A semilinear parabolic system with coupling variable exponents

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Abstract This paper deals with semilinear parabolic equations coupled via variable sources, subject to the homogeneous Dirichlet condition in a bounded domain. Since the variable exponents in the sources are just assumed to be positive, the non-linearities may be non-Lipschitz. We establish the existence–uniqueness with comparison principle of local solutions to the regularized problem at first, and then consider the maximal solutions of the original problem as the limits of the solutions of the regularized problem. Some criteria are established for distinguishing global and non-global solutions of the problem, dependent or independent of initial data. Especially, we prove a Fujita type conclusion that the solutions blow up for any non-trivial initial data under certain assumptions on the variable sources and the size of the domain.

Keywords Variable exponents · Semilinear parabolic system · Maximal solutions · Blow-up · Global solutions · Fujita type conclusion

Mathematics Subject Classification (2000) 35K51 · 35K55

1 Introduction

In this paper, we consider the semilinear parabolic system coupled via variable sources

$$\begin{cases} u_t(x, t) = \Delta u(x, t) + v^{p(x)}(x, t), & (x, t) \in \Omega \times (0, T) = \Omega_T, \\ v_t(x, t) = \Delta v(x, t) + u^{q(x)}(x, t), & (x, t) \in \Omega \times (0, T) = \Omega_T, \\ u(x, t) = v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T) = S_T, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \quad (1.1)$$

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where Ω is a bounded smooth domain in $\mathbb{R}^N$, $p(x), q(x), u_0(x), v_0(x) \in C(\overline{\Omega})$, with $u_0(x), v_0(x) \geq, \neq 0$, compatible on $\partial\Omega$, $p(x), q(x) > 0$, and moreover,

$$\sup_{x \in \Omega} p(x) \cdot \sup_{x \in \Omega} q(x) = \sup_{x \in \Omega} \{p(x)q(x)\}. \quad (1.2)$$

In particular, (1.2) is automatically satisfied whenever one of the $p(x)$ and $q(x)$ are constant. Denote

$$p_1 = \sup_{x \in \Omega} p(x), \quad p_2 = \inf_{x \in \Omega} p(x), \quad q_1 = \sup_{x \in \Omega} q(x), \quad q_2 = \inf_{x \in \Omega} q(x).$$

If $p(x) = q(x), u_0(x) = v_0(x)$, then (1.1) is reduced to the scalar problem

$$\begin{cases} u_t(x, t) = \Delta u(x, t) + u^{p(x)}(x, t), & (x, t) \in \Omega_T, \\ u(x, t) = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \quad (1.3)$$

which was studied by Ferreira, de Pablo, Pérez-Llanos and Rossi recently in [4]. The general problem with $p(x) \equiv p$ and $q(x) \equiv q$ in (1.1) was considered by Escobedo and Herrero [3] (see also [6, 7] for $p, q > 1$).

The aim of this paper is to find criteria for distinguishing global and non-global solutions of (1.1) with the variable exponent functions $p(x), q(x)$. In particular, we will prove a Fujita type conclusion for (1.1) that the solutions blow up for any non-trivial initial data under certain conditions on the variable exponent functions and the size of the domain.

2 Preliminaries

We introduce in this section the local existence of solutions to (1.1) with the related comparison principle as preliminaries. Consider the regularized problem at first.

Lemma 2.1 *There exists $T > 0$ such that the regularized problem*

$$\begin{cases} u_{\varepsilon t}(x, t) = \Delta u_{\varepsilon}(x, t) + v_{\varepsilon}^{p(x)}(x, t), & (x, t) \in \Omega_T, \\ v_{\varepsilon t}(x, t) = \Delta v_{\varepsilon}(x, t) + u_{\varepsilon}^{q(x)}(x, t), & (x, t) \in \Omega_T, \\ u_{\varepsilon}(x, t) = v_{\varepsilon}(x, t) = \varepsilon, & (x, t) \in S_T, \\ u_{\varepsilon}(x, 0) = u_0(x) + \varepsilon, \quad v_{\varepsilon}(x, 0) = v_0(x) + \varepsilon, & x \in \overline{\Omega}. \end{cases} \quad (2.1)$$

has a unique non-negative solution $(u_{\varepsilon}, v_{\varepsilon}) \in (C^{2,1}(\Omega \times [0, T]))^2$, for which the comparison principle is valid.

Proof The arguments are standard (see, e.g., [1, 2]). Construct

$$\begin{cases} f_{\varepsilon}(x, s) = g_{\varepsilon}(x, s) = 0, & s \leq 0, \quad x \in \Omega, \\ f_{\varepsilon}(x, s) = \varepsilon^{p(x)-1}s, \quad g_{\varepsilon}(x, s) = \varepsilon^{q(x)-1}s, & 0 < s < \varepsilon, \quad x \in \Omega, \\ f_{\varepsilon}(x, s) = s^{p(x)}, \quad g_{\varepsilon}(x, s) = s^{q(x)}, & s \geq \varepsilon, \quad x \in \Omega. \end{cases} \quad (2.2)$$

Obviously, f_{ε} and g_{ε} are Lipschitz continuous with respect to s uniformly for $(x, s) \in \Omega \times (0, S_0)$.

