Hölder exponents of the Green functions of planar polynomial Julia sets

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Abstract If the Green function g_E of a compact set $E \subset \mathbb{C}$ is Hölder continuous, then the Hölder exponent of the set E is the supremum over all such α that

$$|g_E(z) - g_E(w)| \leq M|z - w|^\alpha, \quad z, w \in \mathbb{C}.$$

We give a lower bound for the Hölder exponent of the Julia sets of polynomials. In particular, we show that there exist totally disconnected planar sets with the Hölder exponent greater than $1/2$ as well as fat continua with the boundary nowhere smooth and with the Hölder exponent as close to 1 as we wish.

Keywords Complex Green function · Hölder continuity · Polynomials · Iteration · Complex dynamics · Julia sets

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1 Introduction

The continuity of the Green function of a planar compact set (which is actually the Green function of the bounded component of the complement of the compact set to the Riemann sphere) was always the object of an intensive research. In what follows, we investigate only compact planar sets with Hölder continuous Green functions, that is, we take such a compact set $E \subset \mathbb{C}$ that there exist positive constants M and α with

$$|g_E(z) - g_E(w)| \leq M|z - w|^\alpha, \quad z, w \in \mathbb{C}. \quad (1.1)$$

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In this situation, there arises another problem: to find estimates for the exponent α . To this end, we define the *Hölder exponent* of the set E to be

$$\lambda(E) := \sup\{\alpha : (1.1) \text{ holds with the exponent } \alpha\}$$

and look for its estimates.

In this note, we deal with polynomial Julia sets, that is, the Julia sets associated with polynomials of degree $d \geq 2$. They are compact and regular. The proof of the Hölder continuity of their Green functions was provided by Sibony (see [6, Chapter VIII, Theorem 3.2] and the comments near it).

The main result of this paper runs as follows

Main Theorem 1.1 *Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. Denote by $J[p]$ the Julia set of p and by $K[p]$ the filled-in Julia set of p . Let F be a convex compact set containing the set $K[p]$. Then,*

$$\lambda(J[p]) = \lambda(K[p]) \geq \frac{\log d}{\log(\max\{|p'(z)| : z \in F\})}.$$

The most known and investigated are Julia sets $J_c \subset \mathbb{C}$ and their filled-in counterparts K_c defined by the quadratic polynomials of the form $z \mapsto z^2 + c$, $c \in \mathbb{C}$. We will obtain

$$\textbf{Corollary 1.2} \quad \forall c \in \mathbb{C} : \quad \lambda(J_c) = \lambda(K_c) \geq \frac{\log 2}{\log(1 + \sqrt{1 + 4|c|})}.$$

Note that this estimate is sharp in the following sense. Namely $\lambda(K_0) \geq 1$ and $\lambda(K_{-2}) \geq 1/2$ in view of Corollary 1.2. Since $K_0 = \{z \in \mathbb{C} : |z| \leq 1\}$, it is known that $\lambda(K_0) = 1$. On the other hand, $K_{-2} = [-2, 2]$, and hence it is known that $\lambda(K_{-2}) = 1/2$. The estimate given in Theorem 1.2 is especially worthwhile when K_c is totally disconnected or if $|c|$ is small (small enough to give the estimate greater than $1/2$).

What is already known about the Hölder exponent $\lambda(E)$ of a compact set $E \subset \mathbb{C}$ (with the Hölder continuous Green function)? We have always (see [16, Remark 3.7])

$$\lambda(E) \leq 1. \tag{1.2}$$

Furthermore, we have also (see [15, Theorem 2])

$$\lambda(E) \leq \dim_H(E), \tag{1.3}$$

where $\dim_H(E)$ denotes the Hausdorff dimension of the set E .

On the other hand, if $E \subset \mathbb{C}$ is a non-degenerate continuum, then

$$\lambda(E) \geq 1/2 \tag{1.4}$$

(by a continuum, we mean a connected compact set). This inequality remains true if $E \subset \mathbb{C}$ is a compact set satisfying the following condition: there exists a positive constant δ such that each point $a \in E$ belongs to a subset F of E , which is a continuum of the diameter greater than δ . This result follows from Leja Polynomial Lemma (cf. [11]).

