Annali di Matematica (2008) 187:263–321
DOI 10.1007/s10231-007-0045-2

Global regularity for almost minimizers of nonconvex variational problems

M. Foss

Received: 1 February 2006 / Revised: 12 September 2006 / Published online: 5 May 2007

Abstract We prove some global, up to the boundary of a domain $\Omega \subset \mathbb{R}^n$, continuity and Lipschitz regularity results for almost minimizers of functionals of the form

$$\mathbf{u} \mapsto \int_{\Omega} g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}.$$

The main assumption for g is that it be asymptotically convex with respect its third argument. For the continuity results, the integrand is allowed to have some discontinuous behavior with respect to its first and second arguments. For the global Lipschitz regularity result, we require g to be Hölder continuous with respect to its first two arguments.

Keywords Regularity · Asymptotic convexity · Almost minimizer

Mathematics Subject Classification (2000) 49N60 · 35J50 · 35J55

1 Introduction

Let n and N be positive integers, and let $p > 2$ be given. These numbers are fixed for the rest of the paper. In the sequel $\Omega \subset \mathbb{R}^n$ always denotes an open and bounded set. Put $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. Additional notation is collected in Sect. 2.

To facilitate the presentation of the main results and ideas, we will make some simplifying assumptions for this section. Suppose that Ω has a smooth boundary $\partial\Omega$. In Sect. 3, we provide a more precise description of the types of domains that we consider and recall some

M. Foss (✉)
Department of Mathematics, University of Nebraska-Lincoln,
203 Avery Hall, Lincoln, NE 68588-0130, USA
e-mail: mfoss@math.unl.edu

standard material on boundary transformations. Let $g \in \mathcal{C}^\infty(\mathbb{R}^{N\times n};\mathbb{R})$ be given, and define the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \overline{\mathbb{R}}$ by

$$K[\mathbf{u}] := \int_\Omega g(\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Also define $H \in \mathcal{C}^\infty(\mathbb{R}^{N\times n};\mathbb{R})$ by $H(\mathbf{F}) := \left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}}$. The primary assumption that we make for g is that it possesses the following property: for each $\varepsilon > 0$, there exists a $\sigma_\varepsilon < +\infty$ such that

$$\left\| \frac{\partial^2}{\partial\mathbf{F}^2} g(\mathbf{F}) - \frac{\partial^2}{\partial\mathbf{F}^2} H(\mathbf{F}) \right\| < \varepsilon\|\mathbf{F}\|^{p-2}, \tag{A}$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon$. Property (A) implies that g is convex when the norm of its argument is sufficiently large. In fact, for all $\boldsymbol{\xi} \in \mathbb{R}^{N\times n}$, we find that

$$\frac{\partial^2}{\partial\mathbf{F}^2} g(\mathbf{F}) :: \left[\boldsymbol{\xi}\otimes\boldsymbol{\xi}\right] \geq (p-\varepsilon)\left(1+\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\boldsymbol{\xi}\|^2,$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon$. Thus, we say that g is asymptotically convex. Several other implications of assumption (A) are provided in Sect. 5. For our main results, assumption (A) will be significantly relaxed (see Definitions 10, 11), but the essential feature is that we require g to possess some asymptotic convexity.

Given the function g satisfying the condition in (A), one can establish *local* bounds for the norm of the gradient of a minimizer. In [3], Chipot and Evans showed that minimizers for J must belong to $L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, in the case where $p = 2$ (for our results, we require $p > 2$). This result was later generalized, by Giaquinta and Modica [15], to allow $p > 2$. Raymond [23] extended Giaquinta and Modica's work obtaining local bounds for the gradient of a minimizer to functionals of the form

$$\mathbf{u} \mapsto \int_\Omega \{g(\nabla\mathbf{u}(\mathbf{x})) + h(\mathbf{x},\mathbf{u}(\mathbf{x}))\}\,\mathrm{d}\mathbf{x},$$

where $h : \Omega\times\mathbb{R}^N \to \mathbb{R}$ is a Carathédory function satisfying some regularity and growth conditions. Since g is not necessarily convex, or even quasiconvex, minimizers for K may not exist. For the case with $p = 2$, Müller considered perturbed functionals of the form

$$\mathbf{u} \mapsto \int_\Omega \left\{g(\nabla\mathbf{u}(\mathbf{x})) + \varepsilon^2\left(\boldsymbol{\Delta}\mathbf{u}(\mathbf{x})\right)^2\right\}\mathrm{d}\mathbf{x},$$

where $\varepsilon > 0$ is small. For these functionals, one obtains approximate minimizers of K, and Müller [22] provided local Lipschitz estimates, for the approximate minimizers, that are uniform with respect to ε. In [11], Fuchs established local Lipschitz regularity for minimizers of the functional K, where $g(\mathbf{F}) = \|\mathbf{F}\|^p$ outside some ball but is otherwise only required to be Lipschitz continuous. Later, Fuchs and Li [12] considered such functionals as

$$\mathbf{u} \mapsto \int_\Omega \left\{\|\nabla\mathbf{u}(\mathbf{x})\|^2 + f(\|\nabla\mathbf{u}(\mathbf{x})\|) + h(\nabla\mathbf{u}(\mathbf{x}))\right\}\mathrm{d}\mathbf{x},$$

with h a continuous function with compact support and $f : [0,+\infty) \to \mathbb{R}$ a convex, non-decreasing, subquadratic function that is in $\mathcal{C}^1([L,+\infty))$ for some $L > 0$. Fuchs and Li showed that minimizers for such functionals, subject to sufficiently regular boundary conditions, are globally Lipschitz. For all, but this last result, the estimates obtained are local in nature. The results presented in this article are valid up to the boundary for appropriate Dirichlet boundary conditions.

The arguments for our results are similar in spirit to those used in [23] by Raymond. More precisely, we first establish that a minimizer must be in some Morrey space and then show that the minimizer is actually Lipschitz. Ultimately, all the arguments involve making comparisons between minimizers of K and minimizers of the functional

$$\mathbf{v} \mapsto \int_{\mathcal{U}} H(\nabla \mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

with an appropriate domain $\mathcal{U}$ and boundary conditions on $\mathbf{v}$.

There is, however, another strategy that has been successful in establishing local Lipschitz regularity. This strategy is based on the observation that if the relaxed functional

$$\mathscr{K}[\mathbf{u}] := \inf \left\{ \liminf_{k\to+\infty} \int_{\Omega} g(\nabla \mathbf{u}_k(\mathbf{x}))\,\mathrm{d}\mathbf{x} \,\middle|\, \mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N) \right\}$$

can be represented by

$$K^{**}[\mathbf{u}] := \int_{\Omega} g^{**}(\nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

where g^{**} is the convex envelope of g, then a minimizer for K is also a minimizer for K^{**}. In which case, regularity for minimizers of K is obtained by instead establishing regularity for minimizers of K^{**}. In [1], Benedetti and Mascolo follow this approach to obtain local regularity for minimizers of nonconvex, nonhomogeneous functionals satisfying a p–q growth condition (see [4,9] for the convex setting). Under certain conditions, the regularity results actually yield existence of minimizers; for such results in the case where $N = 1$ and additional references, we refer to the work of Fonseca et al. [10]. In [5], Cupini and Migliorini also consider the case where $N = 1$ and establish local Hölder continuity results for minimizers of nonhomogeneous functionals that depend on the minimizer itself as well as its gradient. In many respects, the results in Sect. 8 provide an extension of the results in [5] to the vectorial setting and up to the boundary.

Before describing the main results, we introduce the following notion of an almost minimizer.

Definition 1 Suppose that a functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \overline{\mathbb{R}}$ and a non-decreasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying $\omega(0) = 0$ are given. Further suppose that a family of functions $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ and constants $\{T_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$ are given. We will say that $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a $(K,\omega,\{\nu_\varepsilon,T_\varepsilon\})$-*minimizer at* $\mathbf{x}_0$, if $K[\mathbf{u}] < +\infty$ and there is a $t > 1$ such that for each $\varepsilon > 0$ and every $\rho > 0$, we find that

$$\begin{aligned} K[\mathbf{u}] \le K[\mathbf{u}+\boldsymbol{\varphi}] + (\omega(\rho)+\varepsilon) \int_{\mathrm{supp}(\varphi)} \left(1 + \|\nabla \mathbf{u}\|^p + \|\nabla \boldsymbol{\varphi}\|^p\right)\,\mathrm{d}\mathbf{x} \\ + T_\varepsilon \left(\int_{\mathrm{supp}(\varphi)} \|\nabla \boldsymbol{\varphi}\|^p\,\mathrm{d}\mathbf{x} \right)^t + \int_{\mathrm{supp}(\varphi)} |\nu_\varepsilon(\mathbf{x})|\,\mathrm{d}\mathbf{x} + \rho^n |\nu_\varepsilon(\mathbf{x}_0)|, \end{aligned} \tag{1}$$

for all $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega_{\mathbf{x}_0,\rho}; \mathbb{R}^N)$, whenever $\mathbf{x}_0$ is a Lebesgue point for ν_ε. If a mapping $\mathbf{u}$ is a $(K, \omega, \{\nu_\varepsilon, T_\varepsilon\})$-minimizer at each $\mathbf{x}_0 \in \Omega$, then we will call it a $(K, \omega, \{\nu_\varepsilon, T_\varepsilon\})$*-minimizer*. Moreover, a $(K, \omega, \{\nu_\varepsilon, 0\})$-minimizer will be called a $(K, \omega, \{\nu_\varepsilon\})$*-minimizer*.

We also define

Definition 2 Suppose that a functional $K : W^{1,1}(\Omega; \mathbb{R}^N) \to \overline{\mathbb{R}}$ and a non-decreasing function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying $\omega(0) = 0$ are given. We will say that $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a (K, ω)*-minimizer* at $\mathbf{x}_0$, if $K[\mathbf{u}] < +\infty$ and for every $\rho > 0$, we find that

$$K[\mathbf{u}] \le K[\mathbf{u} + \boldsymbol{\varphi}] + \omega(\rho) \int\limits_{\mathrm{supp}(\varphi)} \left(1 + \|\nabla \mathbf{u}\|^p + \|\nabla \boldsymbol{\varphi}\|^p\right) \, \mathrm{d}\mathbf{x}, \tag{2}$$

for all $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega_{\mathbf{x}_0,\rho}; \mathbb{R}^N)$. Whenever a mapping $\mathbf{u}$ is a (K, ω)-minimizer at each $\mathbf{x}_0 \in \Omega$, then we will simply call it a (K, ω)*-minimizer*. Also, a $(K, 0)$-minimizer will be called a K*-minimizer*, or just minimizer.

Clearly, any $(K, \omega, \{0\})$-minimizer is a (K, ω)-minimizer. Our definition for the (K, ω)-minimizers is similar to Giusti's definition of ω-minimizers in [16], to which we refer for further references.

The first main result is established in Sect. 6. It provides Morrey regularity, up to the boundary of Ω, for the gradient of $(K, \omega, \{\nu_\varepsilon, T_\varepsilon\})$-minimizers. (It should be mentioned that Morrey regularity for a minimizer itself, not its gradient, was considered in [6] in a more general setting.) For simplicity, we state here a special case in terms of the function g and H given above and then describe the actual result.

Theorem 1 *Let* $0 \le \kappa < n$ *and* $\overline{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^N)$*, with* $\nabla \overline{\mathbf{u}} \in L^{p,\kappa}(\Omega; \mathbb{R}^N)$ *be given. Also, let* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ *be given. If* $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *is a* K*-minimizer satisfying* $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,1}(\Omega; \mathbb{R}^N)$*, then* $\nabla \mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N\times n})$ *and* $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^N)$.

Here $L^{p,\kappa}(\Omega; \mathbb{R}^{N\times n})$ and $\mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^N)$ denote, respectively, Morrey and Campanato spaces (Definitions 3, 4). The actual result is stated for mappings that are almost minimizers, in the sense of Definition 1, for a family of functionals parameterized by points in a given subset of $\overline{\Omega}$. Moreover, the integrands for these functionals are only required to satisfy a much weaker version of assumption (A). For a precise statement of the result, we refer to Theorem 8.

With this level of generality, we are able to establish global Morrey regularity for minimizers of non-homogeneous functionals in Sect. 8. To illustrate the types of functionals to which our results are applicable, let κ and $\overline{\mathbf{u}}$, satisfying the hypotheses of the above theorem, be given.

Example 1 Let $q \ge 1$ and $\alpha < \frac{n-\kappa}{q}$ be given. Suppose that $\mathbf{u} \in W^{1,1}(\mathcal{B}; \mathbb{R}^N)$ is a minimizer for the functional

$$\mathbf{u} \mapsto \int\limits_{\mathcal{B}} \left\{ \left(1 + \|\nabla \mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p}{2}} - \frac{1}{\|\mathbf{x}\|^\alpha} \left(1 + \|\nabla \mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p(q-1)}{2q}} \right\} \mathrm{d}\mathbf{x} \tag{3}$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,1}(\mathcal{B}; \mathbb{R}^N)$. Then we find that $\nabla \mathbf{u} \in L^{p,\kappa}(\mathcal{B}; \mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\mathcal{B}; \mathbb{R}^N)$.

To see why we should expect **u** to possess this amount of regularity, observe that for each $\varepsilon > 0$, there is a $C_\varepsilon < +\infty$ such that

$$\int_{\mathcal{B}} \frac{1}{\|\mathbf{x}\|^\alpha} \left(1 + \|\nabla \mathbf{u}\|^2\right)^{\frac{p(q-1)}{2q}} \mathrm{d}\mathbf{x} \leq \varepsilon \int_{\mathcal{B}} \left(1 + \|\nabla \mathbf{u}\|^2\right)^{\frac{p}{2}} \mathrm{d}\mathbf{x} + C_\varepsilon \int_{\mathcal{B}} \frac{1}{\|\mathbf{x}\|^{n-\kappa-\delta}} \mathrm{d}\mathbf{x},$$

for some $\delta > 0$. Thus, the functional in (3) is coercive with respect to the norm in $W^{1,p}(\mathcal{B}; \mathbb{R}^N)$, from which we conclude that $\mathbf{u} \in W^{1,p}(\mathcal{B}; \mathbb{R}^N)$. Moreover, at any point $\mathbf{x}_0 \in \mathcal{B}$ where

$$\limsup_{\rho \to 0^+} \frac{1}{\rho^\kappa} \int_{\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}} \|\nabla \mathbf{u}(\mathbf{x})\|^p \, \mathrm{d}\mathbf{x} = +\infty, \tag{4}$$

we find that

$$\limsup_{\rho \to 0^+} \frac{1}{\rho^\kappa} \int_{\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}} \frac{1}{\|\mathbf{x}\|^{n-\kappa-\delta}} \, \mathrm{d}\mathbf{x} = 0.$$

Therefore, for sufficiently small $\rho > 0$, we have that

$$\int_{\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}} \left\{ \left(1 + \|\nabla \mathbf{u}\|^2\right)^{\frac{p}{2}} - \frac{1}{\|\mathbf{x}\|^\alpha} \left(1 + \|\nabla \mathbf{u}\|^2\right)^{\frac{p(q-1)}{2q}} \right\} \mathrm{d}\mathbf{x} \sim \int_{\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}} H(\nabla \mathbf{u}) \, \mathrm{d}\mathbf{x}$$

and the minimizer **u** is comparable to a minimizer for the functional

$$\mathbf{v} \mapsto \int_{\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}} H(\nabla \mathbf{v}(\mathbf{x})) \, \mathrm{d}\mathbf{x}, \tag{5}$$

satisfying $[\mathbf{v} - \mathbf{u}] \in W_0^{1,p}(\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}; \mathbb{R}^N)$. If $\overline{\mathcal{B}}_{\mathbf{x}_0,\rho} \subset \mathcal{B}$, then a well known result of Uhlenbeck's [25] states that minimizers for this last functional are locally smooth. This result translates into estimates for the minimizer of (3), which preclude (4), thereby yielding the local Morrey regularity of the minimizer **u**. When $\mathcal{B}_{\mathbf{x}_0,\rho} \cap \partial\mathcal{B} \neq \emptyset$, however, Uhlenbeck's result is not directly applicable, since it is only valid up to the boundary under homogeneous Dirichlet conditions. If we have (4), however, then the regularity hypothesis for $\overline{\mathbf{u}}$ implies that the boundary conditions are, in some sense, negligible relative to $\|\nabla \mathbf{u}(\mathbf{x}_0)\|$. Thus, on the set $\mathcal{B} \cap \mathcal{B}_{\mathbf{x}_0,\rho}$, for sufficiently small $\rho > 0$, we may compare the minimizer **u** to a minimizer of the functional in (5) that satisfies homogeneous Dirichlet conditions on $\mathcal{B}_{\mathbf{x}_0,\rho} \cap \partial\mathcal{B}$, and the regularity result of Uhlenbeck again precludes (4). This is roughly the basis for the argument behind the results in Sect. 6.

In Sect. 8, we also apply the results in Sect. 6 to obtain global Morrey regularity for minimizers of functionals that depend on the mapping itself, as well as its gradient. Suppose that $2 < p \leq n$, and let $0 \leq \kappa < n$ and $\overline{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^N)$, with $\nabla \overline{\mathbf{u}} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N \times n})$, be given.

Example 2 Let $q \geq 1$ and $r < \frac{np}{q(n-p)}$ be given. Suppose that $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a minimizer for the functional

$$\mathbf{u} \mapsto \int_{\Omega} \left\{ \|\nabla \mathbf{u}(\mathbf{x})\|^p - (1 + \|\mathbf{u}(\mathbf{x})\|)^r \left(1 + \|\nabla \mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p(q-1)}{2q}} \right\} \mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u}-\overline{\mathbf{u}}] \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. Then we find that $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega;\mathbb{R}^N)$. Thus a minimizer for this functional has at least as much regularity, in the sense of Morrey and Campanato spaces, as is compatible with the boundary data. A local version of Theorem 8, further shows that $\nabla\mathbf{u} \in L^{p,\theta}_{\text{loc}}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\theta}_{\text{loc}}(\Omega;\mathbb{R}^N)$, for each $\theta \in [0,n]$. Hence, $\mathbf{u} \in \mathcal{C}^{0,\beta}(\Omega;\mathbb{R}^N)$ for each $\beta \in (0,1)$. Moreover, if $p+\kappa > n$, then $\mathbf{u} \in \mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\overline{\Omega};\mathbb{R}^N) \cap \mathcal{C}^{0,\beta}(\Omega;\mathbb{R}^N)$, for each $\beta \in (0,1)$.