We claim that the regularized problem (2.1) is equivalent to

$$\begin{cases} u_{\varepsilon t}(x, t) = \Delta u_{\varepsilon}(x, t) + f_{\varepsilon}(x, v_{\varepsilon}(x, t)), & (x, t) \in \Omega_T, \\ v_{\varepsilon t}(x, t) = \Delta v_{\varepsilon}(x, t) + g_{\varepsilon}(x, u_{\varepsilon}(x, t)), & (x, t) \in \Omega_T, \\ u_{\varepsilon}(x, t) = v_{\varepsilon}(x, t) = \varepsilon, & (x, t) \in S_T, \\ u_{\varepsilon}(x, 0) = u_0(x) + \varepsilon, \quad v_{\varepsilon}(x, 0) = v_0(x) + \varepsilon, & x \in \overline{\Omega}. \end{cases} \quad (2.3)$$

Indeed, it follows from either (2.1) or (2.3) that

$$\begin{cases} u_{\varepsilon t}(x, t) \geq \Delta u_{\varepsilon}(x, t), & (x, t) \in \Omega_T, \\ v_{\varepsilon t}(x, t) \geq \Delta v_{\varepsilon}(x, t), & (x, t) \in \Omega_T, \\ u_{\varepsilon}(x, t) = v_{\varepsilon}(x, t) = \varepsilon, & (x, t) \in S_T, \\ u_{\varepsilon}(x, 0), v_{\varepsilon}(x, 0) \geq \varepsilon, & x \in \overline{\Omega}, \end{cases}$$

and hence $u_{\varepsilon} \geq \varepsilon$, $v_{\varepsilon} \geq \varepsilon$ by the comparison principle. Consequently,

$$f_{\varepsilon}(x, v_{\varepsilon}(x, t)) = v_{\varepsilon}^{p(x)}(x, t), \quad g_{\varepsilon}(x, u_{\varepsilon}(x, t)) = u_{\varepsilon}^{q(x)}(x, t).$$

With Lipschitz f_{ε} and g_{ε} in (2.2), the existence and uniqueness of local non-negative solutions to (2.3) are well known (see, e.g., [5]).

Next, we show the comparison principle for the solution $(u_{\varepsilon}, v_{\varepsilon})$ of (2.3). Suppose $(\underline{u}, \underline{v}) \in (C^{2,1}(\Omega \times [0, T]))^2$ is a non-negative subsolution of (2.1), namely,

$$\begin{cases} \underline{u}_t(x, t) \leq \Delta \underline{u}(x, t) + \underline{v}^{p(x)}(x, t), & (x, t) \in \Omega_T, \\ \underline{v}_t(x, t) \leq \Delta \underline{v}(x, t) + \underline{u}^{q(x)}(x, t), & (x, t) \in \Omega_T, \\ \underline{u}(x, t) \leq \varepsilon, \quad \underline{v}(x, t) \leq \varepsilon, & (x, t) \in S_T, \\ \underline{u}(x, 0) \leq u_0(x) + \varepsilon, \quad \underline{v}(x, 0) \leq v_0(x) + \varepsilon, & x \in \overline{\Omega}. \end{cases} \quad (2.4)$$

For $T' \in (0, T)$, denote $\mathbb{A} = \{(x, t) \in \Omega \times [0, T'] : \underline{u} \geq u_{\varepsilon}\}$.

If $(x, t) \in \mathbb{A}$, we have $\underline{u} \geq u_{\varepsilon} \geq \varepsilon$, and thus

$$\varepsilon \leq \theta \underline{u} + (1 - \theta)u_{\varepsilon} \leq \max_{\overline{\Omega}_{T'}} \underline{u} + \max_{\overline{\Omega}_{T'}} u_{\varepsilon}, \quad \theta \in [0, 1].$$

As a result,

$$u_{\varepsilon}^{q(x)} - \underline{u}^{q(x)} = q(x)[\theta \underline{u} + (1 - \theta)u_{\varepsilon}]^{q(x)-1}(u_{\varepsilon} - \underline{u}) \geq C_1(u_{\varepsilon} - \underline{u}), \quad C_1 \geq 0.$$

If $(x, t) \in \Omega \times [0, T'] \setminus \mathbb{A}$, then $\underline{u} < u_{\varepsilon}$, and hence

$$u_{\varepsilon}^{q(x)} - \underline{u}^{q(x)} \geq 0.$$

In summary, we conclude with $K_1(x, t) := C_1 \chi_{\mathbb{A}} \geq 0$ that

$$u_{\varepsilon}^{q(x)} - \underline{u}^{q(x)} \geq K_1(x, t)(u_{\varepsilon} - \underline{u}), \quad (x, t) \in \Omega \times [0, T'], \quad (2.5)$$

and similarly, there exists $K_2(x, t) \geq 0$ such that

$$v_{\varepsilon}^{p(x)} - \underline{v}^{p(x)} \geq K_2(x, t)(v_{\varepsilon} - \underline{v}), \quad (x, t) \in \Omega \times [0, T']. \quad (2.6)$$

It follows from (2.1) and (2.4)–(2.6) that

$$\begin{cases} (u_{\varepsilon} - \underline{u})_t \geq \Delta(u_{\varepsilon} - \underline{u}) + K_2(x, t)(v_{\varepsilon} - \underline{v}), & (x, t) \in \Omega_{T'}, \\ (v_{\varepsilon} - \underline{v})_t \geq \Delta(v_{\varepsilon} - \underline{v}) + K_1(x, t)(u_{\varepsilon} - \underline{u}), & (x, t) \in \Omega_{T'}, \\ (u_{\varepsilon} - \underline{u})(x, t), (v_{\varepsilon} - \underline{v})(x, t) \geq 0, & (x, t) \in S_{T'}, \\ (u_{\varepsilon} - \underline{u})(x, 0), (v_{\varepsilon} - \underline{v})(x, 0) \geq 0, & x \in \overline{\Omega}. \end{cases} \quad (2.7)$$

Consequently, $u_{\varepsilon} \geq \underline{u}$, $v_{\varepsilon} \geq \underline{v}$ hold for $(x, t) \in \Omega \times [0, T']$ by the comparison principle (see, e.g., Proposition 52.21 in [9]), and hence for $(x, t) \in \Omega \times [0, T)$. The case for supersolutions can be treated similarly (even more simply). We omit the details. $\square$

Now deal with the local existence of maximal solutions to (1.1) as well as the comparison principle associated.