Finally, let us mention one of the most important applications of the Hölder exponent in the theory of polynomial inequalities. Namely, if the Green function g_E is Hölder continuous, then E is a *Markov set* (for the background see e.g. [13]), that is, there exist positive constants M, m such that for all polynomials p

$$\|p'\|_E \leq M(\deg p)^m \|p\|_E. \tag{1.5}$$

For such sets, the *Markov exponent*,

$$m(E) := \inf\{m > 0 : (1.5) \text{ holds with the exponent } m\},$$

was defined in [2] (for its importance, good references and some new results see [1]). The following relation is true:

$$\lambda(E) \leq \frac{1}{m(E)}. \quad (1.6)$$

By a *fat* set, we mean a compact set that coincides with the closure of its interior. Let us mention one important well-known fact.

Proposition 1.3 *If E is a fat planar continuum with a piecewise analytic boundary, then $\lambda(E) = 1$.*

Proof It is known that $m(E) = 1 \iff \lambda(E) = 1$ if E is a planar compact set with Hölder continuous Green function. And $m(E) = 1$ for any fat planar continuum E with a piecewise analytic boundary (see [17]). $\square$

Note that while obtaining bounds for the Hölder exponent, we gain also estimates for the Markov exponent by (1.6) and for the Hausdorff dimension of the set by (1.3).

This paper was stimulated by Baran, who asked whether it is possible to find the estimates for the filled-in Julia sets, and also by the paper [15], where some estimates for the Hölder exponent of the Cantor ternary sets were given. The Cantor ternary sets can be viewed (and actually the authors of [15] use this fact) as attractors of iterated function systems. Julia sets can actually be obtained in a similar way even if the methods for getting the estimates are different.

2 Preliminaries

Put $D(a, r) := \{z \in \mathbb{C} : |z - a| \leq r\}$ for $a \in \mathbb{C}$ and $r > 0$. For a compact set $F \subset \mathbb{C}$ and a positive number $r > 0$ define

$$F_r := \{z \in \mathbb{C} : \text{dist}(z, F) \leq r\} = \bigcup_{a \in F} D(a, r).$$

The Green function g_E of the compact set $E \subset \mathbb{C}$ (of positive logarithmic capacity) can be defined in the same way as the function V_E in [10, Chapter 5] or as the Green's function of the unbounded component of $\mathbb{C}_\infty \setminus E$ with pole at ∞ and extended to be zero elsewhere on $\mathbb{C}$ (see e.g. [14, Chapter 4.4]). If $h : \mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$ for some complex number b and non-zero complex number a , then $g_E \circ h = g_{h^{-1}(E)}$ (see e.g. [10, Theorem 5.3.1]).

If a set $E \subset \mathbb{C}$ is compact, its polynomially convex hull is denoted by $\hat{E}$. We have $g_E \equiv g_{\hat{E}}$ by definition (since both can be defined as the Green function of the complement of $\hat{E}$ to $\mathbb{C}_\infty$). Hence, $\lambda(E) = \lambda(\hat{E})$. Moreover, by the harmonicity of the Green function in order to prove the Hölder continuity (with the exponent $\alpha > 0$) of the Green function of a compact set E , it suffices to show that there exist constants $\varrho, M > 0$ such that the following inequality holds

$$g_E(z) \leq M(\text{dist}(z, E))^\alpha, \quad \text{if } \text{dist}(z, E) \leq \varrho \quad (2.1)$$

(cf. [6, Chapter VIII, the proof of Theorem 3.2]). Inequality (2.1) is called the *Hölder Continuity Property*.

Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. The Julia set $J[p]$ is usually defined in the terms of (non-)normality of the family $\{p^n\}$ (see e.g. [3] or [6]). We will use another equivalent way. We define first the filled-in Julia set associated with p to be

$$K[p] := \{z \in \mathbb{C} : (p^n(z))_{n=1}^{\infty} \text{ is bounded}\}$$

and then $J[p]$ to be its boundary. First of all, we have $\widehat{J[p]} = K[p]$. The Julia set $J[p]$ is non-empty; moreover, it is perfect and uncountable (see [3, Theorem 4.2.4]). Furthermore, if $J[p]$ is disconnected, then it has uncountably many components and each point of $J[p]$ is an accumulation point of infinitely many distinct components of $J[p]$ ([3, Theorem 5.7.1]). Both sets are totally invariant under p , that is, $p(J[p]) = J[p] = p^{-1}(J[p])$ and $p(K[p]) = K[p] = p^{-1}(K[p])$. The following transformation formula

$$g_{K[p]} = g_{p^{-1}(K[p])} = \frac{1}{d} g_{K[p]} \circ p. \quad (2.2)$$

can be obtained, for example, from [10, Theorem 5.3.1].