In Sect. 7, under stronger hypotheses than assumed in Sect. 6, we establish global Lipschitz regularity for almost minimizers, in the sense of Definition 2. We state a simplified version here in terms of the functions g and H above; the actual result is contained in Theorem 11.

Theorem 2 *Let $0 < \beta \le 1$ and $\overline{\mathbf{u}} \in \mathcal{C}^{1,\beta}(\Omega;\mathbb{R}^N)$ be given. Also, let the function $\omega \in \mathcal{C}^{0,\beta}([0,\infty);[0,\infty))$ be a non-decreasing function satisfying $\omega(0)=0$. If $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a (K,ω)-minimizer satisfying $[\mathbf{u}-\overline{\mathbf{u}}] \in W_0^{1,1}(\Omega;\mathbb{R}^N)$, then $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^N)$.*

The proof for this result relies on the results in Sect. 6 as well as a comparison to a minimizers for functionals to which Uhlenbeck's result is applicable. Again, the actual result is sufficiently general to allow applications to non-homogeneous functionals in Sect. 8. In fact, Theorem 11 can be applied to Example 2 above. Thus, if in that example $\overline{\mathbf{u}} \in \mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^N)$, then, we find that a minimizer must be Lipschtiz continuous in Ω up to the boundary.

One may also apply the results in Sect. 9 to obtain a partial answer to an open problem recently described in Sect. 4.3 of Mingione's extensive survey paper [21]. We recall the problem here and then describe the result.

Open problem Suppose that $g : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses the following properties:

(i) for each $\mathbf{x} \in \Omega$, we have $g(\mathbf{x},\cdot) \in C^2(\mathbb{R}^{N\times n};\mathbb{R})$;
(ii) there exists $\Lambda_* > 0$ and $\Lambda^* < +\infty$ such that

$$\begin{cases} \Lambda_*\|\mathbf{F}\|^p \le g(\mathbf{x},\mathbf{F}) \le \Lambda^*\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}}; \\ \Lambda_*\|\mathbf{F}\|^{p-2}\|\boldsymbol{\xi}\|^2 \le \dfrac{\partial^2}{\partial\mathbf{F}^2} g(\mathbf{x},\mathbf{F}) :: \boldsymbol{\xi}\otimes\boldsymbol{\xi} \le \Lambda^*\left(1+\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}\|\boldsymbol{\xi}\|^2; \end{cases}$$

for all $\mathbf{x} \in \Omega$ and $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$

(iii) there is a $\delta \in \mathcal{C}([0,+\infty);\mathbb{R})$ that is non-decreasing and satisfies $\delta(0)=0$ such that for each $\mathbf{x},\mathbf{y} \in \Omega$, we have

$$|g(\mathbf{x},\mathbf{F}) - g(\mathbf{y},\mathbf{F})| \le \delta(\|\mathbf{x}-\mathbf{y}\|)\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}},$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$.

Let $\mathbf{u} \in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer for the functional

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x},\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Determine whether or not there is an open set $\Omega_0 \subseteq \Omega$ such that $|\Omega\backslash\Omega_0| = 0$ and $\mathbf{u} \in \mathcal{C}_{\text{loc}}(\Omega_0;\mathbb{R}^N)$.

By local minimizer, we mean that

$$K[\mathbf{u}] \le K[\mathbf{u}+\boldsymbol{\varphi}],$$

for any $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega;\mathbb{R}^N)$ satisfying supp $(\boldsymbol{\varphi})$ is compactly contained in Ω.

The main issue for this problem is that the integrand g is only required to be continuous with respect to the argument $\mathbf{x}$. In Sect. 9, we show that if the function g asymptotically possesses a radial structure, then under suitable boundary conditions, $(K, 0)$-minimizers are actually uniformly continuous in Ω. A local version of this result can also be obtained for local minimizers. It is worth noting that the examples provided by Šverák and Yan [24] show that in general, one can not expect $\Omega_0 = \Omega$.

To conclude our introduction, we mention a couple of potentially interesting directions of future research.

- **Boundary regularity for solutions to elliptic systems** In the context of partial regularity, Growtowski [17] has provided conditions sufficient for the gradient of a weak solution to an elliptic system to be regular (Hölder continuous) in a neighborhood of a boundary point. There is no guarantee, however, that the solution satisfies this condition at any boundary point. More recently, Duzaar et al. [7] have proved that the gradient of solutions to a large class of elliptic systems are regular at almost every point of the boundary, in the sense of the $n-1$ dimensional Hausdorff measure.
  For certain elliptic systems, a subclass of those considered in [7], it may be possible to establish full regularity at the boundary for a weak solution. To obtain local regularity for the gradient of a solution $\mathbf{u}$ to an elliptic system, in divergence form that depends on $\nabla\mathbf{u}$ through its modulus, one of the main steps is to show that $\mathbf{u} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. In this article, conditions ensuring that $\mathbf{u} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ are provided. This may make it possible to prove full boundary regularity results for solutions to these types of elliptic systems.
- **Global regularity for minimizers of nonconvex functionals with general growth** The results produced here ultimately depend on Uhlenbeck's regularity result for solutions to elliptic systems. This appears to be the primary reason for the restrictions on the type of growth, with respect to the gradient variable, allowed for the integrands considered. Esposito et al. [8] and Marcellini and Papi [20] have extended Uhlenbeck's work to allow much flexibility in the way the integrand may grow. Utilizing their results, it seems likely that one can establish analogues of the results in this paper within the setting of functionals with general growth.

2 Notation

We will use C to denote a generic constant that depends only on n, N and p, unless otherwise specified. The value of C may change from line to line. To denote the open ball of radius ρ centered at $\mathbf{x}_0 \in \mathbb{R}^n$, we use $\mathcal{B}_{\mathbf{x}_0,\rho}$. Define $\mathcal{B}_\rho := \mathcal{B}_{\mathbf{0},\rho}$ and $\mathcal{B} := \mathcal{B}_1$. Set

$$\mathcal{H}^+ := \left\{\mathbf{x} \in \mathbb{R}^n \mid x_n > 0\right\} \qquad \text{and} \qquad \mathcal{D} := \left\{\mathbf{x} \in \mathbb{R}^n \mid x_n = 0\right\}.$$

Generally, we will attach a subscript "$\mathbf{x}_0, \rho$" to a set's symbol to denote that set's intersection with $\mathcal{B}_{\mathbf{x}_0,\rho}$ and attach a superscript "$+$" to denote that set's intersection with $\mathcal{H}^+$. Given $\Omega \subset \mathbb{R}^n$, for example, we will write $\Omega_{\mathbf{x}_0,\rho}$ for the set $\Omega \cap \mathcal{B}_{\mathbf{x}_0,\rho}$ and Ω^+ for the set $\Omega \cap \mathcal{H}^+$. Unfortunately, this convention leaves some ambiguity about what $\mathcal{B}_{\mathbf{x}_0,\rho}$ and $\mathcal{B}^+_{\mathbf{x}_0,\rho}$ actually represent; unless otherwise indicated, we will use $\mathcal{B}_{\mathbf{x}_0,\rho}$ to denote the open ball of radius ρ centered at $\mathbf{x}_0$ and $\mathcal{B}^+_{\mathbf{x}_0,\rho} = \mathcal{H}^+ \cap \mathcal{B}_{\mathbf{x}_0,\rho}$. We write $|\mathcal{U}|$ for the Lebesgue measure of $\mathcal{U} \subseteq \mathbb{R}^n$. The support of a mapping $\mathbf{u}$ is denoted by $\operatorname{supp}(\mathbf{u})$. For the characteristic function of $\mathcal{U} \subseteq \mathbb{R}^n$, we use $\chi_{\mathcal{U}}$.

Suppose that $\mathcal{U} \subset \mathbb{R}^n$ is Lebesgue measurable with $|\mathcal{U}| < +\infty$. For the mean value of a function $f \in L^1(\mathcal{U})$ on $\mathcal{U}$, we use

$$\fint_{\mathcal{U}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} := \frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} f(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and for brevity,

$$(f)_{\mathbf{x}_0,\rho} := \fint_{\mathcal{B}_{\mathbf{x}_0,\rho}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \text{and} \quad (f)^+_{\mathbf{x}_0,\rho} := \fint_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Analogous notation will be used for vector-valued mappings.

We next recall the definitions for the Morrey, Campanato and Sobolev–Morrey spaces. For the following definitions, the set $\mathcal{U} \subset \mathbb{R}^n$ is a given measurable set.

Definition 3 For each $p \in [1, +\infty)$ and $0 \le \kappa \le n$, we define the *Morrey space*

$$L^{p,\kappa}(\mathcal{U};\mathbb{R}^N) := \left\{ \mathbf{u} \in L^p(\Omega;\mathbb{R}^N) \,\middle|\, \sup_{\substack{\mathbf{x}_0\in\mathcal{U}\\ \rho>0}} \frac{1}{\rho^\kappa}\int_{\mathcal{U}_{\mathbf{x}_0,\rho}} \|\mathbf{u}(\mathbf{x})\|^p\,\mathrm{d}\mathbf{x} < \infty \right\}.$$

We will write $\mathbf{u} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{U};\mathbb{R}^N)$ if $\mathbf{u} \in L^{p,\kappa}(\mathcal{U}';\mathbb{R}^N)$ for each compact $\mathcal{U}'$ contained in $\mathcal{U}$.

Definition 4 For each $p \in [1, +\infty)$ and $0 \le \kappa \le n + p$, we define the *Campanato space*

$$\mathscr{L}^{p,\kappa}(\mathcal{U};\mathbb{R}^N) := \left\{ \mathbf{u} \in L^p(\Omega;\mathbb{R}^N) \,\middle|\, \sup_{\substack{\mathbf{x}_0\in\mathcal{U}\\ \rho>0}} \frac{1}{\rho^\kappa}\int_{\mathcal{U}_{\mathbf{x}_0,\rho}} \left\|\mathbf{u}(\mathbf{x}) - \mathbf{u}_{\mathbf{x}_0,\rho}\right\|^p\,\mathrm{d}\mathbf{x} < \infty \right\},$$

where $\mathbf{u}_{\mathbf{x}_0,\rho} := \fint_{\mathcal{U}_{\mathbf{x}_0,\rho}} \mathbf{u}\,\mathrm{d}\mathbf{x}$. As above, we will also write $\mathbf{u} \in \mathscr{L}^{p,\kappa}_{\mathrm{loc}}(\mathcal{U};\mathbb{R}^N)$ whenever $\mathbf{u} \in \mathscr{L}^{p,\kappa}(\mathcal{U}';\mathbb{R}^N)$ for each compact $\mathcal{U}'$ contained in $\mathcal{U}$.

Following [2] (see also [6]), we also introduce the Sobolev–Morrey spaces

Definition 5 For each $p \in [1, +\infty)$ and $0 \le \kappa \le n$, we will say that a mapping $\mathbf{u} \in W^{1,p}(\mathcal{U};\mathbb{R}^N)$ belongs to the *Sobolev–Morrey space* $W^{1,(p,\kappa)}(\mathcal{U};\mathbb{R}^N)$ if and only if $\mathbf{u} \in L^{p,\kappa}(\mathcal{U};\mathbb{R}^N)$ and $\nabla\mathbf{u} \in L^{p,\kappa}(\mathcal{U};\mathbb{R}^{N\times n})$.

Whenever convenient, the linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^N$ will be identified with the space of matrices $\mathbb{R}^{N\times n}$. If $\mathbf{A}, \mathbf{B}, \mathbf{C} : \mathbb{R}^n \to \mathbb{R}^N$ are linear mappings, then $\mathbf{A} : \mathbf{B} := \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$ and $\|\mathbf{A}\| := \sqrt{\mathbf{A} : \mathbf{A}}$. The mapping $\mathbf{A} \otimes \mathbf{B} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is defined by

$$[\mathbf{A} \otimes \mathbf{B}]\mathbf{C} := (\mathbf{B} : \mathbf{C})\mathbf{A}.$$

If $g \in \mathcal{C}^1(\mathbb{R}^{N\times n})$, then the matrix of first-order partial derivatives of g is denoted by $\frac{\partial}{\partial \mathbf{F}} g : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ and is that mapping satisfying

$$\left[\frac{\partial}{\partial \mathbf{F}} g(\mathbf{A})\right] : \mathbf{B} = \lim_{\rho\to 0} \frac{g(\mathbf{A} + \rho\mathbf{B}) - g(\mathbf{A})}{\rho}$$

for all $\mathbf{B} \in \mathbb{R}^{N\times n}$. If $g \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, then the fourth-order tensor of second-order partial derivatives of g is denoted by $\frac{\partial^2}{\partial \mathbf{F}^2} g : \mathbb{R}^{N\times n} \to \mathbb{R}^{(N\times n)^2}$ and satisfies

$$\left[\frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{A})\right] :: [\mathbf{C} \otimes \mathbf{B}] = \lim_{\rho\to 0} \frac{1}{\rho}\left[\frac{\partial}{\partial \mathbf{F}} g(\mathbf{A} + \rho\mathbf{C}) : \mathbf{B} - \frac{\partial}{\partial \mathbf{F}} g(\mathbf{A}) : \mathbf{B}\right].$$

3 Boundary transformations

We recall the following regarding the regularity of the boundary of Ω.

Definition 6 We say that a relatively open subset Γ of $\partial\Omega$ is a $\mathcal{C}^{1,\beta}$ portion of $\partial\Omega$, for some $0 \le \beta \le 1$, if for each point $\mathbf{x}_0 \in \Gamma$ there is an $r_{\mathbf{x}_0} > 0$ and a function $\Pi \in \mathcal{C}^{1,\beta}(\mathbb{R}^{n-1}; \mathbb{R})$ such that, after rotating and relabeling the coordinate axes if necessary, we have

$$\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}} = \left\{ \mathbf{x} \in \mathcal{B}_{\mathbf{x}_0, r_0} \mid x_n > \Pi(x_1, \ldots, x_{n-1}) \right\}.$$

If Γ is $\mathcal{C}^{1,\beta}$, then at each point $\mathbf{x}_0 \in \Gamma$, there is a $\mathcal{C}^{1,\beta}$-diffeomorphism that may be used to "straighten-out" the portion of Γ contained in $\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}$. Upon rotating and relabeling the coordinate axes if necessary, we may define such a diffeomorphism $\Psi \in \mathcal{C}^{1,\beta}(\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}; \mathcal{H}^+)$ by

$$\Psi_i := x_i, \quad \text{for } i = 1, \ldots, n-1$$

and

$$\Psi_n := x_n - \Pi(\hat{\mathbf{x}}),$$

with Π provided by Definition 6 above and $\hat{\mathbf{x}} = (x_1, \ldots, x_{n-1})$. Then the inverse map $\Psi^{-1} : \Psi(\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}) \to \mathbb{R}^n$ is given by

$$\Psi_i^{-1} := x_i, \quad \text{for } i = 1, \ldots, n-1$$

and

$$\Psi_n^{-1} := x_n + \Pi(\hat{\mathbf{x}}).$$

So defined, it is clear that $\Psi^{-1} \in \mathcal{C}^{1,\beta}(\Psi(\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}); \mathbb{R}^n)$, and moreover that

$$\|\nabla \boldsymbol{\Psi}(\mathbf{x})\| \le n + \|\nabla \Pi(\hat{\mathbf{x}})\|, \qquad \|\nabla \boldsymbol{\Psi}^{-1}(\mathbf{x})\| \le n + \|\nabla \Pi(\hat{\mathbf{x}})\|$$

and

$$\det \nabla \boldsymbol{\Psi}(\mathbf{x}) = 1,$$

at every $\mathbf{x} \in \Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}$. For each $\mathbf{x}_0 \in \Gamma$, we define $\boldsymbol{\Psi}_{\mathbf{x}_0}$ and $\Pi_{\mathbf{x}_0}$ as above, and put

$$\pi_{\mathbf{x}_0} := \inf_{\mathbf{x} \in \Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}} \frac{1}{n + \|\nabla \Pi_{\mathbf{x}_0}(\hat{\mathbf{x}})\|}.$$

At each $\mathbf{x}_0 \in \Gamma$, we see that $\pi_{\mathbf{x}_0} > 0$.

Fix $\mathbf{x}_0 \in \Gamma$. From the above definitions, we deduce that

$$\|\boldsymbol{\Psi}_{\mathbf{x}_0}(\mathbf{y}) - \boldsymbol{\Psi}_{\mathbf{x}_0}(\mathbf{x})\| \ge \pi_{\mathbf{x}_0} \|\mathbf{y} - \mathbf{x}\|,$$

for every $\mathbf{x}, \mathbf{y} \in \Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}$. Consequently, we find that $\mathcal{B}^+_{\mathbf{z}_0, \pi_{\mathbf{x}_0} r_{\mathbf{x}_0}} \subseteq \boldsymbol{\Psi}_{\mathbf{x}_0}(\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}})$, with $\mathbf{z}_0 = \boldsymbol{\Psi}_{\mathbf{x}_0}(\mathbf{x}_0)$; i.e. we are guaranteed that the half-ball with radius $\pi_{\mathbf{x}_0} r_{\mathbf{x}_0}$ centered at $\mathbf{z}_0$ is contained in the image under $\boldsymbol{\Psi}_{\mathbf{x}_0}$ of $\Omega_{\mathbf{x}_0, r_{\mathbf{x}_0}}$.