Theorem 2.1 *The problem (1.1) possesses a non-negative maximal solution in Ω_T for some $T > 0$, denoted by (u, v) . Moreover, if $(\underline{u}, \underline{v})$ is a subsolution of (1.1) and $(\bar{u}, \bar{v})$ is a supersolution of (1.1) with $\bar{u}, \bar{v} \geq \varepsilon_0 > 0$ and $(\bar{u} - u)|_{t=0}, (\bar{v} - v)|_{t=0} > 0$, then $\underline{u} \leq u \leq \bar{u}, \underline{v} \leq v \leq \bar{v}$ in Ω_T .*

Proof For any $\varepsilon > 0$, let $(u_\varepsilon, v_\varepsilon)$ be the unique solution of the regularized problem (2.1). The comparison principle in Lemma 2.1 implies the monotonicity of $(u_\varepsilon, v_\varepsilon)$. Denote by (u, v) the limit of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \rightarrow 0$. Then (u, v) is a solution of (1.1) (refer to Theorem 2.1 in [2]).

The comparison principle for the regularized problem in Lemma 2.1 implies that the above (u, v) just is the maximal solution of (1.1). Now illustrate the comparison principle for the maximal solution. For $(\underline{u}, \underline{v})$, the subsolution of (1.1), we know by the comparison principle in Lemma 2.1 that $\underline{u} \leq u_\varepsilon, \underline{v} \leq v_\varepsilon$, and hence $\underline{u} \leq u, \underline{v} \leq v$ by letting $\varepsilon \rightarrow 0$. Moreover, since $\bar{u}, \bar{v} \geq \varepsilon_0 > 0$ with $(\bar{u} - u)|_{t=0}, (\bar{v} - v)|_{t=0} > 0$, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $u_\varepsilon \leq \bar{u}, v_\varepsilon \leq \bar{v}$ for $\varepsilon \in (0, \varepsilon_1)$ by Lemma 2.1, and hence $u \leq \bar{u}, v \leq \bar{v}$. $\square$

For simplicity, in the rest of the paper, the solution of (1.1) always means the maximal solution of (1.1).

3 Global existence

Now, we deal with the global solutions to (1.1). The arguments used in this section will follow those in [3] (for the case of small initial data) and [2] (for the case of general initial data). For any function $f \in L^1(\Omega)$, denote by $T(t)f$ the unique solution of

$$\begin{cases} u_t(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = f(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases} \quad (3.1)$$

It is known that the solutions of (1.1) satisfy the following integral system (cf, e.g., [8])

$$\begin{cases} u(x, t) = T(t)u_0 + \int_0^t T(t-s)(v^{p(x)}(x, s))ds, & (x, t) \in \Omega_T, \\ v(x, t) = T(t)v_0 + \int_0^t T(t-s)(u^{q(x)}(x, s))ds, & (x, t) \in \Omega_T. \end{cases} \quad (3.2)$$

At first, suppose $p_1 q_1 \leq 1$, which corresponds to the case where the solutions are global under any initial data. Without loss of generality, assume $p_1 \leq 1$.

Next introduce a preliminary lemma for the global extension of solutions.

Lemma 3.1 *Let (u, v) be a solution of (1.1) in Ω_T . Then the function*

$$w(x, t) = v(x, t) + T(t)u_0^{\frac{1}{p_1}} + T(t)u_0^{\frac{1}{p_2}}$$

satisfies

$$w_t - \Delta w \leq C(1+t)^{q_1} \sum_{i,j=1}^2 w^{p_i q_j} \text{ in } \Omega_T$$

with $C = 2^{(2-p_2)q_1}$.

Proof By (3.2), u satisfies

$$u(x, t) = T(t)u_0 + \int_0^t T(t-s) \left[T(s)v_0 + \int_0^s T(s-r)u^{q(x)}(r)dr \right]^{p(x)} ds.$$

Due to $p_2 \leq p_1 \leq 1$, we obtain by Jensen's inequality that

$$\begin{aligned} u(x, t) &\leq T(t)u_0 + \sum_{i=1}^2 \int_0^t T(t-s) \left[T(s)v_0 + \int_0^s T(s-r)u^{q(x)}(r)dr \right]^{p_i} ds \\ &\leq T(t)u_0 + \sum_{i=1}^2 \int_0^t \left[T(t-s)T(s)v_0 + T(t-s) \int_0^s T(s-r)u^{q(x)}(r)dr \right]^{p_i} ds \\ &\leq T(t)u_0 + \sum_{i=1}^2 t^{1-p_i} \left(\int_0^t \left[T(t)v_0 + T(t-s) \int_0^s T(s-r)u^{q(x)}(r)dr \right] ds \right)^{p_i} \\ &= T(t)u_0 + \sum_{i=1}^2 t^{1-p_i} \left(tT(t)v_0 + \int_0^t \int_r^t T(t-r)u^{q(x)}(r)dsdr \right)^{p_i} \\ &= T(t)u_0 + \sum_{i=1}^2 t^{1-p_i} \left(tT(t)v_0 + \int_0^t (t-r)T(t-r)u^{q(x)}(r)dr \right)^{p_i} \\ &\leq T(t)u_0 + \sum_{i=1}^2 t \left(T(t)v_0 + \int_0^t T(t-r)u^{q(x)}(r)dr \right)^{p_i} \\ &= T(t)u_0 + \sum_{i=1}^2 tv^{p_i} \\ &\leq (1+t)(2T(t)u_0 + \sum_{i=1}^2 v^{p_i}) \\ &\leq 2^{1-p_2}(1+t) \sum_{i=1}^2 (T(t)u_0^{\frac{1}{p_i}} + v)^{p_i} \\ &\leq 2^{1-p_2}(1+t) \sum_{i=1}^2 w^{p_i}, \end{aligned}$$