3 Proof of the main result

Proof (of **Main Theorem 1.1**) Fix $\varepsilon > 0$ and define

$$\begin{aligned} A(\varepsilon) &:= \max\{|p'(z)| : z \in F_\varepsilon\}; \\ \alpha(\varepsilon) &:= \frac{\log d}{\log A(\varepsilon)}; \\ M(\varepsilon) &:= \varepsilon^{-\alpha(\varepsilon)} \max\{(g_{K[p]} \circ p)(z) : z \in F_\varepsilon\}. \end{aligned}$$

Note that

$$(\forall j \in \{0, 1, \dots, n-1\} : p^j(w) \in F_\varepsilon) \implies |(p^n)'(w)| \leq A(\varepsilon)^n \quad (3.1)$$

for any positive integer n .

Since $A(\varepsilon)^{\alpha(\varepsilon)} = d$, we derive from (2.2) by induction that

$$g_{K[p]} = (A(\varepsilon)^{-n})^{\alpha(\varepsilon)} g_{K[p]} \circ p^n, \quad n \in \mathbb{N}. \quad (3.2)$$

Take $z \in F_\varepsilon \setminus K[p]$ and fix $z_0 \in K[p]$ with $|z - z_0| = \text{dist}(z, K[p])$. Then $p^n(z_0) \in K[p] \subset F$, $n \in \mathbb{N}$. Since $z \notin K[p]$, there exists a positive integer n_0 with $p^{n_0}(z) \notin F_\varepsilon$. Note that $[z_0, z] \subset F_\varepsilon$ since F is convex and define $I(n) := p^n([z_0, z])$, $n \in \mathbb{N}$. Take the smallest integer $m = m(z)$ such that $I(m) \not\subset F_\varepsilon$. Then $p^{m-1}(w) \in F_\varepsilon$ for every $w \in [z_0, z]$, there exists, however, a point $z_1 \in [z_0, z]$ with $p^m(z_1) \notin F_\varepsilon$. In view of (3.1) by the mean value property

$$\varepsilon \leq |p^m(z_1) - p^m(z_0)| \leq A(\varepsilon)^m |z_1 - z_0| \leq A(\varepsilon)^m \text{dist}(z, K[p]),$$

and consequently $\varepsilon^{-\alpha(\varepsilon)} (\text{dist}(z, K[p]))^{\alpha(\varepsilon)} \geq (A(\varepsilon)^{-m})^{\alpha(\varepsilon)}$. Hence, in view of (3.2)

$$g_{K[p]}(z) \leq \varepsilon^{-\alpha(\varepsilon)} (g_{K[p]} \circ p)(p^{m-1}(z)) (\text{dist}(z, K[p]))^{\alpha(\varepsilon)}$$

and the definition of $M(\varepsilon)$ yields

$$g_{K[p]}(z) \leq M(\varepsilon) (\text{dist}(z, K[p]))^{\alpha(\varepsilon)}.$$

Note that in the last inequality, there is no dependence on $m = m(z)$.

We have proved thus that $\lambda(K[p]) \geq \alpha(\varepsilon)$. It suffices to note now that

$$\alpha(\varepsilon) \nearrow \alpha := \frac{\log d}{\log(\max\{|p'(z)| : z \in F\})}, \quad \text{as } \varepsilon \searrow 0.$$

□

Remark 3.1 Note that it does not follow from the proof of Main Theorem 1.1 that $g_{K[p]}$ is Hölder continuous with the exponent α , since if $\varepsilon \rightarrow 0$, then $M(\varepsilon)$ tends to infinity.

4 Quadratic polynomials

We start with the announced filled-in Julia sets for quadratic polynomials. For the background see, for example, [6]. It is well known that it suffices to consider the polynomials of the form $Q_c : z \mapsto z^2 + c$, $c \in \mathbb{C}$, since any other quadratic polynomial is conjugated to one of this type. Let J_c denote the Julia set of Q_c and K_c – the filled-in one. We will now prove Corollary 1.2.