We translate and rescale our boundary transformations, so that each one has an image that contains the unit half-ball $\mathcal{B}^+$ and straightens-out a portion of the boundary into the set $\mathcal{D}$. For each $\mathbf{x}_0 \in \Gamma$, we define the $\mathcal{C}^{1,\beta}$-diffeomorphism $\mathbf{\Phi}_{\mathbf{x}_0} : \Omega_{\mathbf{x}_0,r_{\mathbf{x}_0}} \to \mathcal{H}^+$ by

$$\mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{x}) := \frac{1}{\pi_{\mathbf{x}_0} r_{\mathbf{x}_0}} \left[\mathbf{\Psi}_{\mathbf{x}_0}(\mathbf{x}) - \mathbf{\Psi}_{\mathbf{x}_0}(\mathbf{x}_0)\right].$$

Note that

$$\sup_{\mathbf{x}\in\Omega_{\mathbf{x}_0,r_{\mathbf{x}_0}}} \|\nabla \mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{x})\| \leq \frac{1}{\pi_{\mathbf{x}_0}^2 r_{\mathbf{x}_0}}, \quad \text{and} \quad \sup_{\mathbf{x}\in\mathcal{B}^+} \|\nabla \mathbf{\Phi}_{\mathbf{x}_0}^{-1}(\mathbf{x})\| \leq r_{\mathbf{x}_0}.$$

Furthermore, for each $\mathbf{x}_0 \in \Gamma$, we conclude that

$$\det \nabla \mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{x}) = \left(\frac{1}{\pi_{\mathbf{x}_0} r_{\mathbf{x}_0}}\right)^n$$

and

$$\|\mathbf{y} - \mathbf{x}\| \leq r_{\mathbf{x}_0} \|\mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{y}) - \mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{x})\|$$

for each $\mathbf{x}, \mathbf{y} \in \Omega_{\mathbf{x}_0,r_{\mathbf{x}_0}}$.

4 Uhlenbeck's theorem

In this section, we recall a regularity result due to K. Uhlenbeck.

Definition 7 We will say that a function $f \in \mathcal{C}^2(\mathbb{R}^{N\times n})$ has a *p-Uhlenbeck structure* if and only if there are $\Lambda_*, \Lambda^* > 0$ and an $\widetilde{f} \in \mathcal{C}^2([0,\infty))$ such that for every $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$ the following hold

$$\begin{aligned}
&\text{(i)} \quad f(\mathbf{F}) = \widetilde{f}(\|\mathbf{F}\|^2);\\
&\text{(ii)} \quad \Lambda_* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} \leq f(\mathbf{F}) \leq \Lambda^*(1 + \|\mathbf{F}\|^2)^{\frac{p}{2}};\\
&\text{(iii)} \quad \left\|\frac{\partial}{\partial \mathbf{F}} f(\mathbf{F})\right\| \leq \Lambda^*(1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}} \|\mathbf{F}\|;\\
&\text{(iv)} \quad \left\|\frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F})\right\| \leq \Lambda^*(1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}};\\
&\text{(v)} \quad \frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F}) :: [\boldsymbol{\xi} \otimes \boldsymbol{\xi}] \geq \Lambda_*(1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}} \|\boldsymbol{\xi}\|^2.
\end{aligned} \tag{U}$$

With $p > 2$ fixed for the remainder of the paper, define

$$\mathbf{V}(\mathbf{F}) := \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{4}} \mathbf{F} \quad \text{and} \quad H(\mathbf{F}) := \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}}.$$

Later, we will need to deal with families of functions that satisfy the conditions in (U) in some uniform manner.

Definition 8 Given a measurable set $\mathcal{U} \subset \mathbb{R}^N$, we will say that a family of functions $\left\{f_{\mathbf{y}}\right\}_{\mathbf{y}\in\mathcal{U}}$ has a *uniform p-Uhlenbeck* structure if and only if there are measurable $\Lambda_*, \Lambda^* : \mathcal{U} \to (0, +\infty]$ such that $\frac{\Lambda^*+1}{\Lambda_*} \in L^\infty(\mathcal{U})$ and for each $\mathbf{y} \in \mathcal{U}$ the function $f_{\mathbf{y}}$ has a p-Uhlenbeck structure, with $\Lambda_* = \Lambda_*(\mathbf{y})$ and $\Lambda^* = \Lambda^*(\mathbf{y})$.

Remark 1 If $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}}$ has a uniform p-Uhlenbeck structure, then $\frac{1}{\Lambda_*} \in L^\infty(\mathcal{U})$ and $\frac{1}{\Lambda^*} \in L^\infty(\mathcal{U})$. Thus there exists a $l > 0$ such that

$$\operatorname*{ess\,inf}_{\mathbf{y}\in\mathcal{U}} \Lambda_*(\mathbf{y}),\ \operatorname*{ess\,inf}_{\mathbf{y}\in\mathcal{U}} \Lambda^*(\mathbf{y}) \geq l.$$

We start with Uhlenbeck's regularity result for elliptic systems, which can be found in [15] or [18], with suitable modifications.

Theorem 3 *Let $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$, an invertible matrix, and $f \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, with a p-Uhlenbeck structure, be given. Suppose that the mapping $\mathbf{u} \in W^{1,p}(\mathcal{B};\mathbb{R}^N)$ is a minimizer for the functional*

$$J[\mathbf{u}] := \int_{\mathcal{B}} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0)\,\mathrm{d}\mathbf{x}.$$

Then for every $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$, we have that

$$\sup_{\mathbf{x}\in\mathcal{B}_{\mathbf{x}_0,\frac{R}{2}}} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0) \leq c_0' \fint_{\mathcal{B}_{\mathbf{x}_0,R}} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0)\,\mathrm{d}\mathbf{x}. \tag{6}$$

Moreover $\mathbf{u} \in C^{1,\sigma_0}_{\mathrm{loc}}(\mathcal{B};\mathbb{R}^N)$ for each $\sigma_0 \in (0,1)$ and for every $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$ and every $\rho < R$, we have

$$\Phi(\mathbf{x}_0,\rho) \leq c_0' \left(\frac{\rho}{R}\right)^{2\sigma_0} \Phi(\mathbf{x}_0,R), \tag{7}$$

with

$$\Phi(\mathbf{x}_0,\rho) := \fint_{\mathcal{B}_{\mathbf{x}_0,r}} \left\|\mathbf{V}(\nabla\mathbf{u}) - (\mathbf{V}(\nabla\mathbf{u}))_{\mathbf{x}_0,\rho}\right\|^2 \mathrm{d}\mathbf{x}.$$

The constant c_0' depends only upon $\mathbf{G}_0$, σ_0 and the structural parameters n, N, p and $\frac{\Lambda^*}{\Lambda_*}$.

Using a reflection argument and the above theorem, we may state the following version of Uhlenbeck's result.

Theorem 4 *Let $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$, an invertible matrix, and $f \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, with a p-Uhlenbeck structure, be given. Suppose that $\nabla\mathbf{u} \in W^{1,p}(\mathcal{B}^+;\mathbb{R}^N)$ is a minimizer for the functional*

$$J^+[\mathbf{u}] := \int_{\mathcal{B}^+} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0)\,\mathrm{d}\mathbf{x}$$

satisfying $\mathbf{u} = \mathbf{0}$ on $\mathcal{D}_{\mathbf{0},1}$, in the sense of traces. Then for every $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$,

$$\sup_{\mathbf{x}\in\mathcal{B}^+_{\mathbf{x}_0,\frac{R}{2}}} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0) \leq c_0'' \fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} f(\nabla\mathbf{u}(\mathbf{x})\mathbf{G}_0)\,\mathrm{d}\mathbf{x}. \tag{8}$$

Moreover $\mathbf{u} \in C^{1,\sigma_0}_{\mathrm{loc}}(\mathcal{B}^+ \cup \mathcal{D};\mathbb{R}^N)$ for each $\sigma_0 \in (0,1)$ and for each $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$ and every $\rho < R$, we have

$$\Phi^+(\mathbf{x}_0,\rho) \leq c_0'' \left(\frac{\rho}{R}\right)^{2\sigma_0} \Phi^+(\mathbf{x}_0,R), \tag{9}$$

with

$$\Phi^+(\mathbf{x}_0, r) := \fint_{\mathcal{B}^+_{\mathbf{x}_0,r}} \left\| \mathbf{V}(\nabla \mathbf{u}) - (\mathbf{V}(\nabla \mathbf{u}))^+_{\mathbf{x}_0,r} \right\|^2 \, \mathrm{d}\mathbf{x}.$$

The constant c_0' depends only upon $\mathbf{G}_0$, σ_0 and the structural parameters n, N, p and $\frac{\Lambda^*}{\Lambda_*}$. In particular, it does not depend upon dist$(\mathbf{x}_0, \mathcal{D}_{\mathbf{0},1})$.

Using the two theorems above, we finally state the following regularity result, which will be convenient for establishing regularity at the boundary for more general Dirichlet boundary conditions.

Theorem 5 *Let $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$, an invertible matrix, and $f \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, with a p-Uhlenbeck structure, be given. Suppose that $\nabla \mathbf{u} \in W^{1,p}(\mathcal{B}^+_{\mathbf{x}_0,R_0}; \mathbb{R}^N)$ is a minimizer for the functional*

$$J^+_{\mathbf{x}_0,R_0}[\mathbf{u}] := \int_{\mathcal{B}^+_{\mathbf{x}_0,R_0}} f(\nabla \mathbf{u}(\mathbf{x})\mathbf{G}_0) \, \mathrm{d}\mathbf{x}$$

satisfying $\mathbf{u} = \mathbf{0}$ on the set $\mathcal{D}_{\mathbf{x}_0,R_0}$, in the sense of traces. Then for each $\mathcal{B}_{\mathbf{y},R} \subset \mathcal{B}_{\mathbf{x}_0,R_0}$, we have that

$$\sup_{\mathbf{x}\in\mathcal{B}^+_{\mathbf{y},\frac{R}{4}}} f(\nabla \mathbf{u}(\mathbf{x})\mathbf{G}_0) \le c_0 \fint_{\mathcal{B}^+_{\mathbf{y},\frac{R}{2}}} f(\nabla \mathbf{u}(\mathbf{x})\mathbf{G}_0) \, \mathrm{d}\mathbf{x}. \tag{10}$$

Moreover $\mathbf{u} \in \mathcal{C}^{1,\sigma_0}_{\mathrm{loc}}(\mathcal{B}^+_{\mathbf{x}_0,R_0} \cup \mathcal{D}_{\mathbf{x}_0,R_0}; \mathbb{R}^N)$ for each $\sigma_0 \in (0,1)$ and for every $\mathcal{B}_{\mathbf{y},R} \subset \mathcal{B}_{\mathbf{x}_0,R_0}$ and every $\rho < \frac{R}{2}$, we have

$$\Phi^+(\mathbf{y}, \rho) \le c_0 \left(\frac{\rho}{R}\right)^{2\sigma_0} \Phi^+\left(\mathbf{y}, \tfrac{R}{2}\right). \tag{11}$$

The constant c_0 depends only upon $\mathbf{G}_0$, σ_0 and the structural parameters n, N, p and $\frac{\Lambda^*}{\Lambda_*}$. In particular, it does not depend upon dist$(\mathbf{y}, \mathcal{D}_{\mathbf{x}_0,R_0})$.

Proof Put $\hat{\mathbf{y}} := (y_1, \ldots, y_{n-1}, 0)$. Let $\mathbf{y} \in \mathcal{B}^+_{\mathbf{x}_0,R_0}$ and $0 < R < R_0 - \|\mathbf{y} - \mathbf{x}_0\|$ be given. We have two cases. If $\frac{R}{2} \le y_n$, then $\mathcal{B}_{\mathbf{y},\frac{R}{2}} \subset \mathcal{B}^+_{\mathbf{x}_0,R_0}$. Hence (10) and (11) follow from respectively rescaling and translating (6) and (7). If $y_n < \frac{R}{2}$, then $\mathcal{B}^+_{\mathbf{y},\frac{R}{2}} \subset \mathcal{B}^+_{\hat{\mathbf{y}},R} \subset \mathcal{B}^+_{\mathbf{x}_0,R_0}$. For this case (10) and (11) follow from respectively rescaling and translating (8) and (9). □

5 Preliminary lemmata

For this section, recall that $p > 2$ and fix $f \in \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a p-Uhlenbeck structure.

Definition 9 Let us say two functions $g_1, g_2 \in \mathcal{C}^2(\mathbb{R}^{N\times n}; \mathbb{R})$ are *asymptotically related* if and only if for each $\varepsilon > 0$ there exists a $\sigma_\varepsilon < +\infty$ such that

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_1(\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} g_2(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}} \varepsilon \|\mathbf{F}\|^{p-2},$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon$.

Remark 2 Without loss of generality, we will always assume that $\sigma_{\varepsilon_1} \le \sigma_{\varepsilon_2}$ if $\varepsilon_1 \ge \varepsilon_2$.

Remark 3 Clearly, the definition of asymptotically related functions defines an equivalence relation on the set $C^2(\mathbb{R}^{N\times n};\mathbb{R}) \times C^2(\mathbb{R}^{N\times n};\mathbb{R})$. Thus, for example, if $p \geq 4$, then we find that the functions $\mathbf{F} \mapsto \left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}}$ and $\mathbf{F} \mapsto \|\mathbf{F}\|^p$ are asymptotically related with $\sigma_\varepsilon = 1 + \frac{2^{\frac{p}{2}}}{\sqrt{\varepsilon}}\sqrt{p(p-2)(p-4+nN)}$, so any function that is asymptotically related to the function $\mathbf{F} \mapsto \|\mathbf{F}\|^p$ is also asymptotically related to $\mathbf{F} \mapsto \left(1+\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}$. The point is that although the lemmas stated below are for functions that are asymptotically related to the function f which is uniformly convex, they could have also been stated for functions that are asymptotically related to functions that are degenerately convex.

Lemma 1 *Suppose* $\phi : (0,+\infty) \to \mathbb{R}$ *satisfies the following property: there exist* $A > \frac{1}{2}$, $R_0 > 0$, $\alpha > \gamma > \beta \geq 0$ *and* $\mu \geq 0$ *such that for some* $0 \leq \varepsilon \leq \left(\frac{1}{2A}\right)^{\frac{\alpha}{\alpha-\gamma}}$, *we have*

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right]\phi(R) + B\frac{R^{\beta+\mu}}{\rho^\mu}, \tag{12}$$

for each $0 < \rho \leq R \leq R_0$. *Then*

$$\phi(\rho) \leq (2A)^{\frac{\alpha}{\alpha-\gamma}}\left(\frac{\rho}{R}\right)^\gamma \phi(R) + B(2A)^{\frac{\beta+\mu}{\alpha-\gamma}}\left[\frac{(2A)^{\frac{\alpha}{\alpha-\gamma}}}{(2A)^{\frac{\gamma-\beta}{\alpha-\gamma}}-1}+1\right]\rho^\beta$$

for all $0 < \rho \leq R \leq R_0$.

Proof Our proof is a minor modification of the proof for Lemma 2.1 provided on [13, p. 86]. Put $\tau := \left(\frac{1}{2A}\right)^{\frac{1}{\alpha-\gamma}}$. Let ρ and R satisfying $0 < \rho \leq R \leq R_0$ be given. Choose $k \geq 0$ so that $\tau^{k+1}R \leq \rho \leq \tau^k R$.

We first observe that $\varepsilon \leq \tau^\alpha$, so

$$A\left[\tau^\alpha + \varepsilon\right] \leq 2A\tau^\alpha = \tau^\gamma.$$

From (12), we deduce that

$$\begin{aligned}\phi(\rho) &\leq A\left[\left(\frac{\rho}{\tau^k R}\right)^\alpha + \varepsilon\right]\phi(\tau^k R) + B\frac{\left(\tau^k R\right)^{\beta+\mu}}{\rho^\mu}\\ &\leq 2A\phi(\tau^k R) + B\tau^{-(\beta+\mu)}.\end{aligned} \tag{13}$$

Using induction, we next argue that

$$\begin{aligned}\phi(\tau^k R) &\leq A\left[\tau^\alpha + \varepsilon\right]\phi(\tau^{k-1}R) + B\frac{\left(\tau^{k-1}R\right)^{\beta+\mu}}{\left(\tau^k R\right)^\mu}\\ &\leq \tau^\gamma\phi(\tau^{k-1}R) + B\frac{\left(\tau^{k+1}R\right)^\beta}{\tau^{2\beta+\mu}}\\ &\ \ \vdots\\ &\leq \tau^{k\gamma}\phi(R) + B\frac{\left(\tau^{k+1}R\right)^\beta}{\tau^{2\beta+\mu}}\left[\sum_{j=0}^{k-1}\tau^{j(\gamma-\beta)}\right]\\ &\leq \frac{1}{\tau^\gamma}\left(\frac{\rho}{R}\right)^\gamma\phi(R) + \frac{B}{\tau^{\beta+\mu}}\left[\frac{1}{\tau^\beta-\tau^\gamma}\right]\rho^\beta.\end{aligned}$$

Plugging this into (13) yields the Lemma's conclusion. □

The proof for the following lemma can be found in [14] or [18], for instance.

Lemma 2 *For any* $\mathbf{E}, \mathbf{F} \in \mathbb{R}^{N\times n}$, *we have*

$$\frac{1}{c_1} \leq \frac{\|\mathbf{V}(\mathbf{E}) - \mathbf{V}(\mathbf{F})\|^2}{\left(1 + \|\mathbf{E}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\mathbf{E} - \mathbf{F}\|^2} \leq c_1,$$

with $c_1 = p^2 2^{\frac{3p+4}{2}}$.

The first statement in the next lemma follows from the convexity of f, property $(\mathrm{U})_{\mathrm{v}}$, and the previous lemma (see [19] or [23], for example). The second statement follows from the growth properties of the second derivatives of f, property $(\mathrm{U})_{\mathrm{iv}}$, and another application of the lemma above.

Lemma 3 *For any* $\mathbf{E}, \mathbf{F} \in \mathbb{R}^{N\times n}$, *we have*

$$\frac{\Lambda_*}{c_2}\|\mathbf{V}(\mathbf{E}) - \mathbf{V}(\mathbf{F})\|^2 \leq f(\mathbf{E}) - f(\mathbf{F}) - \frac{\partial}{\partial \mathbf{F}} f(\mathbf{F}) : [\mathbf{E} - \mathbf{F}]$$

and

$$c_2 \Lambda^* \|\mathbf{V}(\mathbf{E}) - \mathbf{V}(\mathbf{F})\|^2 \geq f(\mathbf{E}) - f(\mathbf{F}) - \frac{\partial}{\partial \mathbf{F}} f(\mathbf{F}) : [\mathbf{E} - \mathbf{F}],$$

with $c_2 = 2^{\frac{3(p-2)}{2}} (2 + nN)\, c_1$.