and hence

$$\begin{aligned} v_t - \Delta v = u^{q(x)} &\leq 2^{(1-p_2)q(x)}(1+t)^{q(x)}(w^{p_1} + w^{p_2})^{q(x)} \\ &\leq 2^{(1-p_2)q_1}(1+t)^{q_1} 2^{q(x)}(w^{p_1 q(x)} + w^{p_2 q(x)}) \\ &\leq C(1+t)^{q_1}(w^{p_1 q(x)} + w^{p_2 q(x)}) \\ &\leq C(1+t)^{q_1} \sum_{i,j=1}^2 w^{p_i q_j}. \end{aligned}$$

Since $w_t - \Delta w = v_t - \Delta v$, the conclusion follows. $\square$

Theorem 3.1 *If $p_1 q_1 \leq 1$, then every solution of (1.1) is global.*

Proof Consider the following auxiliary problem

$$\begin{cases} w_t(x, t) = \Delta w(x, t) + k \sum_{i,j=1}^2 w^{p_i q_j}, & (x, t) \in \Omega_T, \\ w(x, t) = 0, & (x, t) \in S_T, \\ w(x, 0) = v_0 + u_0^{\frac{1}{p_1}} + u_0^{\frac{1}{p_2}}, & x \in \bar{\Omega}, \end{cases} \quad (3.3)$$

By Lemma 3.1, we know that $\tilde{w}(x, t) = v(x, t) + T(t)u_0^{\frac{1}{p_1}} + T(t)u_0^{\frac{1}{p_2}}$ is a subsolution of (3.3) with $k = 2^{(2-p_2)q_1}(1+T)^{q_1}$. On the other hand, Me^{4kt} is a supersolution of (3.3) provided $M \geq (\|v_0 + u_0^{\frac{1}{p_1}} + u_0^{\frac{1}{p_2}}\|_{L^\infty} + 1)$. Thus,

$$\tilde{w}(x, t) = v(x, t) + T(t)u_0^{\frac{1}{p_1}} + T(t)u_0^{\frac{1}{p_2}} \leq Me^{4kt} \text{ in } \Omega_T,$$

which implies that w can be extended to $\Omega \times [0, \infty)$, and so do v and u by $0 \leq v \leq \tilde{w}$ with (3.2). $\square$

Next consider the global solutions under small initial data with $p_2 q_2 > 1$.

Theorem 3.2 *If $p_2 q_2 > 1$ with $u_0 + v_0$ small enough, then all solutions of (1.1) are globally bounded.*

Proof Let Θ be the solution of the elliptic problem

$$-\Delta \Theta(x, t) = 1 \text{ in } \Omega, \quad \Theta(x) = 0 \text{ on } \partial\Omega. \quad (3.4)$$

Obviously, $1 \leq \Theta + 1 \leq C$ for some $C > 0$. Since $p_2 q_2 > 1$, we can choose $b > 0$ small such that

$$\max \{b^{p_2 q_2 - 1} C^{p_2 q_2 + q_2}, bC, b^{p_2} C^{p_2 + 1}\} \leq 1.$$

Set $(\bar{u}, \bar{v}) = (a(\Theta + 1), b(\Theta + 1))$ with $a = (bC)^{p_2}$. Then

$$\begin{aligned} \bar{u}_t - \Delta \bar{u} - \bar{v}^{p(x)} &= a - (b(\Theta + 1))^{p(x)} \geq a - (bC)^{p(x)} \geq a - (bC)^{p_2} = 0, \\ \bar{v}_t - \Delta \bar{v} - \bar{u}^{q(x)} &= b - (a(\Theta + 1))^{q(x)} \geq b - (aC)^{q(x)} \geq b(1 - b^{p_2 q_2 - 1} C^{p_2 q_2 + q_2}) \geq 0. \end{aligned}$$

Consequently, $(\bar{u}, \bar{v})$ is a time-independent supersolution of (1.1) provided the initial data small that $u_0 < a(\Theta + 1)$, $v_0 < b(\Theta + 1)$. $\square$

If the size of the domain is small enough, then the condition $p_2 q_2 > 1$ in Theorem 3.2 is unnecessary for the global solutions. That is the following theorem.

Theorem 3.3 *If the domain is small that $\bar{\Omega} \subset B_r(x_0)$ with $r < \sqrt{2N}$, then all solutions of (1.1) are globally bounded under small initial data for all $p(x), q(x) \geq 0$.*

Proof Inspired by Theorem 3.8 in [4], construct

$$\omega(x) = \frac{2N - |x - x_0|^2}{2N}.$$

It is easy to check that (ω, ω) is a time-independent supersolution of (1.1) whenever $u_0, v_0 < \omega$. $\square$

4 Blow-up for large initial data

This section is devoted to non-global solutions with large initial data.

Theorem 4.1 *If $p_1 q_1 > 1$, then there exists a blow-up solution for (1.1).*

We begin with two lemmas without proof.