Proof (of Corollary 1.2). It is well known that $J_c \subset D(0, r_c)$ with

$$r_c := \frac{1}{2}(1 + \sqrt{1 + 4|c|})$$

(see [3, Exercise 1.6.4], the explicit proof is given in [4]). Hence, $K_c = \widehat{J}_c \subset D(0, r_c)$ too and Main Theorem 1.1 applied to $F = D(0, r_c)$ yields

$$\lambda(K_c) \geq \frac{\log 2}{\log(2r_c)}.$$

□

Remark 4.1 Note that for $c \leq 0$, we take here the smallest possible value of r with $K_c \subset D(0, r)$ since $-r_c, r_c \in K_c$. Instead of $D(0, r_c)$ we could take F from Main Theorem 1.1 to be the ellipse

$$E(c) = \left\{ x + iy : \frac{x^2}{r_c^2} + \frac{y^2}{r_c + c} \leq 1 \right\}$$

for $c \in [-2, 0]$ and the interval $[-r_c, r_c]$ for $c < -2$ (see [4]) but the result would not change, since

$$\begin{aligned} \max\{|Q'_c(z)| : z \in D(0, r_c)\} &= \max\{|Q'_c(z)| : z \in [-r_c, r_c]\} = 2r_c \\ &= \max\{|Q'_c(z)| : z \in E(c)\} \end{aligned}$$

(here, the last equality is only considered for $c \in [-2, 0]$, the other for all c real).

We shall start with the mentioned special cases

Example 4.2 $K_0 = D(0, 1)$ and $\lambda(K_0) = 1$.

Proof Since $c = 0$, Corollary 1.2 yields $\lambda(K_0) \geq \frac{\log 2}{\log 2} = 1$. On the other hand, we have (1.2) or Proposition 1.3. □

Now we turn to the other special case: the interval.

Example 4.3 If $|c| = 2$, then $\lambda(K_c) \geq 1/2$. In particular, $\lambda([-2, 2]) = \lambda(K_{-2}) = 1/2$.

Proof It follows from Corollary 1.2 that $\lambda(K_c) \geq \frac{\log 2}{\log 4} = \frac{1}{2}$. By the famous Markov inequality, it is known that $m([-2, 2]) = 2$, hence $\lambda(K_{-2}) = 1/2$ by (1.6). $\square$

Remark 4.4 Note that K_{-2} is connected but if $|c| = 2$ and $c \neq -2$, then K_c is totally disconnected. However, its Hölder exponent satisfies the same inequality as all those of connected sets.

Before we list the estimates for some other values of c , let us recall some facts. The Mandelbrot set can be defined in two ways

$$\mathcal{M} := \{c \in \mathbb{C} : Q_c^n(0) \not\rightarrow \infty\} = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

We have $D(0, \frac{1}{4}) \subset \mathcal{M}$, thus in particular if $|c| \leq 1/4$, then the filled-in Julia set K_c is connected. What is more if c is not real and lies in the interior of the main cardioid C_1 (see [8, Fig.17.4]), the Julia set J_c is a simple closed curve that contains no smooth (i.e., of class C^1) arcs (see [7, Proposition 3.6.3 and the Remark after it]). Therefore, if we get the bound greater than $1/2$ of the Hölder exponent, then it is also interesting for such c .

Corollary 4.5 If $|c| < 2$, then $\lambda(K_c) > 1/2$. If $|c| < 2 - \sqrt{2}$, then $\lambda(K_c) > 2/3$. In particular if $c \in \{w \in \mathbb{C} \setminus \mathbb{R} : |w| < 1/4\}$, then J_c is a simple closed curve which contains no smooth arcs and $\lambda(J_c) > 2/3$.

Proof This is the straightforward consequence of Corollary 1.2 and the facts above, since $|c| < 1/4$ yields $c \in C_1$. $\square$

The following result is especially noteworthy in comparison with Proposition 1.3.

Corollary 4.6 There exist fat continua with the boundaries that contain no smooth arcs and with the Hölder exponents as close to 1 as we wish.

Proof The lower bound in Corollary 1.2 depends continuously on $|c|$ and tends to 1 when $|c| \rightarrow 0$. Therefore, it suffices to take, for example, K_{it} with $t \in (0, 1/4)$ small enough. $\square$

The estimate given here is also remarkable for all c outside the Mandelbrot set, since then $J_c = K_c$ is totally disconnected and we cannot use the bound (1.4).

Let us see first one example

Example 4.7 If $|c| = \frac{15}{4}$, then $\lambda(K_c) \geq \frac{\log 2}{\log 5} > 0,43$ and K_c is totally disconnected. $\square$

But this bound is not that impressive, since it is smaller than $1/2$. Recall now that if $c > 1/4$, then $J_c = K_c$ is totally disconnected and has Lebesgue measure 0 [5, Theorem 12.1].