The following lemma is easily verified using $(\mathrm{U})_{\mathrm{ii}}$.

Lemma 4 *For any* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{N\times n}$, *we have*

$$f(\mathbf{B}) \leq c_3 \left\{ \frac{1}{\Lambda_*} f(\mathbf{A}) + \|\mathbf{A} - \mathbf{B}\|^p \right\},$$

with $c_3 = 2^{p-1}\Lambda^*$.

We will use a modified version of Lemma 5.1 in [15].

Lemma 5 *Suppose that* $g \in C^2(\mathbb{R}^{N\times n})$ *is asymptotically related to* f. *Put*

$$L := 1 + 2\Lambda^* \quad \text{and} \quad a := 1 + \max\left\{ \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F}) \right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*} \right\}.$$

Then

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F}) \right\| < L\left(a^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}, \tag{14}$$

for each $\mathbf{F} \in \mathbb{R}^{N\times n}$, *and*

$$\left| \int_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} f(t\mathbf{F}+(1-t)\mathbf{F}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g(t\mathbf{F}+(1-t)\mathbf{F}_0) \right] :: [(\mathbf{F}-\mathbf{F}_0)\otimes(\mathbf{F}-\mathbf{F}_0)]\, \mathrm{d}t \right|$$
$$< \varepsilon\left(\|\mathbf{F}-\mathbf{F}_0\|^2+\lambda^2\right)\left(a^2+\|\mathbf{F}_0\|^2+\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}, \tag{15}$$

whenever

$$\|\mathbf{F}_0\|^2 + \lambda^2 > Q_\varepsilon^2, \tag{16}$$

with $Q_\varepsilon := \left(\frac{2^{p+3}L^2}{\varepsilon^2} + 2\right)^{\frac{1}{2}} \sigma_\varepsilon$.

Proof Our statement is different from that given in [15], but the proof is essentially the same. We first establish the growth estimate for $\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F}) \right\|$. Then we prove the estimate in the lemma.

By Definition 9, whenever $\|\mathbf{F}\| > \sigma_{\Lambda^*}$, we must have

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F}) \right\| \leq \frac{\Lambda^*}{2^{\frac{p}{2}}} \|\mathbf{F}\|^{p-2} + \Lambda^* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}.$$

Thus

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F}) \right\| < L \left(a^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}},$$

for each $\mathbf{F} \in \mathbb{R}^{N \times n}$, which is (14).

We now follow the argument given in [15]. Let $\varepsilon > 0$ and $\mathbf{F}, \mathbf{F}_0 \in \mathbb{R}^{N \times n}$ satisfying the condition in (16) be given, and set

$$I := \{t \in [0, 1] : \|t\mathbf{F} + (1 - t)\mathbf{F}_0\| \leq \sigma_\varepsilon\}.$$

Since f and g are asymptotically related and f has a p-Uhlenbeck structure, we may begin with the estimate

$$\left| \int_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} f(t\mathbf{F}+(1-t)\mathbf{F}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g(t\mathbf{F}+(1-t)\mathbf{F}_0) \right] :: [(\mathbf{F}-\mathbf{F}_0) \otimes (\mathbf{F}-\mathbf{F}_0)]\, dt \right|$$
$$< \left(\frac{1}{2}\varepsilon + 2^{\frac{p-2}{2}} (L+\Lambda^*)|I| \right) \|\mathbf{F}-\mathbf{F}_0\|^2 \left(a + \|\mathbf{F}\|^2 + \|\mathbf{F}_0\|^2\right)^{\frac{p-2}{2}}.$$

We will show that

$$2^{\frac{p-2}{2}} (L + \Lambda^*)|I| \|\mathbf{F} - \mathbf{F}_0\|^2 \leq \frac{1}{2}\varepsilon \left(\|\mathbf{F} - \mathbf{F}_0\|^2 + \lambda^2\right). \tag{17}$$

To this end, let S be the line segment joining $\mathbf{F}$ and $\mathbf{F}_0$. Then

$$|I| = \frac{|S \cap \left\{\mathbf{P} \in \mathbb{R}^{N \times n} : \|\mathbf{P}\| \leq \sigma_\varepsilon\right\}|}{\|\mathbf{F} - \mathbf{F}_0\|} \leq \frac{2\sigma_\varepsilon}{\|\mathbf{F} - \mathbf{F}_0\|}.$$

If

$$\|\mathbf{F} - \mathbf{F}_0\| > \frac{2^{\frac{p+2}{2}} L}{\varepsilon} \sigma_\varepsilon,$$

then since $L \geq \Lambda^*$, we deduce that

$$|I| < \frac{\varepsilon}{2^{\frac{p}{2}} (2L)},$$

which implies (17). It only remains to consider the case where

$$\|\mathbf{F} - \mathbf{F}_0\| \leq \frac{2^{\frac{p+2}{2}} L}{\varepsilon} \sigma_\varepsilon.$$

In this case, condition (16) implies that either

$$\|\mathbf{F}_0\| > Q_\varepsilon \quad \text{or} \quad \lambda > Q_\varepsilon.$$

If $\|\mathbf{F}_0\| > Q_\varepsilon$, then for each $t \in [0, 1]$, we argue that

$$\|t\mathbf{F} + (1-t)\mathbf{F}_0\| \geq Q_\varepsilon - \|\mathbf{F} - \mathbf{F}_0\| > \sigma_\varepsilon,$$

which implies that $|I| = 0$ yielding (17). On the other hand, if $\lambda > Q_\varepsilon$, then

$$2^{\frac{p-2}{2}}(L + 2\Lambda^*(2 + nN))|I|\|\mathbf{F} - \mathbf{F}_0\|^2 \leq 2^{\frac{p}{2}} L\sigma_\varepsilon\|\mathbf{F} - \mathbf{F}_0\| < \frac{2^{p+1}L^2\sigma_\varepsilon^2}{\varepsilon Q_\varepsilon^2}\lambda^2 < \frac{1}{2}\varepsilon\lambda^2.$$

In this case we also arrive at (17), and the lemma is proved. □

The following lemma is an easy consequence of Lemma 5 (see [23]).

Lemma 6 *Suppose that* $g \in C^2(\mathbb{R}^{N\times n})$ *is asymptotically related to* f. *Put*

$$L := 1 + 2\Lambda^* \quad \textit{and} \quad a := 1 + \max\left\{\left\|\frac{\partial^2}{\partial\mathbf{F}^2}g(\mathbf{F})\right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*}\right\}.$$

Then

$$|f(\mathbf{F}) - g(\mathbf{F})| < \varepsilon 2^{\frac{p+2}{2}}\left(a^p + \frac{1}{\Lambda_*}f(\mathbf{F})\right),$$

whenever

$$\|\mathbf{F}\| > \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}}|f(\mathbf{0}) - g(\mathbf{0})|^{\frac{1}{p}} + \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p-1}}\left\|\frac{\partial}{\partial\mathbf{F}}g(\mathbf{0})\right\|^{\frac{1}{p-1}} + Q_\varepsilon,$$

with $Q_\varepsilon = \left(\frac{2^{p+3}L^2}{\varepsilon^2} + 2\right)^{\frac{1}{2}}\sigma_\varepsilon$.

Lemma 6 implies that asymptotically convex functions must be coercive.

Lemma 7 *Suppose that* $g \in C^2(\mathbb{R}^{N\times n})$ *is asymptotically related to* f. *Then there exists a constant* $c_4 < +\infty$ *such that*

$$g(\mathbf{F}) \geq \frac{\Lambda_*}{2}\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} - c_4.$$

Proof Put

$$L := 1 + 2\Lambda^* \quad \text{and} \quad a := 1 + \max\left\{\left\|\frac{\partial^2}{\partial\mathbf{F}^2}g(\mathbf{F})\right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*}\right\}.$$

For each $\varepsilon > 0$, define

$$k_\varepsilon := \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}}|f(\mathbf{0}) - g(\mathbf{0})|^{\frac{1}{p}} + \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p-1}}\left\|\frac{\partial}{\partial\mathbf{F}}g(\mathbf{0})\right\|^{\frac{1}{p-1}} + Q_\varepsilon$$

with $Q_\varepsilon = \left(\frac{2^{p+3}L^2}{\varepsilon^2} + 2\right)^{\frac{1}{2}}\sigma_\varepsilon$.

We may use property $(U)_{ii}$ and Lemma 6, to deduce that for each $\varepsilon > 0$, if $\|\mathbf{F}\| > k_\varepsilon$, then

$$\begin{aligned}\Lambda_* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} &\leq f(\mathbf{F}) = f(\mathbf{F}) - g(\mathbf{F}) + g(\mathbf{F}) \\ &\leq \varepsilon 2^{\frac{p+2}{2}} \left(a^p + \frac{1}{\Lambda_*} f(\mathbf{F})\right) + g(\mathbf{F}) \\ &\leq \varepsilon 2^{\frac{p+2}{2}} \frac{\Lambda^*}{\Lambda_*} \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} + \varepsilon 2^{\frac{p+2}{2}} a^p + g(\mathbf{F}).\end{aligned}$$

Fixing $\varepsilon^* = \frac{\Lambda_*^2}{2^{\frac{p+4}{2}} \Lambda^*}$, we conclude that

$$g(\mathbf{F}) > \frac{\Lambda_*}{2} \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} - \frac{\Lambda_*^2}{2\Lambda^*} a^p,$$

whenever $\|\mathbf{F}\| > k_{\varepsilon^*}$. Defining

$$c_4 := \frac{\Lambda_*}{2} \left(1 + k_{\varepsilon^*}^2\right)^{\frac{p}{2}} + \frac{\Lambda_*^2}{2\Lambda^*} a^p - \min\left\{g(\mathbf{F}) : \|\mathbf{F}\| \leq k_{\varepsilon^*}\right\}, \tag{18}$$

the Lemma is proved. □

Remark 4 To facilitate the proof of Lemma 8 below, we collect and examine some of the constants defined in Lemmas 5 through 7. Suppose that $g \in C^2(\mathbb{R}^{N\times n})$ is asymptotically related to a function $f \in C^2(\mathbb{R}^{N\times n})$ with a p-Uhlenbeck structure. First, we defined

$$L := 1 + 2\Lambda^* \quad \text{and} \quad a := 1 + \max\left\{\left\|\frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F})\right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*}\right\}.$$

So defined, in Lemma 5, we showed that

$$\left\|\frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{F})\right\| \leq L\left(a^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}},$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$. Thus

$$\left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{F})\right\| \leq L\left(a^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} + \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|$$

and

$$|g(\mathbf{F})| \leq 2L\left(a^2 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} + \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{p}{p-1}} + |g(\mathbf{0})| \tag{19}$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$. We also defined

$$Q_\varepsilon := \left(\frac{2^{p+3} L^2}{\varepsilon^2} + 2\right)^{\frac{1}{2}} \sigma_\varepsilon.$$

Plugging in the definition for L, we see that

$$Q_\varepsilon \leq C\left(\frac{1 + \Lambda^*}{\varepsilon} + 1\right) \sigma_\varepsilon. \tag{20}$$

In the proof for Lemma 7, we defined

$$k_\varepsilon := \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}} |f(\mathbf{0}) - g(\mathbf{0})|^{\frac{1}{p}} + \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p-1}} \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{1}{p-1}} + Q_\varepsilon$$

and $\varepsilon^* = \frac{\Lambda_*^2}{2^{\frac{p+4}{2}}\Lambda^*}$. Property $(\mathrm{U})_{\mathrm{ii}}$ and our estimate in (20) imply

$$k_\varepsilon \le \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}} \left\{(\Lambda^*)^{\frac{1}{p}} + |g(\mathbf{0})|^{\frac{1}{p}}\right\} + \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p-1}} \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{1}{p-1}} + C\left(\frac{1+\Lambda^*}{\varepsilon} + 1\right)\sigma_\varepsilon.$$

Thus

$$k_{\varepsilon^*} \le C\left(\frac{\Lambda^*}{\Lambda_*^2}\right)^{\frac{1}{p}} \left\{(\Lambda^*)^{\frac{1}{p}} + |g(\mathbf{0})|^{\frac{1}{p}}\right\} + C\left(\frac{\Lambda^*}{\Lambda_*^2}\right)^{\frac{1}{p-1}} \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{1}{p-1}} + C\left(\frac{1+\Lambda^*}{\Lambda_*}\right)^2 \sigma_{\varepsilon^*}. \tag{21}$$

Finally the constant c_4 in Lemma 7 is defined by

$$c_4 := \frac{\Lambda_*}{2}\left(1 + k_{\varepsilon^*}^2\right)^{\frac{p}{2}} + \frac{\Lambda_*^2}{2\Lambda^*} a^p - \min\left\{g(\mathbf{F}) : \|\mathbf{F}\| \le k_{\varepsilon^*}\right\}.$$

Using the inequalities in (19) and (21), we conclude that

$$c_4 \le C(1+\Lambda_*)\left\{\left(\frac{\Lambda^*}{\Lambda_*^2}\right)\left(\Lambda^* + |g(\mathbf{0})|\right) + \left(\frac{\Lambda^*}{\Lambda_*^2}\right)^{\frac{p}{p-1}} \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{p}{p-1}} + \left(\frac{1+\Lambda^*}{\Lambda_*}\right)^{2p} \sigma_{\varepsilon^*}^p + \left(\frac{\Lambda_*^2}{\Lambda^*} + \Lambda^*\right) a^p + \left\|\frac{\partial}{\partial \mathbf{F}} g(\mathbf{0})\right\|^{\frac{p}{p-1}} + |g(\mathbf{0})|\right\}.$$

In each estimate above, the constant C depends only on n, N and p.

6 Global Morrey regularity

For this section, let $\omega \in \mathcal{C}^0([0,+\infty))$ be a non-decreasing function satisfying $\omega(0) = 0$. Also fix $0 \le \kappa < n$. We will prove a global Morrey regularity result for almost minimizers of asymptotically convex functionals.

Definition 10 Given a measurable set $\mathcal{U} \subset \mathbb{R}^n$, we will say that a family of functions $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ is $L^{p,\kappa}$*-asymptotically related* to a family of functions $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ if and only if for each $\varepsilon > 0$ there exists a $\sigma_\varepsilon \in L^{p,\kappa}(\mathcal{U})$ such that for every $\mathbf{y} \in \mathcal{U}$, we have

$$\left\|\frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f_{\mathbf{y}}(\mathbf{F})\right\| < \frac{1}{2^{\frac{p}{2}}}\varepsilon\|\mathbf{F}\|^{p-2},$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon(\mathbf{y})$.

For this section, given a measurable set $\mathcal{U} \subset \mathbb{R}^n$ and a family of functions $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ that is $L^{p,\kappa}$-asymptotically related to a family of functions $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a uniform p-Uhlenbeck structure, we will assume the following growth properties:

(i) there exists an a $\in \mathrm{L}^{\mathrm{p},\kappa}(\mathcal{U})$ such that

$$a(\mathbf{y}) \geq 1 + \max\left\{ \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) \right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*(\mathbf{y})}(\mathbf{y}) \right\}.$$

for each $\mathbf{y} \in \mathcal{U}$; (G)

(ii) there exists a b $\in \mathrm{L}^{\mathrm{p},\kappa}(\mathcal{U})$ such that

$$b(\mathbf{y}) \geq |g_{\mathbf{y}}(\mathbf{0})|^{\frac{1}{p}} + \left\| \frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}(\mathbf{0}) \right\|^{\frac{1}{p-1}}.$$

Recall that

$$H(\mathbf{F}) := \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}}$$

for all $\mathbf{F} \in \mathbb{R}^{N\times n}$.

Lemma 8 *Suppose that* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *has a uniform p-Uhlenbeck structure and that* $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *possesses the growth properties* (G) *and is* $L^{p,\kappa}$-*asymptotically related to* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$. *For each* $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$, *define* $R_{\mathbf{y}} := 1 - \|\mathbf{y}\|$. *Let* $\mathbf{x}_0 \in \mathcal{B}^+ \cup \mathcal{D}$ *be given. Let* $\mathbf{A} \in L^{p,\kappa}(\mathcal{B}^+_{\mathbf{x}_0,R_{\mathbf{x}_0}}; \mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^0(\overline{\mathcal{B}^+_{\mathbf{x}_0,R_{\mathbf{x}_0}}}; \mathbb{R}^{n\times n})$, *with a matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^0(\overline{\mathcal{B}^+_{\mathbf{x}_0,R_{\mathbf{x}_0}}}; \mathbb{R}^{n\times n})$, *be given. Define the functional* $K^+_{\mathbf{y},R_{\mathbf{y}}} : W^{1,1}(\mathcal{B}^+_{\mathbf{y},R_{\mathbf{y}}}; \mathbb{R}^N) \to \overline{\mathbb{R}}$ *by*

$$K^+_{\mathbf{y},R_{\mathbf{y}}}[\mathbf{w}] := \int_{\mathcal{B}^+_{\mathbf{y},R_{\mathbf{y}}}} g_{\mathbf{y}}([\nabla \mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

and let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B}^+_{\mathbf{x}_0,R_0})$ *and* $\{T_\varepsilon\}_{\varepsilon>0} \subseteq [0,\infty)$ *be given. Suppose that the mapping* $\mathbf{w} \in W^{1,p}(\mathcal{B}^+; \mathbb{R}^N)$, *satisfying* $\mathbf{w} = \mathbf{0}$ *on* $\mathcal{D}_{\mathbf{0},1}$, *in the sense of traces, is a* $(K^+_{\mathbf{y},R_{\mathbf{y}}}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$-*minimizer at each* $\mathbf{y} \in \mathcal{B}^+_{\mathbf{x}_0,R_{\mathbf{x}_0}}$. *Then there are constants* c_5 *and* c_6, *independent of* $dist(\mathbf{x}_0, \mathcal{D}_{\mathbf{0},1})$, *such that for every* $0 < \rho \leq R \leq R_{\mathbf{x}_0}$, *we have*

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} H(\nabla \mathbf{w}\mathbf{G}(\mathbf{x}_0))\,\mathrm{d}\mathbf{x} \leq c_5 \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} H(\nabla \mathbf{w}\mathbf{G}(\mathbf{x}_0))\,\mathrm{d}\mathbf{x} + c_6\rho^\kappa. \tag{22}$$

Proof To begin, we have several preliminary definitions and observations to make. By Lemma 7 and Definition 1, the result is trivial if $\kappa = 0$, so we assume that $0 < \kappa < n$.