Lemma 4.1 ([3]) *For any non-negative and integrable functions f, g , it holds that*

$$\begin{aligned} T(t)(fg) &\leq \|f\|_\infty (T(t)g^r)^{1/r}, & r \geq 1, \\ T(t)(fg) &\geq \|f\|_\infty (T(t)g^r)^{1/r}, & r \leq 1. \end{aligned}$$

Lemma 4.2 (Corollary 3.2 of [4]) *Let μ be a positive measure in $B \subset \mathbb{R}^N$ with $\int_B d\mu = 1$, $f \in L^\gamma(B, d\mu)$, and $1 \leq \delta \leq p(x) \leq \gamma$ for $x \in B$. If $\int_B f d\mu \geq 1$, then*

$$\int_B |f|^{p(x)} d\mu \geq c \left(\int_B |f| d\mu \right)^\delta$$

for some $c > 0$. □

Proof of Theorem 4.1. Due to $p_1 q_1 > 1$, we claim that there exists a ball $B_r(x_0)$ such that either (i) $p(x) \geq q(x) > 1$ ($q(x) \geq p(x) > 1$) for all $x \in B_r(x_0)$ or (ii) $p(x) > 1 \geq q(x)$ ($q(x) > 1 \geq p(x)$) with $p(x)q(x) > 1$ for all $x \in B_r(x_0)$. The claim is obviously true if $p_1 \neq q_1$. Otherwise, there is $x_0 \in \Omega$ such that $p_1 = p(x_0) = q_1 = q(x_0) > 1$ by (1.2), and hence either there exists $x_1 \in \Omega$ with $x_1 \neq x_0$ such that $p(x_1) > q(x_1) > 1$ ($q(x_1) > p(x_1) > 1$), or $p(x) \equiv q(x) > 1$ in a neighborhood of x_0 , which have been covered by the case (i).

Denote

$$\begin{aligned} p_1^* &= \sup_{B_r(x_0)} p(x) \leq p_1, & p_2^* &= \inf_{B_r(x_0)} p(x) \geq p_2, \\ q_1^* &= \sup_{B_r(x_0)} q(x) \leq q_1, & q_2^* &= \inf_{B_r(x_0)} q(x) \geq q_2. \end{aligned}$$

We will prove the conclusion for the two cases in turn, by following a technique for the proof of Theorem 32.5 in [9] (also see the earlier [3]).

Case (i) Without loss of generality, suppose $p(x) \geq q(x) > 1$ for all $x \in B_r(x_0)$. Let λ be the first eigenvalue of

$$-\Delta \varphi = \lambda \varphi \text{ in } B_r, \quad \varphi = 0 \text{ on } \partial B_r, \tag{4.1}$$

and φ the corresponding eigenfunction, normalized by $\varphi > 0$ in B_r , $\int_{B_r} \varphi dx = 1$. Set

$$I(t) = \int_{B_r(x_0)} u \varphi dx, \quad J(t) = \int_{B_r(x_0)} v \varphi dx.$$

We have from (1.1) with Jensen's inequality that

$$\begin{aligned} I'(t) &= \int_B \varphi \Delta u + v^{p(x)} \varphi dx \geq \int_B u \Delta \varphi + (v^{q_2^*} - v) \varphi dx \\ &= -\lambda I(t) - J(t) + \int_B v^{q_2^*} \varphi dx \\ &\geq -\lambda I(t) - J(t) + J^{q_2^*}(t), \end{aligned}$$

and similarly,

$$J'(t) \geq -\lambda J(t) - I(t) + I^{q_2^*}(t).$$

Thus, $K(t) = I(t) + J(t)$ satisfies

$$\begin{aligned} K'(t) &= I'(t) + J'(t) \geq -(1 + \lambda)K(t) + I^{q_2^*}(t) + J^{q_2^*}(t) \\ &\geq -(1 + \lambda)K(t) + 2^{1-q_2^*} K^{q_2^*}(t). \end{aligned}$$

This yields $T < +\infty$ whenever $K(0) > \max\{1, 2(\lambda + 1)^{\frac{1}{q_2^*-1}}\}$.

Case (ii) Without loss of generality, only consider $q(x) > 1 \geq p(x)$ with $p(x)q(x) > 1$ for all $x \in B_r(x_0)$. Let u_0 or v_0 be so large that

$$\int_B u_0 \varphi dx + \frac{1}{8} \int_B v_0^{p_2^*} \varphi dx > 2e^\lambda \left(\int_B \Theta \varphi dx + 1 \right), \quad (4.2)$$

where Θ is the unique solution of (3.4) in $B_r(x_0)$. We have

$$(u + \Theta)_t - \Delta(u + \Theta) = v^{p(x)} + 1 \geq v^{p_2^*}.$$

By Lemma 4.1 with (3.2), we deduce

$$\begin{aligned} u + \Theta &\geq T(t)(u_0 + \Theta) + \int_0^t T(t-s)v^{p_2^*}(s)ds \\ &\geq T(t)u_0 + \frac{1}{2} \int_0^t T(t-s)(T(s)v_0)^{p_2^*}ds \\ &\quad + \frac{1}{2} \int_0^t T(t-s) \left(\int_0^s T(s-r)u^{q(x)}(r)dr \right)^{p_2^*} ds \\ &\geq T(t)u_0 + \frac{1}{2} \int_0^t T(t-s)(T(s)v_0^{p_2^*})ds \\ &\quad + \frac{1}{2} \int_0^t s^{p_2^*} T(t-s) \left(\int_0^s \frac{1}{s} T(s-r)u^{q(x)}(r)dr \right)^{p_2^*} ds \end{aligned}$$

