = \lim (2|\lambda_n|)^{|\alpha|} \widehat{D}_\alpha(\beta_n) = \lim (2|\lambda_n|)^{|\alpha|} \widehat{D}_\alpha((2|\lambda_n|)^{-1} \beta) = (\text{top}(\widehat{D}_\alpha))(\beta)$$

where $\beta := \lim(2|\lambda_n|\beta_n)$. On the other hand, using Lemma A.1,

$$\begin{aligned} (\text{top}(\widehat{D}_\alpha))(\beta) &= \lim(\text{top}(\widehat{D}_\alpha))(2|\lambda_n|\beta_n) \\ &= \lim(2|\lambda_n|)^{|\alpha|} (\text{top}(\widehat{D}_\alpha))(\beta_n) \\ &= \lim(2|\lambda_n|)^{|\alpha|} p_\alpha(v_{\beta_n}) = \lim p_\alpha(\sqrt{2|\lambda_n|} v_{\beta_n}). \end{aligned}$$

□

Revised Proof of Theorem 1.2 Assume that the multiplicity free action $K:V$ is well behaved. Let $(\varphi_n)_{n=1}^\infty$ be a sequence in $\Delta(K, H_V)$, $\varphi \in \Delta(K, H_V)$, and write $\mathcal{K}_n = \Psi(\varphi_n)$, $\mathcal{K} = \Psi(\varphi)$. We will show that

$$\varphi_n \longrightarrow \varphi \text{ in } \Delta(K, H_V) \iff \mathcal{K}_n \longrightarrow \mathcal{K} \text{ in } \mathfrak{h}_V^*/K.$$

The Heisenberg fan model shows that $\varphi_n \longrightarrow \varphi$ if and only if $\widehat{L}(\varphi_n) \longrightarrow \widehat{L}(\varphi)$ for every $L \in \mathbb{D}_K(H_V)$. By considering subsequences, we may assume that either every φ_n is a spherical function of Type 2 or every φ_n is of Type 1.

($\Rightarrow$): First assume that $\varphi_n \longrightarrow \varphi$. We will show $\mathcal{K}_n \longrightarrow \mathcal{K}$.

Case 1: Suppose that each φ_n is a spherical function of Type 2. For some points $w_n \in V$, one has $\varphi_n = \eta_{K \cdot w_n}$ and $\mathcal{K}_n = (K \cdot w_n) \times \{0\}$. As $\widehat{T}\varphi = \lim \widehat{T}\varphi_n = 0$, it follows that φ is also of Type 2. So now $\varphi = \eta_{K \cdot w}$ and $\mathcal{K} = (K \cdot w) \times \{0\}$ for some $w \in V$. As $\widehat{L}_\alpha(\varphi_n) = |w_n|^2$ [see (4.2)] converges to $\widehat{L}_\alpha(\varphi) = |w|^2$, it follows that (w_n) is a bounded sequence. Passing to a subsequence, we may assume that (w_n) converges in V , with $\lim w_n = w'$ say. Now for each $\alpha \in \Lambda$, we observe that

$$p_\alpha(w') = \lim p_\alpha(w_n) = \lim \widehat{L}_\alpha(\varphi_n) = \widehat{L}_\alpha(\varphi) = p_\alpha(w).$$

As $\{p_\alpha : \alpha \in \Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$ and the invariants for a compact linear action separate orbits it follows that $K \cdot w' = K \cdot w$. Hence, $\mathcal{K}_n$ converges to $\mathcal{K}$ in $\mathfrak{h}_V^*/K$.

Case 2: Suppose that each φ_n is a spherical function of Type 1,

$$\varphi_n = \phi_{\beta_n, \lambda_n} \quad \text{and} \quad \mathcal{K}_n = \sqrt{2|\lambda_n|} (K \cdot v_{\beta_n}) \times \{\lambda_n\}$$

say. Let

$$\lambda := \widehat{T}\varphi = \lim \widehat{T}\varphi_n = \lim \lambda_n.$$

Case 2(a): If $\lambda \neq 0$ then $\varphi = \phi_{\beta, \lambda}$ and $\mathcal{K} = \sqrt{2|\lambda|} (K \cdot v_\beta) \times \{\lambda\}$ for some $\beta \in \Lambda$. As $\widehat{L}_\alpha(\varphi_n) = 2|\lambda_n||\beta_n|$ converges to $\widehat{L}_\alpha(\varphi) = 2|\lambda||\beta|$, it follows that $\lim |\beta_n| = |\beta|$. As $\{\alpha \in \Lambda : |\alpha| = |\beta|\}$ is a finite set we can assume, by passing to a subsequence, that $\beta_n = \beta$ for every n . So now $\mathcal{K}_n = \sqrt{2|\lambda_n|} (K \cdot v_\beta) \times \{\lambda_n\}$ with $\lambda_n \longrightarrow \lambda$ and thus $\mathcal{K}_n \longrightarrow \mathcal{K}$ as desired.