We first rescale several quantities related to the families $\{f_{\mathbf{y}}\}$ and $\{g_{\mathbf{y}}\}$. For each $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$, define

$$f^*_{\mathbf{y}}(\mathbf{F}) := \frac{1}{\Lambda^*(\mathbf{y})} f_{\mathbf{y}}(\mathbf{F}) \quad \text{and } g^*_{\mathbf{y}}(\mathbf{F}) := \frac{1}{\Lambda^*(\mathbf{y})} g_{\mathbf{y}}(\mathbf{F}),$$

and for each $\varepsilon > 0$ define $\sigma^*_\varepsilon \in L^{p,\kappa}(\mathcal{B}^+)$ by $\sigma^*_\varepsilon(\mathbf{y}) := \sigma_{\varepsilon\Lambda^*(\mathbf{y})}(\mathbf{y})$. So defined, from Definition 10 we see that for each $\varepsilon > 0$, we have

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g^*_{\mathbf{y}}(\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f^*_{\mathbf{y}}(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}} \varepsilon \|\mathbf{F}\|^{p-2}, \tag{23}$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon^*(\mathbf{y})$. We further point out that by the growth conditions in $(\mathrm{U})_{\mathrm{ii}}$,

$$\frac{\Lambda_*(\mathbf{y})}{\Lambda^*(\mathbf{y})}\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}} \le f_{\mathbf{y}}^*(\mathbf{F}) \le \left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}},$$

for each $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$.

Put

$$L^*(\mathbf{y}) := \frac{1}{\Lambda^*(\mathbf{y})} + 2, \tag{24}$$

so $L^* \in L^\infty(\mathcal{B}^+)$. For each $\varepsilon > 0$, define $Q_\varepsilon^* \in L^{p,\kappa}(\mathcal{B}^+)$ by

$$Q_\varepsilon^*(\mathbf{y}) := \left(\frac{2^{p+3}\left(L^*(\mathbf{y})\right)^2}{\varepsilon^2} + 2\right)^{\frac{1}{2}} \sigma_\varepsilon^*(\mathbf{y}).$$

Let $a^* \in L^{p,\kappa}(\mathcal{B}^+)$ be given by

$$a^*(\mathbf{y}) := 1 + \frac{a(\mathbf{y})}{\Lambda^*(\mathbf{y})}.$$

Using (14) in Lemma 5 and $(\mathrm{G})_{\mathrm{ii}}$, for every $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$ and $\mathbf{F} \in \mathbb{R}^{N\times n}$ we have

$$\left\|\frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}^*(\mathbf{F})\right\| \le L^*(\mathbf{y})\left(a^*(\mathbf{y})^2 + \|\mathbf{F}\|^2\right)^{\frac{p-1}{2}} + \frac{b(\mathbf{y})^{p-1}}{\Lambda^*(\mathbf{y})} \tag{25}$$

and

$$|g_{\mathbf{y}}^*(\mathbf{F})| \le 2L^*(\mathbf{y})\left(a^*(\mathbf{y})^2 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} + 2\frac{b(\mathbf{y})^p}{\Lambda^*(\mathbf{y})}. \tag{26}$$

Furthermore, from (18) in Lemma 7, we deduce that there is a $c^* \in L^{1,\kappa}(\mathcal{B}^+)$ such that

$$g_{\mathbf{y}}^*(\mathbf{F}) \ge \frac{\Lambda_*(\mathbf{y})}{2\Lambda^*(\mathbf{y})}\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}} - c^*(\mathbf{y}). \tag{27}$$

Indeed, from Remark 4, we find that there is a constant C that depends only on n, N and p such that

$$\begin{aligned}|c^*| \le C\Bigg\{&\left(\frac{\Lambda^*}{\Lambda_*}\right)^2\left(1+\frac{|g_{\mathbf{y}}(\mathbf{0})|}{\Lambda^*}\right) + \left(\frac{\Lambda^*}{\Lambda_*^2}\right)^{\frac{p}{p-1}}\left\|\frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}(\mathbf{0})\right\|^{\frac{p}{p-1}} \\ &+ \left(\frac{\Lambda^*}{\Lambda_*}\right)^{2p}\left(\sigma_{\varepsilon_0}^*\right)^p + \left(a^*\right)^p\Bigg\},\end{aligned}$$

with $\varepsilon_0 = \left(\frac{\Lambda_*}{\Lambda^*}\right)^2 \frac{1}{2^{\frac{p+4}{2}}}$, which is essentially bounded from below by a positive number since $\frac{\Lambda^*}{\Lambda_*} \in L^\infty(\mathcal{B}^+)$.

Let $\mu \in \mathcal{C}^0((0,1))$ be the modulus of continuity for $\mathbf{G}$; i.e. μ is non-decreasing, concave, $\mu(0) = 0$ and for each $\mathbf{x}, \mathbf{y} \in \mathcal{B}^+$, we have that

$$\|\mathbf{G}(\mathbf{y}) - \mathbf{G}(\mathbf{x})\| \le \mu(\|\mathbf{y} - \mathbf{x}\|). \tag{28}$$

Finally, put

$$M_1 := \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{G}(\mathbf{x})\| + \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{G}^{-1}(\mathbf{x})\| + 1,$$

and for each $\varepsilon > 0$, define $k^*_\varepsilon \in L^{p,\kappa}(\mathcal{B}^+)$ by

$$k^*_\varepsilon(\mathbf{y}) := \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}} \left(1 + \left(\frac{|g_{\mathbf{y}}(\mathbf{0})|}{\Lambda^*}\right)^{\frac{1}{p}}\right) + \left(\frac{2}{\varepsilon\Lambda^*}\right)^{\frac{1}{p-1}} \left\| \frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}(\mathbf{0}) \right\|^{\frac{1}{p-1}} + Q^*_\varepsilon(\mathbf{y}).$$

The proof of the lemma, will be split into three steps. Ultimately, we want to use Lemma 1. In the first step, we make a comparison between $\mathbf{w}$ and a minimizer for the functional involving the integrand H. The resulting estimate involves the point-wise values of functions belonging to Morrey spaces, such as ν_ε. To take advantage of the fact that these functions are in a Morrey space, for Step 2, we integrate the estimate from Step 1. In the last step we apply Lemma 1.

Step 1 For this step, fix $\mathbf{y} \in \mathcal{B}^+_{\mathbf{x}_0, R_{\mathbf{x}_0}}$, and put $\mathbf{G}_{\mathbf{y}} := \mathbf{G}(\mathbf{y})$. We will establish the following: there is a constant C depending on only n, N and p such that for each $0 < \varepsilon \le 1$ and every $0 < \rho < \frac{R}{4} < R \le \frac{R_{\mathbf{y}}}{2}$, we have

$$\begin{aligned} \int_{\mathcal{B}^+_{\mathbf{y},\rho}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}})\, d\mathbf{x} \le c'_5 \left[\left(\frac{\rho}{R}\right)^n + \{\varepsilon + \mu(R) + \omega(R) + T_\varepsilon \lambda(R)\}\right] \int_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}})\, d\mathbf{x} \\ + c''_5 \left\{1 + k^*_\varepsilon(\mathbf{y})^p + a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*)^{\frac{p}{p-1}}} + c^*(\mathbf{y})\right\} R^n \\ + c''_5 \left\{ \frac{\|\mathbf{A}\|^p_{L^{p,\kappa}}}{\varepsilon^{p-1}} R^\kappa + \frac{\|\nu_\varepsilon\|_{L^{1,\kappa}}}{\Lambda^*} R^\kappa + \frac{|\nu_\varepsilon(\mathbf{y})|}{\Lambda^*} R^n \right\}. \end{aligned} \tag{29}$$

Here, we have put

$$c'_5 := C M_1^{2p} \|1 + c_0\|_{L^\infty} \left\| \frac{\Lambda^* + 1}{\Lambda_*} \right\|^2_{L^\infty}, \qquad c''_5 := C M_1^{2p}\, (1 + \mu(1)) \left\| \frac{\Lambda^* + 1}{\Lambda_*} \right\|^2_{L^\infty}$$

and

$$\lambda(R) := M_1^{tp} \left\| \frac{\Lambda^* + 1}{\Lambda_*} \right\|^t_{L^\infty} \left[\sup_{\mathbf{z} \in \mathcal{B}^+_{\mathbf{x}_0, R_0}} \left(\int_{\mathcal{B}^+_{\mathbf{x}_0, R_0} \cap \mathcal{B}_{\mathbf{z},R}} H(\nabla \mathbf{w} \mathbf{G}(\mathbf{z}))\, d\mathbf{x} \right)^{t-1} \right], \tag{30}$$

with $t > 1$ provided by Definition 1.

Fix $R \le \frac{R_{\mathbf{y}}}{2}$. Let $\mathbf{v} \in W^{1,p}(\mathcal{B}^+_{\mathbf{y},R_{\mathbf{y}}}; \mathbb{R}^N)$ be the minimizer for the functional

$$J^+_{\mathbf{y},R}[\mathbf{v}] := \int_{\mathcal{B}^+_{\mathbf{y},R}} f^*_{\mathbf{y}}(\nabla \mathbf{v}(\mathbf{x}) \mathbf{G}_{\mathbf{y}})\, d\mathbf{x}$$

satisfying $[\mathbf{v} - \mathbf{w}] \in W^{1,p}_0(\mathcal{B}^+_{\mathbf{y},R}; \mathbb{R}^N)$. Let $0 < \rho \le \frac{R}{4}$ and $0 < \varepsilon \le 1$ be given.

Using Lemma 4 and (10) from Theorem 5, we deduce that

$$\int_{\mathcal{B}^+_{\mathbf{y},\rho}} f^*_{\mathbf{y}}(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}})\, d\mathbf{x} \le c_0 c_3 \left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{\mathbf{y},\frac{R}{2}}} f^*_{\mathbf{y}}(\nabla \mathbf{v} \mathbf{G}_{\mathbf{y}})\, d\mathbf{x} + c_3 \int_{\mathcal{B}^+_{\mathbf{y},\frac{R}{2}}} \|[\nabla \mathbf{v} - \nabla \mathbf{w}] \mathbf{G}_{\mathbf{y}}\|^p\, d\mathbf{x}.$$

Since $\mathbf{v}$ is a $J^+_{\mathbf{y},R}$-minimizer, the above inequality is preserved with $\nabla\mathbf{v}$ replaced with $\nabla\mathbf{w}$ in the first integral on the right-hand side. Furthermore, $\mathbf{v}$ satisfies the Euler–Lagrange system associated with $J^+_{\mathbf{y},R}$, so Lemmas 2 and 3 give us

$$\begin{aligned}\frac{\Lambda_*}{\Lambda^*}\int_{\mathcal{B}^+_{\mathbf{y},\rho}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x} &\le c_0c_3\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x}\\ &\quad+Cc_1c_3\int_{\mathcal{B}^+_{\mathbf{y},R}}\|\mathbf{V}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})-\mathbf{V}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})\|^2\,\mathrm{d}\mathbf{x}\\ &\le c_0c_3\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x}\\ &\quad+Cc_1c_2c_3\int_{\mathcal{B}^+_{\mathbf{y},R}}\left\{f^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})-f^*_\mathbf{y}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})\right\}\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{31}$$

Note that due to the definition of $f^*_\mathbf{y}$, the constants c_1, c_2 and c_3 actually only depend upon n, N and p. Moreover the constant c_0 from Theorem 5, which depends on $\mathbf{y}$, can be uniformly bounded in terms of n, N, p and $\left\|\frac{1+\Lambda^*}{\Lambda_*}\right\|_{L^\infty}$.

We will now work to estimate the last integral above. By (1) in our definition of a $(K^+_{\mathbf{y},R_\mathbf{y}},\omega,\{\nu_\varepsilon,T_\varepsilon\})$-minimizer at $\mathbf{y}$, for some $t>1$ we have that

$$\begin{aligned}&\int_{\mathcal{B}^+_{\mathbf{y},R}}\left\{f^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})-f^*_\mathbf{y}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})\right\}\,\mathrm{d}\mathbf{x}\\ &\quad\le\int_{\mathcal{B}^+_{\mathbf{y},R}}\left\{f^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})-g^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\right\}+\left\{g^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})-g^*_\mathbf{y}([\nabla\mathbf{w}+\mathbf{A}]\mathbf{G})\right\}\mathrm{d}\mathbf{x}\\ &\qquad+\int_{\mathcal{B}^+_{\mathbf{y},R}}\left\{g^*_\mathbf{y}([\nabla\mathbf{v}+\mathbf{A}]\mathbf{G})-g^*_\mathbf{y}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})\right\}+\left\{g^*_\mathbf{y}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})-f^*_\mathbf{y}(\nabla\mathbf{v}\mathbf{G}_\mathbf{y})\right\}\mathrm{d}\mathbf{x}\\ &\qquad+\frac{\omega(R)+\varepsilon}{\Lambda^*}\int_{\mathcal{B}^+_{\mathbf{y},R}}\left(1+\|\nabla\mathbf{w}\|^p+\|\nabla\mathbf{w}-\nabla\mathbf{v}\|^p\right)\mathrm{d}\mathbf{x}+\frac{1}{\Lambda^*}\int_{\mathcal{B}^+_{\mathbf{y},R}}|\nu_\varepsilon|\,\mathrm{d}\mathbf{x}+\frac{R^n}{\Lambda^*}|\nu_\varepsilon(\mathbf{y})|\\ &\qquad+T_\varepsilon\left(\int_{\mathcal{B}^+_{\mathbf{y},R}}\|\nabla\mathbf{w}-\nabla\mathbf{v}\|^p\,\mathrm{d}\mathbf{x}\right)^t\\ &\quad\le I_1+I_2+I_3+I_4+I_5+\frac{R^\kappa}{\Lambda^*}\|\nu_\varepsilon\|_{L^{1,\kappa}}+\frac{R^n}{\Lambda^*}|\nu_\varepsilon(\mathbf{y})|\\ &\qquad+T_\varepsilon\left(\int_{\mathcal{B}^+_{\mathbf{y},R}}\|\nabla\mathbf{w}-\nabla\mathbf{v}\|^p\,\mathrm{d}\mathbf{x}\right)^t.\end{aligned}\tag{32}$$

For I_1, we deduce from (23), Lemma 6, $(\mathrm{U})_{\mathrm{ii}}$ and (27) that

$$I_1 \le \int\limits_{\{\mathbf{x}\in\mathcal{B}^+_{\mathbf{y},R}:\,\|\nabla\mathbf{w}\mathbf{G}_\mathbf{y}\|>k^*_\varepsilon\}} \left\{f^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y}) - g^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\right\}\mathrm{d}\mathbf{x} + \int\limits_{\{\mathbf{x}\in\mathcal{B}^+_{\mathbf{y},R}:\,\|\nabla\mathbf{w}\mathbf{G}_\mathbf{y}\|\le k^*_\varepsilon\}} \left\{f^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y}) - g^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\right\}\mathrm{d}\mathbf{x}$$

$$\le \varepsilon C\frac{\Lambda^*}{\Lambda_*}\int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x} + \varepsilon C a^*(\mathbf{y})^p + C\left\{1+k^*_\varepsilon(\mathbf{y})^p + c^*(\mathbf{y})\right\}|\mathcal{B}|R^n.$$

Since $\mathbf{v}$ is a $J^+_{\mathbf{y},R}$-minimizer, with (26), we similarly estimate

$$I_4 \le \varepsilon C\frac{\Lambda^*}{\Lambda_*}\int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x} + \varepsilon C a^*(\mathbf{y})^p + CL^*\left\{1+k^*_\varepsilon(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{\Lambda^*}\right\}|\mathcal{B}|R^n.$$

Now for I_2, we have by (25) that

$$I_2 = \int\limits_{\mathcal{B}^+_{\mathbf{y},R}}\int\limits_0^1 \frac{\partial}{\partial\mathbf{F}} g^*_\mathbf{y}(\nabla\mathbf{w}\mathbf{G} + t\nabla\mathbf{w}(\mathbf{G}_\mathbf{y}-\mathbf{G}) + (1-t)\mathbf{A}\mathbf{G}) : [\nabla\mathbf{w}(\mathbf{G}_\mathbf{y}-\mathbf{G}) - \mathbf{A}\mathbf{G}]\,\mathrm{d}t\,\mathrm{d}\mathbf{x}$$

$$\le CM_1^{2p-1}L^*\int\limits_{\mathcal{B}^+_{\mathbf{y},R}}\left\{a^*(\mathbf{y})^{p-1} + \frac{b(\mathbf{y})^{p-1}}{\Lambda^*} + \|\nabla\mathbf{w}\mathbf{G}_\mathbf{y}\|^{p-1} + \|\mathbf{A}\|^{p-1}\right\}$$

$$\times\left(\|\nabla\mathbf{w}\mathbf{G}_\mathbf{y}\|\|\mathbf{G}-\mathbf{G}_\mathbf{y}\| + \|\mathbf{A}\|\right)\mathrm{d}\mathbf{x}.$$

Young's inequality and the modulus of continuity for $\mathbf{G}$, defined in (28), yields

$$I_2 \le CM_1^{2p-1}L^*(\varepsilon+\mu(R))\int\limits_{\mathcal{B}^+_{\mathbf{y},R}}\left(1+\|\nabla\mathbf{w}\mathbf{G}_\mathbf{y}\|^2\right)^{\frac{p}{2}}\,\mathrm{d}x$$

$$+ CM_1^{2p-1}L^*(1+\mu(R))\left(a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*)^{\frac{p}{p-1}}}\right)|\mathcal{B}|R^n$$

$$+ CM_1^{2p-1}L^*\left(\frac{1}{\varepsilon^{p-1}} + \mu(R)\right)\int\limits_{\mathcal{B}^+_{\mathbf{y},R}}\|\mathbf{A}\|^p\,\mathrm{d}x.$$