$$\begin{aligned}
&\geq T(t)u_0 + \frac{1}{2}tT(t)v_0^{p_2^*} + \frac{1}{2}\int_0^t s^{p_2^*}T(t-s)\int_0^s \frac{1}{s}\left(T(s-r)u^{q(x)}(r)\right)^{p_2^*}drds \\
&\geq T(t)u_0 + \frac{1}{8}T(t)v_0^{p_2^*} + \frac{1}{2}t^{p_2^*-1}\int_0^t T(t-s)\int_0^s \left(T(s-r)u^{q(x)}(r)\right)^{p_2^*}drds \\
&\geq T(t)u_0 + \frac{1}{8}T(t)v_0^{p_2^*} \\
&\quad + \frac{1}{2}t^{p_2^*-1}\int_0^t T(t-s)\left(\int_0^s T(s-r)u^{p_2^*q(x)}(r)dr\right)ds, \quad t \geq \frac{1}{4},
\end{aligned}$$

and hence

$$u + \Theta \geq T(t)u_0 + \frac{1}{8}T(t)v_0^{p_2^*} + \frac{1}{2}t^{p_2^*-1}\int_0^t \int_0^s T(t-r)u^{p_2^*q(x)}(r)drds, \quad t \geq \frac{1}{4} \quad (4.3)$$

by Fubini's theorem. Multiplying (4.3) with φ and denoting $y(t) = \int_{B_r} u\varphi dx$, we obtain

$$\begin{aligned}
y(t) &\geq (T(t)u_0 + \frac{1}{8}T(t)v_0^{p_2^*}, \varphi) - \int_{B_r} \Theta\varphi dx \\
&\quad + \frac{1}{2}t^{p_2^*-1}\int_0^t \int_0^s (T(t-r)u^{p_2^*q(x)}(r), \varphi)drds.
\end{aligned}$$

Consider the ODE problem

$$\begin{cases} w'(t) + \lambda w(t) = 0, & t \in [r, \infty), \\ w(r) = \int_{B_r} u^{p_2^*q(x)}(r)\varphi(x)dx. \end{cases} \quad (4.4)$$

Denote $\int_{B_r} fgdx = (f, g)_{B_r}$. The uniqueness of solutions to (4.4) implies that

$$(T(t-r)u^{p_2^*q(x)}(r), \varphi)_{B_r} = e^{-\lambda(t-r)} (u^{p_2^*q(x)}(r), \varphi)_{B_r},$$

and similarly,

$$(T(t)u_0, \varphi)_{B_r} = e^{-\lambda t} (u_0, \varphi)_{B_r}, \quad (T(t)v_0^{p_2^*}, \varphi)_{B_r} = e^{-\lambda t} (v_0^{p_2^*}, \varphi)_{B_r}.$$

By using Lemma 4.2 with (4.2), we have

$$\begin{aligned}
y(t) &\geq e^{-\lambda t} \left(u_0 + \frac{1}{8}v_0^{p_2^*}, \varphi\right)_{B_r} - \int_B \Theta\varphi dx + \frac{1}{2}t^{p_2^*-1}e^{-\lambda t}\int_0^t \int_0^s (u^{p_2^*q(x)}(r), \varphi)_{B_r} drds \\
&\geq \frac{1}{2}e^{-\lambda} (u_0, \varphi)_{B_r} + 1 + \frac{1}{2}e^{-\lambda t}\int_0^t \int_0^s (u^{p_2^*q(x)}(r), \varphi)_{B_r} drds \\
&\geq \frac{1}{2}e^{-\lambda} (u_0, \varphi)_{B_r} + 1 + c e^{-\lambda}\int_0^t \int_0^s y^{p_2^*q_2^*}(r)drds, \quad \frac{1}{4} \leq t < 1,
\end{aligned}$$

and hence

$$y(t) \geq c_1 y(0) + c_1 \int_{\frac{1}{4}}^t \int_{\frac{1}{4}}^s y^{p_2^* q_2^*}(r) dr ds =: h(t), \quad \frac{1}{4} \leq t < 1.$$

We know with $\gamma = p_2^* q_2^*$ that

$$h''(t) = c_1 y^\gamma \geq c_1 h^\gamma, \quad \frac{1}{4} \leq t < 1. \quad (4.5)$$

Multiplying (4.5) by $h' \geq 0$ and then integrating from $\frac{1}{4}$ to t , we have

$$h'^2(t) \geq c_2 h^{\gamma+1}(t) - c_3 y^{\gamma+1}(0), \quad \frac{1}{4} \leq t < 1. \quad (4.6)$$

On the other hand,

$$\begin{aligned} z(t) &:= (v^{p_2^*}(t), \varphi)_{B_r} = \int_{B_r} v^{p_2^*} \varphi dx \geq \int_{B_r} (T(t)v_0)^{p_2^*} \varphi dx \\ &\geq \int_{B_r} (T(t)v_0^{p_2^*}) \varphi dx = e^{-\lambda t} z(0), \end{aligned}$$

and in addition,

$$\begin{aligned} y(t) &\geq \frac{1}{2} e^{-\lambda} (u_0, \varphi)_{B_r} + \int_0^t (T(t-s)v^{p_2^*}(s), \varphi)_{B_r} ds \\ &\geq \frac{1}{2} e^{-\lambda} (u_0, \varphi)_{B_r} + \int_0^t e^{-\lambda(t-s)} (v^{p_2^*}(s), \varphi)_{B_r} ds \\ &\geq c_4 (y(0) + tz(0)), \quad \frac{1}{4} \leq t < 1. \end{aligned}$$

Since $h'' = c_1 y^\gamma$, we have

$$h\left(\frac{1}{2}\right) \geq c_5 (y(0) + z(0))^\gamma.$$

Due to $\gamma \geq 1$, if $y(0) + z(0) \geq M$ with M properly large, then it follows from (4.6) that $h' \geq c_6 h^{\frac{\gamma+1}{2}}$ on $(\frac{1}{2}, 1)$, and hence $h(t)$ would blow up before $t = 1$ with M large enough. $\square$

5 Blow-up for any initial data

This section discusses a Fujita type conclusion for the coupled system (1.1) with variable sources, motivated by the significant result for the scalar case (1.3) (see Theorem 3.7 in [4]). We need two lemmas.