Case 2(b): If $\lambda = 0$ then $\varphi = \eta_{K \cdot w}$ and $\mathcal{K} = (K \cdot w) \times \{0\}$ for some $w \in V$. Moreover, $\widehat{L}_\alpha(\varphi_n) = 2|\lambda_n||\beta_n| = 2|\lambda_n||v_{\beta_n}|^2$ converges to $\widehat{L}_\alpha(\varphi) = |w|^2$, and thus $\sqrt{2|\lambda_n|} v_{\beta_n}$ is a bounded sequence. By passing to a subsequence, we may assume this converges in V and

write $v := \lim \sqrt{2|\lambda_n|} v_{\beta_n}$. Applying the moment map, it follows that $2|\lambda_n|\beta_n$ converges to a point $\beta \in \mathfrak{a}^*$ with $\beta_{\mathfrak{t}} = \tau(v)$. Now Lemma A.2 yields

$$p_{\alpha}(w) = \widehat{L}_{\alpha}(\varphi) = \lim \widehat{L}_{\alpha}(\phi_{\beta_n, \lambda_n}) = \lim p_{\alpha}(\sqrt{2|\lambda|} v_{\beta_n}) = p_{\alpha}(v)$$

for each $\alpha \in \Lambda$. As in Case 1, this implies that $K \cdot w = K \cdot v$ and thus $\mathcal{K} = \lim \mathcal{K}_n$.

($\Leftarrow$): Next assume conversely that $\mathcal{K}_n \rightarrow \mathcal{K}$. We will show $\varphi_n \rightarrow \varphi$.

Case 1: Suppose that each φ_n is a spherical function of Type 2. Hence $\mathcal{K}_n \subset V \times \{0\}$ for all n and as $\mathcal{K}_n \rightarrow \mathcal{K}$, it follows that $\mathcal{K} \subset V \times \{0\}$ and that φ is of Type 2. So

$$\varphi_n = \eta_{K \cdot w_n}, \quad \mathcal{K}_n = (K \cdot w_n) \times \{0\}; \quad \varphi = \eta_{K \cdot w}, \quad \mathcal{K} = (K \cdot w) \times \{0\}$$

say. Now $K \cdot w_n \rightarrow K \cdot w$ and hence $(\widehat{L}_{\alpha}(\varphi_n) = p_{\alpha}(w_n)) \rightarrow (p_{\alpha}(w) = \widehat{L}_{\alpha}(\varphi))$ for each $\alpha \in \Lambda$. It follows that $\varphi_n \rightarrow \varphi$.

Case 2: Suppose that each φ_n is a spherical function of Type 1, and write

$$\varphi_n = \phi_{\beta_n, \lambda_n} \quad \text{and} \quad \mathcal{K}_n = \sqrt{2|\lambda_n|} (K \cdot v_{\beta_n}) \times \{\lambda_n\}.$$

As $(\mathcal{K}_n)$ converges so does (λ_n) . Let $\lambda := \lim \lambda_n$.

Case 2(a): If $\lambda \neq 0$ then $\varphi = \phi_{\beta, \lambda}$ and $\mathcal{K} = \sqrt{2|\lambda|} (K \cdot v_{\beta}) \times \{\lambda\}$ for some $\beta \in \Lambda$ with $K \cdot v_{\beta_n} \rightarrow K \cdot v_{\beta}$. As Λ is a discrete set it follows that the sequence (β_n) is eventually constant. Thus $\varphi = \phi_{\beta, \lambda} = \lim \phi_{\beta, \lambda_n} = \lim \varphi_n$.

Case 2(b): If $\lambda = 0$ then $\varphi = \eta_{K \cdot w}$ and $\mathcal{K} = (K \cdot w) \times \{0\}$ for some $w \in V$. As $\mathcal{K}_n \rightarrow \mathcal{K}$, we have $\sqrt{2|\lambda_n|} (K \cdot v_{\beta_n}) \rightarrow K \cdot w$ and by passing to a subsequence we may assume that $\sqrt{2|\lambda_n|} v_{\beta_n}$ converges to a point $v \in K \cdot w$. Applying the moment map, it follows that $2|\lambda_n|\beta_n$ converges in $\mathfrak{a}^*$. Again using Lemma A.2, we obtain

$$\lim \widehat{L}_{\alpha}(\varphi_n) = \lim p_{\alpha}(\sqrt{2|\lambda_n|} v_{\beta_n}) = p_{\alpha}(v) = p_{\alpha}(w) = \widehat{L}_{\alpha}(\varphi)$$

for each $\alpha \in \Lambda$ and hence $\varphi_n \rightarrow \varphi$ as claimed. $\square$

Finally, we note that without the functions ε_L given by Proposition 4.2 we do not obtain an explicit homeomorphism between the orbital and Heisenberg fan models for $\Delta(K, H_V)$, as claimed in Corollary 4.4.