The definition of H and the hypothesis that $\mathbf{A}\in L^{p,\kappa}(\mathcal{B}^+_{\mathbf{x}_0,R_0};\mathbb{R}^{N\times n})$ allow us to write

$$I_2 \le CM_1^{2p-1}L^*(\varepsilon+\mu(R))\int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla\mathbf{w}\mathbf{G}_\mathbf{y})\,\mathrm{d}\mathbf{x}$$

$$+ CM_1^{2p-1}L^*(1+\mu(R))\left(a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*)^{\frac{p}{p-1}}}\right)|\mathcal{B}|R^n$$

$$+ CM_1^{2p-1}L^*\left(\frac{1}{\varepsilon^{p-1}} + \mu(R)\right)\|\mathbf{A}\|^p_{L^{p,\kappa}}R^\kappa.$$

Since $\mathbf{v}$ is a minimizer for $J^+_{\mathbf{y},R}$, the same estimate is valid for $\frac{\Lambda_*}{\Lambda^*} I_3$. Turning to I_5 in (32), we once more use the definition of H and the fact that $\mathbf{v}$ is a $J^+_{\mathbf{y},R}$-minimizer to get

$$
\begin{aligned}
I_5 &\leq C \frac{M_1^p}{\Lambda^*} (\omega(R) + \varepsilon) \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} \left\{ \left(1 + \|\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}\|^2\right)^{\frac{p}{2}} \, \mathrm{d}\mathbf{x} + \left(1 + \|\nabla \mathbf{v} \mathbf{G}_{\mathbf{y}}\|^2\right)^{\frac{p}{2}} \right\} \mathrm{d}\mathbf{x} \\
&\leq C \frac{M_1^p}{\Lambda_*} (\omega(R) + \varepsilon) \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x}.
\end{aligned}
$$

Putting our estimates for $I_1, \ldots, I_5$ into (32), we conclude that

$$
\begin{aligned}
&\frac{\Lambda_*}{1 + \Lambda^*} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} \left\{ f^*_{\mathbf{y}}(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) - f^*_{\mathbf{y}}(\nabla \mathbf{v} \mathbf{G}_{\mathbf{y}}) \right\} \mathrm{d}\mathbf{x} \\
&\leq C M_1^{2p-1} L^* \{\varepsilon + \mu(R) + \omega(R)\} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x} \\
&\quad + C M_1^{2p-1} L^* (1 + \mu(R)) \left\{ 1 + k^*(\mathbf{y})^p_{\varepsilon} + a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*)^{\frac{p}{p-1}}} + c^*(\mathbf{y}) \right\} R^n \\
&\quad + C M_1^{2p-1} L^* \left(\frac{1}{\varepsilon^{p-1}} + \mu(R) \right) \|\mathbf{A}\|^p_{L^{p,\kappa}} R^\kappa + \|\nu_\varepsilon\|_{L^{1,\kappa}} R^\kappa + |\nu_\varepsilon(\mathbf{y})| R^n \\
&\quad + T_\varepsilon \left(\int\limits_{\mathcal{B}^+_{\mathbf{y},R}} \|\nabla \mathbf{w} - \nabla \mathbf{v}\|^p \, \mathrm{d}\mathbf{x} \right)^t.
\end{aligned} \tag{33}
$$

For the last integral above, we have

$$
\begin{aligned}
\int\limits_{\mathcal{B}^+_{\mathbf{y},R}} \|\nabla \mathbf{w} - \nabla \mathbf{v}\|^p \, \mathrm{d}\mathbf{x} &\leq C M^p \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x} + C M^p \frac{\Lambda^*}{\Lambda_*} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} f^*(\nabla \mathbf{v} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x} \\
&\leq C M^p \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x} + C M^p \frac{\Lambda^*}{\Lambda_*} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} f^*(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x} \\
&\leq C M^p \left(1 + \frac{\Lambda^*}{\Lambda_*}\right) \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{y}}) \, \mathrm{d}\mathbf{x},
\end{aligned} \tag{34}
$$

where the fact that $\mathbf{v}$ is a minimizer for $J^+_{\mathbf{y},R}$ was used. Since $0 < R \leq \frac{R_{\mathbf{y}}}{2} \leq 1$ and μ is non-decreasing, we arrive at (29) upon incorporating the estimate in (34), the definition of L^* in (24) and the definition of λ in (30) into the estimate in (33).

Step 2 The inequality in (29) is valid for each $\mathbf{y} \in \mathcal{B}^+_{\mathbf{x}_0, R_{\mathbf{x}_0}}$, so we may integrate it, with respect to the center $\mathbf{y}$, obtaining

$$\begin{aligned}
&\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \int\limits_{\mathcal{B}^+_{\mathbf{y},\rho}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}(\mathbf{y}))\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le c_5' \left[\left(\frac{\rho}{R}\right)^n + \{\varepsilon + \mu(R) + \omega(R) + T_\varepsilon \lambda(R)\}\right] \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}(\mathbf{y}))\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\quad + c_5'' R^n \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \left\{ k^*_\varepsilon(\mathbf{y})^p + a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*(\mathbf{y}))^{\frac{p}{p-1}}} + c^*(\mathbf{y}) \right\} \mathrm{d}\mathbf{y} \\
&\quad + c_5'' \left\{ \frac{\|\mathbf{A}\|^p_{L^{p,\kappa}}}{\varepsilon^{p-1}} + \frac{2}{\Lambda^*} \|\nu_\varepsilon\|_{L^{1,\kappa}} \right\} R^{n+\kappa}.
\end{aligned} \tag{35}$$

For the iterated integrals, we apply Fubini's theorem. We estimate the integral on the left from below as follows:

$$\begin{aligned}
&\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \int\limits_{\mathcal{B}^+_{\mathbf{y},\rho}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}(\mathbf{y}))\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\ge \frac{1}{M_1^{2p}} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}(\mathbf{x}_0)) \chi_{\mathcal{B}^+_{\mathbf{y},\rho}}(\mathbf{x}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,\frac{\rho}{2}}}(\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\ge \frac{1}{M_1^{2p}} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}_{\mathbf{x}_0}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,\frac{\rho}{2}}}(\mathbf{x}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,\frac{\rho}{2}}}(\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\ge \frac{\rho^n}{C M_1^{2p}} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\frac{\rho}{2}}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}_{\mathbf{x}_0})\, \mathrm{d}\mathbf{x},
\end{aligned} \tag{36}$$

where $C < +\infty$ depends only on n. For the iterated integral on the right of the inequality in (35), we estimate from above:

$$\begin{aligned}
&\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \int\limits_{\mathcal{B}^+_{\mathbf{y},R}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}(\mathbf{y}))\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le M_1^{2p} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}_{\mathbf{x}_0}) \chi_{\mathcal{B}^+_{\mathbf{y},R}}(\mathbf{x}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,\rho}}(\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le M_1^{2p} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}_{\mathbf{x}_0}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,R+\rho}}(\mathbf{x}) \chi_{\mathcal{B}^+_{\mathbf{x}_0,\rho}}(\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le C M_1^{2p} \rho^n \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,2R}} H(\nabla \mathbf{w}(\mathbf{x})\mathbf{G}_{\mathbf{x}_0})\, \mathrm{d}\mathbf{x}
\end{aligned} \tag{37}$$

For the last integral in (35), recall that $a^*, b \in L^{p,\kappa}(\mathcal{B}^+)$, $c^* \in L^{1,\kappa}(\mathcal{B}^+)$, $\frac{1}{\Lambda^*} \in L^\infty(\mathcal{B}^+)$ and $k_\varepsilon^* \in L^{p,\kappa}(\mathcal{B}^+)$ for each $0 < \varepsilon \leq 1$. Therefore, we may write

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \left\{ k_\varepsilon^*(\mathbf{y})^p + a^*(\mathbf{y})^p + \frac{b(\mathbf{y})^p}{(\Lambda^*(\mathbf{y}))^{\frac{p}{p-1}}} + c^*(\mathbf{y}) \right\} \mathrm{d}\mathbf{y} \leq \left(\|k_\varepsilon^*\|^p_{L^{p,\kappa}} + \|a^*\|^p_{L^{p,\kappa}} + \left\| \frac{1}{\Lambda^*} \right\|^{\frac{p}{p-1}}_{L^\infty} \|b\|^p_{L^{p,\kappa}} + \|c^*\|_{L^{1,\kappa}} \right) \rho^\kappa. \tag{38}$$

Upon collecting our estimates (36–38) into (35), we obtain

$$\begin{aligned} &\int_{\mathcal{B}^+_{\mathbf{x}_0,\frac{\rho}{2}}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{x}_0})\, \mathrm{d}\mathbf{x} \\ &\leq C M_1^{4p} c_5' \left[\left(\frac{\rho}{R}\right)^n + \{\varepsilon + \mu(R) + \omega(R) + T_\varepsilon \lambda(R)\} \right] \int_{\mathcal{B}^+_{\mathbf{x}_0,2R}} H(\nabla \mathbf{w} \mathbf{G}_0)\, \mathrm{d}\mathbf{x} \\ &\quad + C M_1^{2p} c_5'' \left(\|k_\varepsilon^*\|^p_{L^{p,\kappa}} + \|a^*\|^p_{L^{p,\kappa}} + \left\| \frac{1}{\Lambda^*} \right\|^{\frac{p}{p-1}}_{L^\infty} \|b\|^p_{L^{p,\kappa}} + \|c^*\|_{L^{1,\kappa}} \right) \frac{R^{n+\kappa}}{\rho^n} \\ &\quad + C M_1^{2p} c_5'' \left(\frac{\|\mathbf{A}\|^p_{L^{p,\kappa}}}{\varepsilon^{p-1}} R^\kappa + \left\| \frac{1}{\Lambda^*} \right\|_{L^\infty} \|\nu_\varepsilon\|_{L^{1,\kappa}} \frac{R^{n+\kappa}}{\rho^n} \right). \end{aligned} \tag{39}$$

Step 3 In this step we use Lemma 1. First, note that our estimate in (29) extends to allow any ρ and R satisfying $0 < \rho \leq R \leq \frac{R_{\mathbf{x}_0}}{2}$. Since μ, ω and λ are non-decreasing, we deduce that

$$\begin{aligned} \int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{x}_0})\, \mathrm{d}\mathbf{x} &\leq C M_1^{4p} c_5' \left[\left(\frac{\rho}{R}\right)^n + \{\varepsilon + \mu(R) + \omega(R) + T_\varepsilon \lambda(R)\} \right] \\ &\quad \times \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} H(\nabla \mathbf{w} \mathbf{G}_{\mathbf{x}_0})\, \mathrm{d}\mathbf{x} + C c'_{6,\varepsilon} \frac{R^{n+\kappa}}{\rho^n}, \end{aligned}$$

with C depending only on n, N and p and

$$\begin{aligned} c'_{6,\varepsilon} := M_1^{4p} c_1^2 (1+\mu(1)) \left\| \frac{\Lambda^*+1}{\Lambda_*} \right\|_{L^\infty} \Bigg(&\frac{\|\mathbf{A}\|^p_{L^{p,\kappa}}}{\varepsilon^{p-1}} + \|k_\varepsilon^*\|^p_{L^{p,\kappa}} + \|a^*\|^p_{L^{p,\kappa}} \\ &+ \left\| \frac{1}{\Lambda^*} \right\|^{\frac{1}{p-1}}_{L^\infty} \|b\|^p_{L^{p,\kappa}} + \|c^*\|_{L^{1,\kappa}} + \|\nu_\varepsilon\|_{L^{1,\kappa}} \Bigg) \end{aligned}$$

Put $\gamma := \frac{\kappa+n}{2}$. Fix $\varepsilon^* = \frac{1}{2c_5'} \left(\frac{1}{2CM_1^{4p}} \right)^{\frac{n}{n-\gamma}}$. Since μ, ω and λ are each continuous and satisfy $\mu(0) = \omega(0) = \lambda(0) = 0$, we may define

$$\widetilde{R} := \min \left\{ R \in (0, R_{\mathbf{x}_0}] : \mu(R) + \omega(R) + T_{\varepsilon^*} \lambda(R) \geq \frac{1}{2c_5'} \left(\frac{1}{2CM_1^{4p}} \right)^{\frac{n}{n-\gamma}} \right\}.$$

According to Lemma 1, for each $0 < \rho \le R \le \widetilde{R}$, we have

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} H(\nabla \mathbf{w}\mathbf{G}_{\mathbf{x}_0})\,\mathrm{d}\mathbf{x} \le \left(2CM_1^{4p}\right)^{\frac{n}{n-\gamma}} \left(\frac{\rho}{R}\right)^{\gamma} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} H(\nabla \mathbf{w}\mathbf{G}_{\mathbf{x}_0})\,\mathrm{d}\mathbf{x}$$

$$+ Cc'_{6,\varepsilon^*}\left(2CM_1^{4p}\right)^{\frac{n+\kappa}{n-\gamma}} \left[\frac{\left(2CM_1^{4p}\right)^{\frac{\gamma}{n-\gamma}}}{\left(2CM_1^{4p}\right)^{\frac{\gamma-\kappa}{n-\gamma}} - 1} + 1\right]\rho^{\kappa}.$$

Defining, for an appropriate C depending on n, N and p only,

$$c_5 := C^{\frac{n}{n-\kappa}} M_1^{\frac{8np}{n-\kappa}} \left(\frac{R_{\mathbf{x}_0}}{\widetilde{R}}\right)^{\kappa} \qquad \text{and} \qquad c_6 := c'_{6,\varepsilon^*} C^{\frac{3(n+\kappa)}{n-\kappa}} M_1^{\frac{12p(n+\kappa)}{n-\kappa}},$$

we obtain the conclusion of the Lemma, with c_5 and c_6 depending on the structural parameters for the families $\{f_{\mathbf{y}}\}$ and $\{g_{\mathbf{y}}\}$, as well as κ, $R_{\mathbf{x}_0}$, ω, $\|\nu_{\varepsilon^*}\|_{L^{1,\kappa}}$, T_{ε^*} $\|\mathbf{G}\|_{L^\infty}$, $\|\mathbf{G}^{-1}\|_{L^\infty}$, $\|\mathbf{A}\|_{L^{p,\kappa}}$, and the modulus of continuity for $\mathbf{G}$, but they do not depend upon $\operatorname{dist}(\mathbf{x}_0, \mathcal{D}_{\mathbf{0},1})$. □

Theorem 6 *Suppose that $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ has a uniform p-Uhlenbeck structure and that $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ possesses the growth properties* (G) *and is $L^{p,\kappa}$-asymptotically related to $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$. Let $\mathbf{A} \in L^{p,\kappa}(\mathcal{B}^+;\mathbb{R}^{N\times n})$ and $\mathbf{G} \in \mathcal{C}^0(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$, with a point-wise matrix inverse $\mathbf{G}^{-1} \in \mathcal{C}^0(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$, be given. For each $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$, define the functional $K^+_{\mathbf{y}} : W^{1,1}(\mathcal{B}^+;\mathbb{R}^N) \to \overline{\mathbb{R}}$ by*

$$K^+_{\mathbf{y}} := \int_{\mathcal{B}^+} g_{\mathbf{y}}([\nabla \mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B}^+)$ and $\{T_\varepsilon\}_{\varepsilon>0} \subseteq [0,\infty)$ be given. If $\mathbf{w} \in W^{1,1}(\mathcal{B}^+;\mathbb{R}^N)$ satisfies $\mathbf{w} = \mathbf{0}$ on $\mathcal{D}_{\mathbf{0},1}$, in the sense of traces, and $\mathbf{w}$ is a $(K^+_{\mathbf{y}}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$-minimizer at each $\mathbf{y} \in \mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}$, then we find that $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1};\mathbb{R}^{N\times n})$ and $\mathbf{w} \in \mathscr{L}^{p,p+\kappa}_{\mathrm{loc}}(\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1};\mathbb{R}^N)$.

Proof First, we argue that $\mathbf{w} \in W^{1,p}(\mathcal{B}^+;\mathbb{R}^N)$. Since $\mathbf{w}$ is a $(K^+_{\mathbf{0}}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$ minimizer, the coercivity property (27) of $g_{\mathbf{0}}$ implies $\nabla\mathbf{w} \in L^p(\mathcal{B}^+;\mathbb{R}^{N\times n})$. Extend $\mathbf{w}$ to $\mathcal{B}$ by reflection; i.e. define the extension of $\mathbf{w}$ by

$$\widetilde{\mathbf{w}}(\mathbf{x}) := \begin{cases} \mathbf{w}(\mathbf{x}), & \mathbf{x} \in \mathcal{B}^+; \\ -\mathbf{w}(-\mathbf{x}), & \mathbf{x} \in \mathcal{B}\backslash\mathcal{B}^+. \end{cases}$$

Due to the hypothesis that $\mathbf{w} = \mathbf{0}$, in the sense of traces, on $\mathcal{D}_{\mathbf{0},1}$, we see that $\nabla\widetilde{\mathbf{w}} \in L^p(\mathcal{B};\mathbb{R}^{N\times n})$. Moreover $(\widetilde{\mathbf{w}})_{\mathbf{0},1} = \mathbf{0}$, and therefore Poincaré's inequality implies

$$\int_{\mathcal{B}^+} \|\mathbf{w}\|^p\,\mathrm{d}\mathbf{x} = \frac{1}{2}\int_{\mathcal{B}} \|\widetilde{\mathbf{w}} - (\widetilde{\mathbf{w}})_{\mathbf{0},1}\|^p\,\mathrm{d}\mathbf{x} \le 2^{np}\int_{\mathcal{B}^+} \|\nabla\mathbf{w}\|^p\,\mathrm{d}\mathbf{x} < +\infty.$$

Thus $\mathbf{w} \in W^{1,p}(\mathcal{B}^+;\mathbb{R}^N)$ as claimed.

Now, the result is already proved if $\kappa = 0$, so we assume $0 < \kappa < n$. Let $\mathbf{y} \in \mathcal{B}^+$ be given. With $R_\mathbf{y} = 1 - \|\mathbf{y}\|$, we may use (22) in Lemma 8 to conclude that for every $0 < \rho \leq R_\mathbf{y}$ we have

$$\int\limits_{\mathcal{B}^+_{\mathbf{y},\rho}} H(\nabla\mathbf{w}\mathbf{G}(\mathbf{y}))\,\mathrm{d}\mathbf{x} \leq \left[c_5 \left(\frac{1}{R_\mathbf{y}}\right)^\kappa \int\limits_{\mathcal{B}^+_{\mathbf{y},R_\mathbf{y}}} H(\nabla\mathbf{w}\mathbf{G}(\mathbf{y}))\,\mathrm{d}\mathbf{x} + c_6 \right] \rho^\kappa, \tag{40}$$

with c_5 and c_6 independent of $\mathrm{dist}(\mathbf{y}, \mathcal{D}_{\mathbf{0},1})$. Thus $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}; \mathbb{R}^{N\times n})$.