Lemma 5.1 *Assume that $0 < p(x)q(x) < 1$ with either $p_1 < q(x) < 1$ or $p_1 < 1 \leq q(x)$. Then for any $\Omega_1 \subset \subset \Omega$ and $t_0 \in (0, 1)$, there exists a positive constant c_0 independent of*

u_0, v_0 such that

$$u(x, t_0), v(x, t_0) \geq c_0, \quad x \in \Omega_1.$$

Proof Since $0 < p(x)q(x) < 1$, we know by Theorem 3.1 that the solution (u, v) is global. The proof of the lemma is motivated by Lemma 2.3 in [3].

Consider the case of $0 < p(x) < q(x) < 1$ only. The other case of $p_1 < 1 \leq q(x)$ can be treated similarly. Without loss of generality, assume $0 < u_0, v_0 \leq 1$, due to the comparison principle. Denote by G_Ω the Dirichlet heat kernel, namely,

$$(T(t)f)(x) = \int_{\Omega} G_\Omega(x, y, t) f(y) dy \quad \text{for } f \in L^p(\Omega), \quad 1 \leq p \leq \infty.$$

For $0 < u_0 \leq 1$, we have by the comparison principle with $r > 0$ that

$$T(t)u_0^r \leq 1,$$

and hence

$$(T(t)u_0)^{q(x)} \geq (T(t)u_0)^{q_1} \geq T(t)u_0^{q_1} \quad (5.1)$$

by Lemma 4.1 with $0 < q(x) \leq q_1 < 1$. Since $u(x, t) \geq T(t)u_0$ by (3.2), it follows again by (3.2) with (5.1) that

$$v(x, t) \geq \int_0^t T(t-s)(T(s)u_0^{q_1}) ds \geq tT(t)u_0^{q_1},$$

and in turns,

$$u(x, t) \geq \int_0^t T(t-s)(s(T(s)u_0^{q_1}))^{p_1} ds \geq \frac{t^{p_1+1}}{p_1+1} T(t)u_0^{p_1q_1}, \quad 0 < t < 1.$$

Therefore, by the similar arguments used for the proof of Lemma 2.3 in [3], we can arrive at

$$u(x, t) \geq C_1 t^{(p_1+1)/(1-p_1q_1)} S_1(t), \quad 0 < t < 1,$$

where $C_1 = C_1(p_1, q_1) > 0$, and $h = S_1(t)$ solves

$$h_t - \Delta h = 0 \text{ in } \Omega \times (0, \infty), \quad h(x, 0) = 1 \text{ in } \Omega, \quad h = 0 \text{ on } \partial\Omega \times (0, \infty).$$

Denote $c_1 = C_1 t_0^{(p_1+1)/(1-p_1q_1)} \inf_{\Omega_1} S_1(t_0) > 0$. Then $u(x, t_0) \geq c_1$ for $x \in \Omega_1$. Similarly, $v(x, t_0) \geq c_2, x \in \Omega_1$. Let $c_0 = \min(c_1, c_2)$. The proof is complete. $\square$

Corollary 5.1 *Under the hypotheses of Lemma 5.1, for any $\Omega_0 \subset\subset \Omega$, there exists $c > 0$ such that*

$$u(x, t), v(x, t) \geq c, \quad (x, t) \in \Omega_0 \times [t_0, \infty).$$

Proof Denote $\Omega_1 := \{x \in \Omega; \text{dist}(x, \partial\Omega) > \varepsilon\}$ with $\varepsilon = \text{dist}(\Omega_0, \partial\Omega)/2$. Then $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. By Lemma 5.1, we have

$$u(x, t_0), v(x, t_0) \geq c, \quad x \in \Omega_1.$$

Therefore, there exists $\delta_0 \in (0, 1)$ such that

$$u(x, t_0) \geq \delta^{\alpha_0} \varphi, \quad v(x, t_0) \geq \delta^{\beta_0} \varphi, \quad \delta \in (0, \delta_0], \quad x \in \Omega_1,$$

where $\alpha_0, \beta_0 \geq 1$ with $p_1 < \alpha_0/\beta_0 < 1/q_1$, and φ is the solution of (4.1) with Ω_1 replacing B_r .

A simple calculation shows that $((\delta\varphi)^{\alpha_0}, (\delta\varphi)^{\beta_0})$ with δ small enough is a subsolution of

$$\begin{cases} U_t = \Delta U + V^{p(x)}, & (x, t) \in \Omega_1 \times (t_0, \infty), \\ V_t = \Delta V + U^{q(x)}, & (x, t) \in \Omega_1 \times (t_0, \infty), \\ U = V = 0, & (x, t) \in \partial\Omega_1 \times (t_0, \infty), \\ U(x, t_0) = u(x, t_0), \quad V(x, t_0) = v(x, t_0), & x \in \overline{\Omega}_1, \end{cases} \quad (5.2)$$

and the conclusion of the corollary is obtained with $c = \min\{\inf_{\Omega_0} (\delta\varphi)^{\alpha_0}, \inf_{\Omega_0} (\delta\varphi)^{\beta_0}\}$. $\square$

Lemma 5.2 Assume that $p(x), q(x)$ admit the assumptions of Lemma 5.1 in a ball $B_R(x_0) \subset \Omega$ with $R \geq 1$. Then there exists $t_1 > 0$ such that the solutions of (1.1) satisfy

$$u(x, t) \geq cR^\alpha, \quad v(x, t) \geq cR^\beta \quad \text{whenever } (x, t) \in B_{R/2}(x_0) \times [t_1, \infty),$$

with $(\alpha, \beta) = \left(\frac{2(1+p_2^*)}{1-p_2^*q_2^*}, \frac{2(1+q_2^*)}{1-p_2^*q_2^*}\right)$ and $c > 0$ independent of R , where $p_2^* = \inf\{p(x) : x \in B_R(x_0)\}$, $q_2^* = \inf\{q(x) : x \in B_R(x_0)\}$.