Corollary 4.8 There exist totally disconnected sets with the Hölder exponent greater than $3/5$ (in particular greater than $1/2$).

Moreover, for all $\varepsilon > 0$, there exist totally disconnected sets with the Hölder exponents bigger than

$$\frac{\log 2}{\log(1 + \sqrt{2})} - \varepsilon.$$

Remark 4.9 Just let us recall that $(\log 2)/(\log(1 + \sqrt{2})) > 0,785$.

Proof For the first assertion, it suffices to take the sets $K_c = J_c$ for $c \in (1/4, 2\sqrt[3]{2} - \sqrt[3]{4}]$ (in particular, it is enough to take $c \in (0, 25; 0, 925]$).

The second statement follows from the continuity of the lower bound in Corollary 1.2 with respect to $c \in [0, 25; \infty)$ and the fact that $\lambda(K_{0,25}) \geq (\log 2)/(\log(1 + \sqrt{2}))$ (also obtained from Corollary 1.2). $\square$

We refer the readers to see the pictures of a few of such sets: $K_{0,251}$ is in [6, Chapter VIII.1, Fig.5], $K_{0,255}$ is in [8, Fig.16.10] and $K_{0,5}$ in [8, Fig.16.13].

One can also visit, for example, [9] for pictures of Julia sets for polynomials Q_c , $c \in \mathbb{C}$.

5 Cubic polynomials

Every cubic polynomial can be conjugated to one of the form

$$C_{a,b} : \mathbb{C} \ni z \mapsto z^3 + az + b \in \mathbb{C}, \quad a, b \in \mathbb{C},$$

thus it suffices to consider cubic polynomials of this type. Denote by $J_{a,b}$ and $K_{a,b}$ the Julia set and the filled-in Julia set of $C_{a,b}$.

Corollary 5.1 *Let a, b be complex numbers. Then,*

$$\lambda(J_{a,b}) = \lambda(K_{a,b}) \geq \frac{\log 3}{\log [3 \max(|b|^2, |a| + 2)) + |a|]}.$$

Proof From the Escape Criterion for Cubics [8, Chapter 18.2] conclude that $K_{a,b} \subset D(0, \max(|b|, \sqrt{|a| + 2}))$ and apply Main Theorem 1.1. $\square$

We will improve this result in some special cases.

Proposition 5.2 *Let b be a complex number. Then,*

$$\lambda(K_{0,b}) \geq \frac{\log 3}{\log \left(3 \left(1 + \sqrt[3]{|b|} \right)^2 \right)}.$$

Proof Define $r := 1 + \sqrt[3]{|b|}$. For any $\varepsilon > 0$ put $r_\varepsilon := \sqrt{1 + \varepsilon} + \sqrt[3]{|b|}$. Note that $r_\varepsilon > 1$ and $r_\varepsilon \searrow r$ if $\varepsilon \searrow 0$.

Fix an $\varepsilon > 0$ and consider the function $h(t) = t^3 - (1 + \varepsilon)t - |b|$, which is increasing in $[\sqrt{(1 + \varepsilon)/3}, \infty)$. Since $r_\varepsilon > \sqrt{(1 + \varepsilon)/3}$ and $h(r_\varepsilon) > 0$, we have

$$|C_{0,b}(z)| \geq |z|^3 - |b| > (1 + \varepsilon)|z|,$$

whenever $|z| > r_\varepsilon$. We conclude that $K_{0,b} \subset D(0, r)$.

Main Theorem 1.1 yields

$$\lambda(K_{0,b}) \geq \frac{\log 3}{\log(3r^2)}.$$

$\square$

The estimate from Proposition 5.1 is better than the one from Corollary 5.1 if $|b| < 5\sqrt{2} - 7$ or $|b| \geq 2\sqrt{2}$. Recall that if $b \in [-2\sqrt{3}/9, 2\sqrt{3}/9]$ then $J_{0,b}$ is connected, and for other real values of b , the Julia set $J_{0,b}$ is totally disconnected and has Lebesgue measure 0 [5, Theorem 13.3].

The following proposition improves the result from Corollary 5.1 if a is real and $b = 0$. Recall here that if $a \in [-3, 3]$, then $J_{a,0}$ is connected, and for other real values of a , the Julia set $J_{a,0}$ is totally disconnected and has linear measure 0 [5, Theorem 13.1].