By Poincaré's inequality, we see that

$$\int\limits_{\mathcal{B}^+_{\mathbf{y},\rho}} \|\mathbf{w} - (\mathbf{w})^+_{\mathbf{y},\rho}\|^p \mathrm{d}\mathbf{x} \leq 2^{pn}\rho^p \int\limits_{\mathcal{B}^+_{\mathbf{y},\rho}} \|\nabla\mathbf{w}\|^p \mathrm{d}\mathbf{x} \leq 2^{pn}\left[c_6\left(\frac{1}{R_\mathbf{y}}\right) K^+_\mathbf{y}[\mathbf{w}] + c_7\right]\rho^{p+\kappa}.$$

Hence $\mathbf{w} \in \mathscr{L}^{p,p+\kappa}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}; \mathbb{R}^N)$. □

Remark 5 We note that if $\mathbf{A} \in L^\infty(\mathcal{B}^+; \mathbb{R}^{N\times n})$ and $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^\infty(\mathcal{B}^+)$, then we find that $\nabla\mathbf{w} \in L^{p,\nu}_{\mathrm{loc}}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}; \mathbb{R}^{N\times n})$ and $\mathbf{w} \in \mathscr{L}^{p,p+\nu}_{\mathrm{loc}}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}; \mathbb{R}^N)$, for each $0 \leq \nu < n$.

The following may be proved in a similar fashion, using an obvious analogue of Lemma 8 for the set $\mathcal{B}$.

Theorem 7 *Suppose that* $\{f_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *has a uniform p-Uhlenbeck structure and that* $\{g_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *possesses the growth properties* (G) *and is* $L^{p,\kappa}$*-asymptotically related to the family* $\{f_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}}$. *Let* $\mathbf{A} \in L^{p,\kappa}(\mathcal{B}; \mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^0(\overline{\mathcal{B}}; \mathbb{R}^{n\times n})$, *with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^0(\overline{\mathcal{B}}; \mathbb{R}^{n\times n})$, *be given. For each* $\mathbf{y} \in \mathcal{B}$, *define the functional* $K_\mathbf{y} : W^{1,1}(\mathcal{B}; \mathbb{R}^N) \to \overline{\mathbb{R}}$ *by*

$$K_\mathbf{y}[\mathbf{w}] := \int\limits_{\mathcal{B}} g_\mathbf{y}([\nabla\mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B})$ *and* $\{T_\varepsilon\}_{\varepsilon>0} \subseteq [0,\infty)$ *be given. If* $\mathbf{w} \in W^{1,1}(\mathcal{B}; \mathbb{R}^N)$ *is a* $(K_\mathbf{y}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$*-minimizer at each* $\mathbf{y} \in \mathcal{B}$, *then* $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}; \mathbb{R}^{N\times n})$. *We also find that* $\mathbf{w} \in \mathscr{L}^{p,p+\kappa}_{\mathrm{loc}}(\mathcal{B}; \mathbb{R}^N)$.

Now we use the theorems above and a standard argument to "straighten out" smooth portions of the boundary to prove the main result for this section.

Theorem 8 *Suppose that* $\Omega \subset \mathbb{R}^n$ *and that* Γ *is a* $C^{1,0}$ *portion of* $\partial\Omega$. *Suppose that* $\{f_\mathbf{y}\}_{\mathbf{y}\in\Omega\cup\Gamma} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *has a uniform p-Uhlenbeck structure. Suppose also that* $\{g_\mathbf{y}\}_{\mathbf{y}\in\Omega\cup\Gamma} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *possesses the growth properties* (G) *and is* $L^{p,\kappa}$*-asymptotically related to* $\{f_\mathbf{y}\}_{\mathbf{y}\in\Omega\cup\Gamma}$. *Let the mappings* $\mathbf{A} \in L^{p,\kappa}_{\mathrm{loc}}(\Omega\cup\Gamma; \mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^0(\Omega \cup \Gamma; \mathbb{R}^{n\times n})$, *with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^0(\Omega \cup \Gamma; \mathbb{R}^{n\times n})$, *be given. For each* $\mathbf{y} \in \Omega \cup \Gamma$, *define* $K_\mathbf{y} : W^{1,1}(\Omega; \mathbb{R}^N) \to \overline{\mathbb{R}}$ *by*

$$K_\mathbf{y}[\mathbf{w}] := \int\limits_{\Omega} g_\mathbf{y}([\nabla\mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}_{\mathrm{loc}}(\Omega\cup\Gamma)$ *and* $\{T_\varepsilon\}_{\varepsilon>0} \subseteq [0,\infty)$ *be given. If* $\mathbf{w} \in W^{1,1}(\Omega;\mathbb{R}^N)$ *satisfies* $\mathbf{w} = \mathbf{0}$ *on* Γ*, in the sense of traces, and* $\mathbf{w}$ *is a* $(K_{\mathbf{y}}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$*-minimizer at each* $\mathbf{y} \in \Omega\cup\Gamma$*, then* $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\Omega\cup\Gamma;\mathbb{R}^{N\times n})$*. In addition, we find that* $\mathbf{w} \in \mathscr{L}^{p,p+\kappa}_{\mathrm{loc}}(\Omega\cup\Gamma;\mathbb{R}^N)$.

Proof Let $\Omega' \subset\subset \Omega\cup\Gamma$ be given, and set $\Gamma' := \overline{\Omega'}\cap\partial\Omega$. We note that Γ' is a compact set in $\mathbb{R}^n$ and $\Gamma' \subset \Gamma$. Since Γ is a $C^{1,0}$ portion of $\partial\Omega$, for each $\mathbf{x}_0 \in \Gamma'$, there is a $\mathcal{C}^{1,0}$-diffeomorphism $\mathbf{\Phi}_{\mathbf{x}_0}$ and a relatively open neighborhood in Ω' of $\mathbf{x}_0$, say $\mathcal{N}_{\mathbf{x}_0}$, such that $\mathbf{\Phi}_{\mathbf{x}_0}(\mathcal{N}_{\mathbf{x}_0}) = \mathcal{B}^+$.

Suppose that $\Gamma' \neq \emptyset$, and fix $\mathbf{x}_0 \in \Gamma'$. Put $S := 1 + \left(\frac{1}{\pi^2_{\mathbf{x}_0} r_{\mathbf{x}_0}}\right)^p$ and $U := 1 + \frac{1}{\pi^n_{\mathbf{x}_0}}$. Recalling that $t > 1$ is provided by Definition 1, for each $\mathbf{x} \in \mathcal{N}_{\mathbf{x}_0}$ and $\varepsilon > 0$ define

$$\widetilde{\mathbf{x}} := \mathbf{\Phi}_{\mathbf{x}_0}(\mathbf{x}), \quad \widetilde{\mathbf{A}}(\widetilde{\mathbf{x}}) := \mathbf{A}(\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}}))\nabla_{\widetilde{\mathbf{x}}}\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}}), \quad \widetilde{T}_\varepsilon := U^t T_{\frac{\varepsilon}{S}},$$

$$\widetilde{\mathbf{G}}(\widetilde{\mathbf{x}}) := \left[\nabla_{\widetilde{\mathbf{x}}}\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})\right]^{-1}\mathbf{G}(\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})), \quad \widetilde{\nu}_\varepsilon(\widetilde{\mathbf{x}}) := U\nu_{\frac{\varepsilon}{S}}(\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})),$$

and

$$\widetilde{\mathbf{w}}(\widetilde{\mathbf{x}}) := \mathbf{w}(\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})).$$

Define $\widetilde{\omega} \in \mathcal{C}([0,+\infty))$ by

$$\widetilde{\omega}(t) := S\omega(r_{\mathbf{x}_0} t).$$

Then we find that $\widetilde{\mathbf{A}} \in L^{p,\kappa}(\mathcal{B}^+;\mathbb{R}^{N\times n})$, $\widetilde{\mathbf{G}} \in \mathcal{C}^0(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$, $\{\widetilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B}^+)$ and $\widetilde{\mathbf{w}} \in W^{1,p}(\mathcal{B}^+;\mathbb{R}^N)$ with

$$\nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x}) = \nabla_{\widetilde{\mathbf{x}}}\widetilde{\mathbf{w}}(\widetilde{\mathbf{x}})\left[\nabla_{\widetilde{\mathbf{x}}}\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})\right]^{-1}.$$

Furthermore, we see that $\sup_{\widetilde{\mathbf{x}}\in\mathcal{B}^+}\|\nabla\mathbf{\Phi}^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{x}})\| \le r_{\mathbf{x}_0}$ and that $\det\nabla\mathbf{\Phi}_{\mathbf{x}_0} = \left(\frac{1}{\pi_{\mathbf{x}_0} r_{\mathbf{x}_0}}\right)^n$ throughout $\mathcal{N}_{\mathbf{x}_0}$. Hence, the mapping $\widetilde{\mathbf{w}}$ is a $(\widetilde{K}^+_{\widetilde{\mathbf{y}}}, \widetilde{\omega}, \{\widetilde{\nu}_\varepsilon, \widetilde{T}_\varepsilon\})$-minimizer at each $\widetilde{\mathbf{y}} \in \mathcal{B}^+$ for

$$\widetilde{K}^+_{\widetilde{\mathbf{y}}}[\widetilde{\mathbf{w}}] := \int\limits_{\mathcal{B}^+} g_{\Phi^{-1}_{\mathbf{x}_0}(\widetilde{\mathbf{y}})}\left([\nabla\widetilde{\mathbf{w}}(\widetilde{\mathbf{x}}) + \widetilde{\mathbf{A}}(\widetilde{\mathbf{x}})]\widetilde{\mathbf{G}}(\widetilde{\mathbf{x}})\right)\mathrm{d}\widetilde{\mathbf{x}}$$

and satisfies $\widetilde{\mathbf{w}} = \mathbf{0}$ on $\mathcal{D}_{\mathbf{0},1}$, in the sense of traces. By Theorem 6, we conclude that $\nabla\widetilde{\mathbf{w}} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1};\mathbb{R}^{N\times n})$, and consequently $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{N}_{\mathbf{x}_0};\mathbb{R}^{N\times n})$.

If $\Gamma' = \emptyset$, then set $\mathcal{N} = \emptyset$. Otherwise, consider the collection of $\mathcal{N}_{\mathbf{x}}$ for all $\mathbf{x} \in \Gamma'$. Since Γ' is compact, there exists a finite subcollection $\{\mathcal{N}_{\mathbf{x}_j}\}^m_{j=1}$ such that $\Gamma' \subset \bigcup^m_{j=1}\mathcal{N}_{\mathbf{x}_j}$. Setting $\mathcal{N} = \bigcup^m_{j=1}\mathcal{N}_{\mathbf{x}_j}$, we find $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{N},\mathbb{R}^{N\times n})$.

Next we use Theorem 7 to complete the proof. Put

$$d := \frac{1}{2}\left(\inf_{\mathbf{x}\in\Omega'\setminus\mathcal{N}}\mathrm{dist}(\mathbf{x},\partial\Omega)\right),$$

which is positive since $\Omega'\setminus\mathcal{N}$ is compactly contained in Ω. We use Theorem 7 to deduce for each $\mathbf{x} \in \overline{(\Omega'\setminus\mathcal{N})}$ that $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}_{\mathbf{x},d};\mathbb{R}^{N\times n})$. Since $\overline{(\Omega'\setminus\mathcal{N})}$ is compact, there is a finite collection $\{\mathcal{B}_{\mathbf{x}_{j,d}}\}^m_{j=1}$, with $\mathbf{x}_j \in \overline{(\Omega'\setminus\mathcal{N})}$, such that $\overline{(\Omega'\setminus\mathcal{N})} \subset \bigcup^m_{j=1}\mathcal{B}_{\mathbf{x}_j,d}$ and $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\bigcup^m_{j=1}\mathcal{B}_{\mathbf{x}_j,d};\mathbb{R}^{N\times n})$. We conclude that $\nabla\mathbf{w} \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{N}\cup\bigcup^m_{j=1}\mathcal{B}_{\mathbf{x}_j,d};\mathbb{R}^{N\times n})$. Since $\Omega' \subset\subset \mathcal{N}\cup\bigcup^m_{j=1}\mathcal{B}_{\mathbf{x}_j,d}$, the proof is complete. □

As a corollary to the above theorem, we have global Morrey regularity for the gradients of almost minimizers for asymptotically convex variational problems satisfying Dirichlet conditions that are compatible with a mapping that has a gradient belonging to the Morrey space $L^{p,\kappa}$.

Corollary 1 *Let $\Omega \subset \mathbb{R}^n$, with a $\mathcal{C}^{1,0}$ boundary, and $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ be given. Suppose that $g \in \mathcal{C}^2(\mathbb{R}^{N\times n})$ is asymptotically related to f. If a mapping $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a (K,ω)-minimizer for the functional*

$$K[\mathbf{u}] := \int_\Omega g(\nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

satisfying $[\mathbf{u}-\overline{\mathbf{u}}] \in W_0^{1,p}(\Omega;\mathbb{R}^N)$, then $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n}) \cap L^{p,\theta}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega;\mathbb{R}^N) \cap \mathscr{L}^{p,p+\theta}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, for each $\theta \in [0,n)$.

Proof Define $\mathbf{w} \in W_0^{1,1}(\Omega;\mathbb{R}^N)$ by $\mathbf{w} := \mathbf{u}-\overline{\mathbf{u}}$. Then $\mathbf{w}$ is a $(K,\omega,0)$-minimizer for the functional

$$\overline{K}[\mathbf{w}] := \int_\Omega g(\nabla \mathbf{w}(\mathbf{x}) + \nabla\overline{\mathbf{u}}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Since $\nabla\overline{\mathbf{u}} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$, Theorem 8 implies that $\nabla\mathbf{w} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$. Hence $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega;\mathbb{R}^N)$. The interior regularity follows from a local version of Theorem 8 for local minimizers of $\overline{K}$, for which we may take $\mathbf{A}=\mathbf{0}$. □

7 Global Lipschitz regularity

Our objective in this section is to prove global bounds for the gradient of an almost minimizer of an asymptotically convex functional. As in the previous section, we initially work on the half-ball $\mathcal{B}^+$ and establish bounds for the gradient up to the set $\mathcal{D}_{\mathbf{0},1}$, provided that the almost minimizer is $\mathbf{0}$ on $\mathcal{D}_{\mathbf{0},1}$. A similar argument gives local bounds for almost minimizers on a full ball with general boundary values. The standard program followed in the previous section then yields the global bounds for almost minimizers with boundary values compatible with a mapping with a Hölder continuous gradient.

Let $\omega_0 \geq 0$ and $0 < \beta \leq 1$ be given and define the function $\omega \in \mathcal{C}^{0,\beta}([0,\infty))$ by $\omega(r) := \omega_0 r^\beta$.

Definition 11 Given a measurable set $\mathcal{U} \subset \mathbb{R}^n$, we will say that a family of functions $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ is *uniformly asymptotically related* to a family of functions $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n};)$ if and only if for each $\varepsilon > 0$ there exists a $\sigma_\varepsilon < +\infty$ such that for every $\mathbf{y} \in \mathcal{U}$, we have

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f_{\mathbf{y}}(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}}\varepsilon \|\mathbf{F}\|^{p-2},$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon$.

For this section, given a measurable set $\mathcal{U} \subset \mathbb{R}^n$ and a family of functions $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^3(\mathbb{R}^{N\times n})$ that is uniformly asymptotically related to $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}}$ with a uniform p-Uhlenbeck structure, we will assume the following growth properties:

(i) there is an $L_1 < +\infty$ such that

$$\sup_{\mathbf{y}\in\mathcal{U}} \left\| \frac{\partial^3}{\partial \mathbf{F}^3} g_{\mathbf{y}}(\mathbf{F}) \right\| \le L_1 \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-3}{2}}$$

for all $\mathbf{F} \in \mathbb{R}^{N\times n}$; (G′)

(ii) there is a $b < +\infty$ such that

$$b \ge \sup_{\mathbf{y}\in\mathcal{U}} \left(|g_{\mathbf{y}}(\mathbf{0})| + \left\| \frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}(\mathbf{0}) \right\| + \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{0}) \right\| \right).$$

Properties $(\mathrm{G}')_{\mathrm{i,ii}}$ imply that

$$\sup_{\mathbf{y}\in\mathcal{U}} \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) \right\| \le L_2 \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}, \tag{41}$$

$$\sup_{\mathbf{y}\in\mathcal{U}} \left\| \frac{\partial}{\partial \mathbf{F}} g_{\mathbf{y}}(\mathbf{F}) \right\| \le L_2 \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-1}{2}} \tag{42}$$

and

$$\sup_{\mathbf{y}\in\mathcal{U}} |g_{\mathbf{y}}(\mathbf{F})| \le L_2 \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}}, \tag{43}$$

for all $\mathbf{F} \in \mathbb{R}^{N\times n}$, with

$$L_2 := L_1 + b.$$

With

$$a := 1 + L_2^{\frac{1}{p-2}} \sup_{\mathbf{y}\in\mathcal{U}} \left\{ \left(1 + \sigma^2_{\Lambda^*(\mathbf{y})}\right)^{\frac{p-2}{2}} \right\},$$

we have

$$a \ge 1 + \sup_{\mathbf{y}\in\mathcal{U}} \left\{ \max \left\{ \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) \right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \le \sigma_{\Lambda^*(\mathbf{y})} \right\} \right\}. \tag{44}$$

Recall that since $\{f_{\mathbf{y}}\}$ has a uniform p-Uhlenbeck structure, we must have that $\operatorname{ess\,inf}_{\mathbf{y}\in\mathcal{U}} \Lambda^*(\mathbf{y})$ is strictly positive and thus $\sup_{\mathbf{y}\in\mathcal{U}} \sigma_{\Lambda^*(\mathbf{y})}$, and a, is finite.