Proof Without loss of generality, assume $x_0 = 0$. Construct

$$(\bar{u}(x, t), \bar{v}(x, t)) = (R^{-\alpha}u(Rx, R^2t), R^{-\beta}v(Rx, R^2t)), \quad (x, t) \in B_1(0) \times [0, \infty).$$

We have

$$\begin{aligned} \bar{u}_t - \Delta \bar{u} &= R^{2-\alpha}v^{p(Rx)}(Rx, R^2t) = R^{2-\alpha}R^{\beta p(Rx)}\bar{v}^{p(Rx)} \\ &\geq R^{2-\alpha+\beta p_2^*}\bar{v}^{p(Rx)}(x, t) = \bar{v}^{p(Rx)}, \end{aligned}$$

and similarly, $\bar{v}_t - \Delta \bar{v} \geq \bar{u}^{q(Rx)}$. Therefore $(\bar{u}, \bar{v})$ is a supersolutions of

$$\begin{cases} U_t = \Delta U + V^{p(Rx)}, & (x, t) \in B_1 \times (0, \infty), \\ V_t = \Delta V + U^{q(Rx)}, & (x, t) \in B_1 \times (0, \infty), \\ U = V = 0, & (x, t) \in \partial B_1 \times (0, \infty), \\ U(x, 0) = R^{-\alpha}u_0(Rx), \quad V(x, 0) = R^{-\beta}v_0(Rx), & x \in \overline{\Omega}, \end{cases} \quad (5.3)$$

Moreover, notice $p(x), q(x)$ satisfy the assumptions of Lemma 5.1, and so do $p(Rx), q(Rx)$. By Corollary(5.1), we conclude that

$$\bar{u}, \bar{v} \geq c, \quad (x, t) \in B_{\frac{1}{2}}(0) \times [t_0, \infty)$$

for some $c > 0$ independent of R . The conclusion is obtained with $y = Rx, s = R^2t$. $\square$

Theorem 5.1 Suppose $\bar{p}(x)$ and $\bar{q}(x)$ are functions defined in the unit ball $B_1(0)$ with $\bar{p}(x)\bar{q}(x) - 1$ changing sign. If Ω is large that $\Omega \supset B_L(x_0)$ with $L > 0$ large enough, then the solutions of (1.1) blow up in finite time for any non-trivial non-negative initial data provided $p(x) \equiv \bar{p}(\frac{x-x_0}{L}), q(x) \equiv \bar{q}(\frac{x-x_0}{L})$ for all $x \in B_L(x_0)$.

Proof The proof follows the idea for the proof of Theorem 3.7 in [4].

Assume for simplicity $x_0 = 0$. Since $\bar{p}(x)\bar{q}(x) - 1$ changes sign and L is large enough, there exists a ball $B_{\lambda_1 L}(x_1) \subset B_L(0)$, such that $0 < p(x)q(x) < 1$ and $p_1 < q(x) < 1$ or $p_1 < 1 \leq q(x)$ for all $x \in B_{\lambda_1 L}(x_1)$, and a ball $B_{\lambda_2 L}(x_2) \subset B_L(0)$ such that $p(x)q(x) > 1$

for all $x \in B_{\lambda_2 L}(x_2)$. Suppose for contradiction that the solution (u, v) of (1.1) is global for any $L > 0$. By Lemma 5.2, for some $t_1 > 0$ and $R_1 = \lambda_1 L$, we have

$$u(x, t) \geq cR_1^\alpha, \quad v(x, t) \geq cR_1^\beta, \quad (x, t) \in B_{R_1/2}(x_1) \times [t_1, \infty).$$

It is not difficult to check that the solution component u is a supersolution to the scalar problem

$$\begin{cases} w_t = \Delta w & \text{in } \Omega \setminus \overline{B}_{R_1/2}(x_1) \times (t_1, \infty), \\ w = 0 & \text{on } \partial\Omega \times (t_1, \infty), \\ w = cR_1^\alpha & \text{on } \partial B_{R_1/2}(x_1) \times (t_1, \infty), \\ w(x, 0) = u(x, t_1) & \text{in } \Omega \setminus B_{R_1/2}(x_1). \end{cases} \quad (5.4)$$

Let $r(x)$ be the stationary solution to the problem (5.4). Obviously, $w(x, t)$ converges uniformly to $r(x)$, and hence $u(x, t) \geq w(x, t) \geq r(x) - 1$ when $t > t_2$ with some $t_2 > t_1$. Denote $R_2 = \frac{\lambda_2 L}{2}$, and φ the first normalized eigenfunction of (4.1) in B_{R_2} . Since $r(x) \geq C_L > 0$ on $B_{R_2}(x_2)$ with C_L monotonously tends to ∞ as $L \rightarrow \infty$, we know that $\int_{B_{R_2}(x_2)} u(x, t) \varphi(x) dx \geq \int_{B_{R_2}(x_2)} r(x) \varphi(x) dx - 1$ can be arbitrarily large for $t \geq t_2$ whenever L is sufficiently large. The parallel result is true for $\int_{B_{R_2}(x_2)} v \varphi dx$. We conclude from Theorem 4.1 that the solution of (1.1) blows up in a finite time provided L large enough. $\square$

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