Proposition 5.3 *Let $a \in \mathbb{R}$. Then,*

$$\lambda(K_{a,0}) \geq \frac{\log 3}{\log(3 + 2|a|)}.$$

Proof In view of [12, Theorem 3.1 and Corollary 3.2]

$$\begin{aligned} a \leq -3 &\implies J_{a,0} \subset E(a) := \left[-\sqrt{1-a}, \sqrt{1-a}\right] \\ -3 < a \leq 0 &\implies J_{a,0} \subset E(a) := \left\{x + iy : \frac{x^2}{1-a} + \frac{y^2}{1+\frac{a}{3}} \leq 1\right\} \\ 0 \leq a < 3 &\implies J_{a,0} \subset E(a) := \left\{x + iy : \frac{x^2}{1-\frac{a}{3}} + \frac{y^2}{1+a} \leq 1\right\} \\ a \geq 3 &\implies J_{a,0} \subset E(a) := \left[-i\sqrt{1+a}, i\sqrt{1+a}\right]. \end{aligned}$$

Apply Main Theorem 1.1. We have $\max\{|3z^2 + a| : z \in E(a)\} = 3 + 2|a|$. $\square$

Some pictures of cubic Julia sets can be find in [12].

Let us finally count some other examples

Example 5.4 $J[z \mapsto z^2 - z^3/9]$ has infinitely many non-degenerate components and $\lambda(J[z \mapsto z^2 - z^3/9]) > 3/8$.

Proof The first statement is from [3, Section 11.4]. The polynomial $z \mapsto z^2 - z^3/9$ is conjugated to $C_{3,-i}$, so it suffices to apply Corollary 5.1. $\square$

Example 5.5 $J[z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)]$ has infinitely many non-degenerate components and $\lambda(J[z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)]) > 3/8$.

Proof The first statement is from [3, Section 11.5]. The polynomial

$$z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)$$

is conjugated to $C_{-3\sqrt{3}/2, \sqrt{6}\sqrt[4]{3}/2}$, so it suffices to apply Corollary 5.1. $\square$

Example 5.6 $L := J[z \mapsto z^3 - 12z^2 + 36z]$ is totally disconnected and $\lambda(L) > 2/7$.

Proof The first statement is from [3, Section 11.5]. The polynomial $z \mapsto z^3 - 12z^2 + 36z$ is conjugated to $C_{-12,12}$, so it suffices to apply Corollary 5.1. $\square$

6 Other polynomials

Let us note first

Remark 6.1 $K[z \mapsto z^k] = D(0, 1)$ for any $k \in \{2, 3, 4, \dots\}$. By Main Theorem 1.1 for $r = 1$ and $K = K[z \mapsto z^k]$, we obtain once again $\lambda(D(0, 1)) \geq \frac{\log k}{\log k} = 1$.

Let us also recall that $\forall n \in \mathbb{N} : J[p] = J[p^n]$ ([7, Proposition 3.5.4]), thus in the previous sections, we already considered the Julia sets of some polynomials of higher degrees too.

We propose some more examples whose proofs we omit. They are more or less based on the same argument for a quadratic function.

Proposition 6.2

$$\forall k \in \mathbb{N} \forall c \in \mathbb{C} : \lambda(K[z \mapsto z^{2k} + c]) \geq \frac{\log(2k)}{\log\left(2k \left(\frac{1+\sqrt{1+4|c|}}{2}\right)^{\frac{2k-1}{k}}\right)}.$$

Proposition 6.3 *If $k \in \mathbb{N}$ and $c \in \mathbb{C}$, then $\lambda(K[z \mapsto z^k(z^2 + c)]) \geq$*

$$\geq \frac{\log(2+k)}{\log\left(\left(\frac{1+\sqrt{1+4|c|}}{2}\right)^{k-1} \left((k+2)\frac{1+\sqrt{1+4|c|}}{2} + 2(k+1)|c|\right)\right)}.$$

Proposition 6.4 *If $k \in \mathbb{N}$ and $c_1, \dots, c_k \in \mathbb{C}$ with $a := \max(|c_1|, \dots, |c_k|)$, then*

$$\lambda\left(K\left[z \mapsto \prod_{j=1}^k (z^2 + c_j)\right]\right) \geq \frac{\log(2k)}{\log(k(1+4a + (1+2a)\sqrt{1+4a}))}.$$

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