For each $\varepsilon > 0$, we also define

$$Q_\varepsilon := \left(\frac{2^{p+3} L_2^2}{\varepsilon^2} + 2 \right)^{\frac{1}{2}} \sigma_\varepsilon.$$

Lemma 9 *Suppose that* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *has a uniform p-Uhlenbeck structure. Suppose also that* $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^3(\mathbb{R}^{N\times n})$ *possesses the growth properties* (G′) *and is uniformly asymptotically related to* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$. *Let the mappings* $\mathbf{A} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$, *with a matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$, *be given. For each* $0 < \alpha < 1$, *there exist* $R_\alpha, \varepsilon_\alpha > 0$ *with the following property: for every* $0 < \varepsilon \le \varepsilon_\alpha$ *and* $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$, *with* $0 < R \le R_\alpha$, *if* $\mathbf{v} \in W^{1,p}(\mathcal{B}^+_{\mathbf{x}_0,R}; \mathbb{R}^N)$ *is a minimizer for*

$$J^+_{\mathbf{x}_0,R}[\mathbf{v}] := \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} f_{\mathbf{x}_0}(\nabla \mathbf{v}(\mathbf{x}) \mathbf{G}(\mathbf{x}_0))\, d\mathbf{x}$$

and $\mathbf{w} \in W^{1,1}(\mathcal{B}^+ \cap \mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$ *is a* $(K^+_{\mathbf{x}_0,R}, \omega)$*-minimizer for*

$$K^+_{\mathbf{x}_0,R}[\mathbf{w}] := \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} g_{\mathbf{x}_0}([\nabla \mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\, \mathrm{d}\mathbf{x},$$

satisfying $[\mathbf{w} - \mathbf{v}] \in W^{1,1}_0(\mathcal{B}^+_{\mathbf{x}_0,R}; \mathbb{R}^N)$ *and* $\mathbf{w} = \mathbf{0}$*, in the sense of traces, on* $\mathcal{D}_{\mathbf{x}_0,R}$*, then*

$$\|\mathbf{F}_{\mathbf{x}_0,R}\mathbf{G}(\mathbf{x}_0)\|^2 + \lambda_{\mathbf{x}_0,R}(\mathbf{F}_{\mathbf{x}_0,R})^2 > Q^2_\varepsilon + \frac{1}{\varepsilon^2} \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}(\mathbf{x})\|^2 \tag{45}$$

implies

$$\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2\, \mathrm{d}\mathbf{x} \le \alpha \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2\, \mathrm{d}\mathbf{x}$$
$$+ \frac{c_8 L_2}{\Lambda_*(\mathbf{x}_0)} R^\beta \left(R^n + \frac{1}{\Lambda_*(\mathbf{x}_0)} J^+_{\mathbf{x}_0,R}[\mathbf{w}] \right), \tag{46}$$

with c_8 *depending only on* $\mathbf{A}$*,* $\mathbf{G}$*,* $\mathbf{G}^{-1}$ *and the structural constants for* $\{f_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$ *and* $\{g_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$*. In particular, the constant* c_8 *is independent of* α *and* R*.*

Here, for each $\mathcal{B}_{\mathbf{x}_0,r} \subseteq \mathcal{B}$*, the function* $\lambda_{\mathbf{x}_0,r} : \mathbb{R}^{N\times n} \to \mathbb{R}$ *provides the unique positive solution to*

$$|\mathcal{B}_r|\lambda^p_{\mathbf{x}_0,r} + |\mathcal{B}_r|\lambda^2_{\mathbf{x}_0,r}\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} = \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,r}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2\, \mathrm{d}\mathbf{x} \tag{47}$$

and

$$\mathbf{F}_{\mathbf{x}_0,r} := \mathbf{V}^{-1}((\mathbf{V}(\nabla \mathbf{w}))^+_{\mathbf{x}_0,r}). \tag{48}$$

Proof Let $\mathbf{x}_0 \in \mathcal{B}^+$ and $0 < R < 1 - \|\mathbf{x}_0\|$ be given. Put

$$M_2 := \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{G}(\mathbf{x})\| + \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{G}^{-1}(\mathbf{x})\| + \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}(\mathbf{x})\| + 1,$$

$$\mathbf{G}_0 := \mathbf{G}(\mathbf{x}_0) \quad \text{and} \quad \mathbf{A}_0 := \mathbf{A}(\mathbf{x}_0).$$

Since $\mathbf{G} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ and $\mathbf{A} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{N\times n})$, there exists a constant $\mu_0 < +\infty$ such that

$$\sup_{\mathbf{x}\in\mathcal{B}^+_{\mathbf{x}_0,R}} (\|\mathbf{A}(\mathbf{x}) - \mathbf{A}_0\| + \|\mathbf{G}(\mathbf{x}) - \mathbf{G}_0\|) \le \mu_0 R^\beta. \tag{49}$$

For convenience, we will write λ for $\lambda_{\mathbf{x}_0,R}(\mathbf{F}_{\mathbf{x}_0,R})$, $\mathbf{F}$ for $\mathbf{F}_{\mathbf{x}_0,R}$, f for $f_{\mathbf{x}_0}$ and g for $g_{\mathbf{x}_0}$. Let $0 < \alpha < 1$ be given, and assume that (45) holds. We will show (46), provided that $0 < \varepsilon \le \varepsilon_\alpha$ and $0 < R \le R_\alpha$, where ε_α and R_α will be determined later.

By Lemma 3, we may write

$$\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\nabla \mathbf{v})\|^2\, \mathrm{d}\mathbf{x} \le \frac{c_2 M_2^p}{\Lambda_*} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \{f(\nabla \mathbf{w}\mathbf{G}_0) - f(\nabla \mathbf{v}\mathbf{G}_0)\}\, \mathrm{d}\mathbf{x},$$

where we used the fact that $\mathbf{v}$ is a minimizer for $J^+_{\mathbf{x}_0,R}$ in the last line. Since $\mathbf{w}$ is a $(K^+_{\mathbf{x}_0,R},\omega)$-minimizer, we continue with

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\nabla\mathbf{v})\|^2 \mathrm{d}\mathbf{x} \le \frac{c_2M_2^p}{\Lambda_*}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\{f(\nabla\mathbf{w}\mathbf{G}_0)-g([\nabla\mathbf{w}+\mathbf{A}]\mathbf{G})\}\,\mathrm{d}\mathbf{x} \\ +\frac{c_2M_2^p}{\Lambda_*}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\{g([\nabla\mathbf{v}+\mathbf{A}]\mathbf{G}_0)-f(\nabla\mathbf{v}\mathbf{G}_0)\}\,\mathrm{d}\mathbf{x} \\ +\frac{c_2M_2^p}{\Lambda_*}\omega(R)\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\left(1+\|\nabla\mathbf{w}\|^p+\|\nabla\mathbf{w}-\nabla\mathbf{v}\|^p\right)\mathrm{d}\mathbf{x}. \tag{50}$$

Hence

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\nabla\mathbf{v})\|^2\,\mathrm{d}\mathbf{x} \le \frac{c_2M_2^p}{\Lambda_*}(I_1+I_2+I_3+\omega(R)I_4), \tag{51}$$

where

$$I_1 := \int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\Bigg\{f(\nabla\mathbf{w}\mathbf{G}_0)-f(\mathbf{F}\mathbf{G}_0)-\frac{\partial}{\partial\mathbf{F}}f(\mathbf{F}\mathbf{G}_0):[\nabla\mathbf{w}-\mathbf{F}]\mathbf{G}_0 \\ -g([\nabla\mathbf{w}+\mathbf{A}]\mathbf{G})+g([\mathbf{F}+\mathbf{A}]\mathbf{G})+\frac{\partial}{\partial\mathbf{F}}g([\mathbf{F}+\mathbf{A}]\mathbf{G}):[\nabla\mathbf{w}-\mathbf{F}]\mathbf{G}\Bigg\}\mathrm{d}\mathbf{x},$$

$$I_2 := \int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\Bigg\{g([\nabla\mathbf{v}+\mathbf{A}]\mathbf{G})-g([\mathbf{F}+\mathbf{A}]\mathbf{G})-\frac{\partial}{\partial\mathbf{F}}g([\mathbf{F}+\mathbf{A}]\mathbf{G}):[\nabla\mathbf{v}-\mathbf{F}]\mathbf{G} \\ -f(\nabla\mathbf{v}\mathbf{G}_0)+f(\mathbf{F}\mathbf{G}_0)+\frac{\partial}{\partial\mathbf{F}}f(\mathbf{F}\mathbf{G}_0):[\nabla\mathbf{v}-\mathbf{F}]\mathbf{G}_0\Bigg\}\mathrm{d}\mathbf{x},$$

$$I_3 := \int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\left\{\frac{\partial}{\partial\mathbf{F}}g([\mathbf{F}+\mathbf{A}_0]\mathbf{G}_0)\mathbf{G}_0^{\mathrm{T}}-\frac{\partial}{\partial\mathbf{F}}g([\mathbf{F}+\mathbf{A}]\mathbf{G})\mathbf{G}^{\mathrm{T}}\right\}:[\nabla\mathbf{w}-\nabla\mathbf{v}]\,\mathrm{d}\mathbf{x},$$

and I_4 is the last integral in (50). We will analyze each of the above integrals in turn.

For I_1, we write

$$I_1 = \int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\int_0^1\Bigg\{\frac{\partial^2}{\partial\mathbf{F}^2}f([t\nabla\mathbf{w}+(1-t)\mathbf{F}]\mathbf{G}_0)::([\nabla\mathbf{w}-\mathbf{F}]\mathbf{G}_0)\otimes([\nabla\mathbf{w}-\mathbf{F}]\mathbf{G}_0) \\ -\frac{\partial^2}{\partial\mathbf{F}^2}g([t\nabla\mathbf{w}+(1-t)\mathbf{F}+\mathbf{A}]\mathbf{G})::([\nabla\mathbf{w}-\mathbf{F}]\mathbf{G})\otimes([\nabla\mathbf{w}-\mathbf{F}]\mathbf{G})\Bigg\}\mathrm{d}t\mathrm{d}\mathbf{x},$$

and we further split I_1 as follows:

$$
\begin{aligned}
I_1 =& \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} f([t\nabla \mathbf{w} + (1-t)\mathbf{F}]\mathbf{G}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{w} + (1-t)\mathbf{F}]\mathbf{G}_0) \right] \\
&\quad :: ([\nabla \mathbf{w} - \mathbf{F}]\mathbf{G}_0) \otimes ([\nabla \mathbf{w} - \mathbf{F}]\mathbf{G}_0)\, \mathrm{d}t \mathrm{d}\mathbf{x} \\
&\quad + \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{w} + (1-t)\mathbf{F}]\mathbf{G}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{w} + (1-t)\mathbf{F} + \mathbf{A}]\mathbf{G}) \right] \\
&\quad :: ([\nabla \mathbf{w} - \mathbf{F}]\mathbf{G}_0) \otimes ([\nabla \mathbf{w} - \mathbf{F}]\mathbf{G}_0)\, \mathrm{d}t \mathrm{d}\mathbf{x} \\
&\quad + \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{w} + (1-t)\mathbf{F} + \mathbf{A}]\mathbf{G}) \\
&\quad :: ([\nabla \mathbf{w} - \mathbf{F}]) \otimes ([\nabla \mathbf{w} - \mathbf{F}](\mathbf{G}_0\mathbf{G}_0^{\mathrm{T}} - \mathbf{G}\mathbf{G}^{\mathrm{T}}))\, \mathrm{d}t \mathrm{d}\mathbf{x} \\
=& I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned} \tag{52}
$$

For the term $I_{1,1}$, assumption (45) implies that $\|\mathbf{F}\mathbf{G}_0\|^2 + \lambda^2 > Q_\varepsilon^2$, and thus (41) and Lemma 5 imply that

$$
|I_{1,1}| \le \varepsilon \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(\|[\nabla \mathbf{w} - \mathbf{F}]\mathbf{G}_0\|^2 + \lambda^2\right) \left(a^2 + \|\nabla \mathbf{w}\mathbf{G}_0\|^2 + \|\mathbf{F}\mathbf{G}_0\|^2\right)^{\frac{p-2}{2}} \mathrm{d}\mathbf{x} \tag{53}
$$

Since $p > 2$ and $a \ge 1$, we may write

$$
\begin{aligned}
|I_{1,1}| \le& \varepsilon M_2^p a^{p-2} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(1 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \mathrm{d}\mathbf{x} \\
&+ \varepsilon C M_2^{p-2} a^{p-2} \lambda^2 \left\{ |\mathcal{B}_R| (1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}} + |\mathcal{B}_R|^{\frac{2}{p}} \left(\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w}\|^p \mathrm{d}\mathbf{x} \right)^{\frac{p-2}{p}} \right\}.
\end{aligned}
$$

Using Lemma 2, we get

$$
\begin{aligned}
|I_{1,1}| \le& \varepsilon c_1 M_2^p a^{p-2} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \mathrm{d}\mathbf{x} + \varepsilon C M_2^{p-2} a^{p-2} \lambda^2 |\mathcal{B}_R| (1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}} \\
&+ \varepsilon C M_2^{p-2} a^{p-2} |\mathcal{B}_R|^{\frac{2}{p}} \lambda^2 \left(\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w} - \mathbf{F}\|^p \mathrm{d}\mathbf{x} + |\mathcal{B}_R|\, \|\mathbf{F}\|^p \right)^{\frac{p-2}{p}}.
\end{aligned}
$$

Hence

$$|I_{1,1}| \leq \varepsilon c_1 M_2^p a^{p-2} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \mathrm{d}\mathbf{x} + \varepsilon C M_2^{p-2} a^{p-2} \lambda^2 |\mathcal{B}_R| (1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}}$$
$$+ \varepsilon C M_2^{p-2} a^{p-2} |\mathcal{B}_R|^{\frac{2}{p}} \lambda^2 \left(\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w} - \mathbf{F}\|^p \,\mathrm{d}\mathbf{x} \right)^{\frac{p-2}{p}}.$$

Since $p > 2$, we may use Young's inequality on the last term above; then another application of Lemma 2 yields

$$|I_{1,1}| \leq \varepsilon C c_1 M_2^p a^{p-2} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ \varepsilon C M_2^{p-2} a^{p-2} \left\{ |\mathcal{B}_R| \lambda^p + |\mathcal{B}_R| \lambda^2 (1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}} \right\}. \tag{54}$$

The definition of λ in (54) now implies

$$|I_{1,1}| \leq \varepsilon C c_1 M_2^p a^{p-2} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_R)\|^2 \,\mathrm{d}\mathbf{x}. \tag{55}$$

For the term $I_{1,2}$, the growth condition $(\mathrm{G}')_{\mathrm{i}}$ and the conditions that $a \geq 1$ and $L_2 \geq L_1$ allow us to write

$$|I_{1,2}| \leq L_2 \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\mathbf{G}\|^2 + \|\nabla \mathbf{w}\mathbf{G}_0\|^2 + \|\mathbf{F}\mathbf{G}\|^2 + \|\mathbf{F}\mathbf{G}_0\|^2 + \|\mathbf{A}\mathbf{G}\|^2\right)^{\frac{p-3}{2}}$$
$$\times \left[(\|\nabla \mathbf{w}\| + \|\mathbf{F}\|) \|\mathbf{G} - \mathbf{G}_0\| + \|\mathbf{A}\mathbf{G}\| \right] \|\nabla \mathbf{w} - \mathbf{F}\|^2 \|\mathbf{G}_0\|^2 \mathrm{d}\mathbf{x}. \tag{56}$$

There are two cases that we consider: $2 < p \leq 3$ and $3 < p$.

Case 1 If $2 < p \leq 3$, then we note that

$$\|\nabla \mathbf{w}\| \leq \|\nabla \mathbf{w}\mathbf{G}\| \|\mathbf{G}^{-1}\| \leq M_2 \|\nabla \mathbf{w}\mathbf{G}\| \quad \text{and} \quad \|\mathbf{F}\| \leq M_2 \|\mathbf{F}\mathbf{G}\|.$$

Hence from (56), we deduce that

$$|I_{1,2}| \leq C L_2 M_2^p \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \|\mathbf{G} - \mathbf{G}_0\| \,\mathrm{d}\mathbf{x}$$
$$+ C L_2 M_2^p \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{A}\|^{p-2} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x},$$

where we have used the fact that in this case, $\frac{p-3}{2} < 0$. Recalling the Hölder continuity of $\mathbf{G}$ captured in (49), we may write

$$\frac{|I_{1,2}|}{L_2 M_2^p} \leq C\mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \, \mathrm{d}\mathbf{x}$$
$$+ C\left(\sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2\right)^{\frac{p-2}{2}} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \, \mathrm{d}\mathbf{x}.$$

Assumption (45), implies that $\sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2 \leq \varepsilon^2 \left(\|\mathbf{F}\mathbf{G}_0\|^2 + \lambda^2\right)$. Consequently,

$$\frac{|I_{1,2}|}{L_2 M_2^p} \leq C\mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \mathrm{d}\mathbf{x}$$
$$+ \varepsilon^{p-2} C \left(\|\mathbf{F}\mathbf{G}_0\|^2 + \lambda^2\right)^{\frac{p-2}{2}} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \mathrm{d}\mathbf{x}$$
$$\leq C\left(\mu_0 R^\beta + M_2^{p-2}\varepsilon^{p-2}\right) \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \mathrm{d}\mathbf{x}$$
$$+ \varepsilon^{p-2} C \lambda^{p-2} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \mathrm{d}\mathbf{x}.$$

Put $\widetilde{c}_1 := a^{p-2} c_1$. Apply Young's inequality to the last integral and then use Lemma 2 to get

$$\frac{|I_{1,2}|}{L_2 M_2^p} \leq C\widetilde{c}_1 \left\{\mu_0 R^\beta + M_2^{p-2}\varepsilon^{p-2}\right\} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \, \mathrm{d}\mathbf{x} + \varepsilon^{p-2} C \lambda^p |\mathcal{B}_R|, \quad (57)$$

and thus

$$|I_{1,2}| \leq C\widetilde{c}_1 L_2 M_2^{2p-2} \left(\mu_0 R^\beta + \varepsilon^{p-2}\right) \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \, \mathrm{d}\mathbf{x} \quad (58)$$

follows from the definition of λ and M_2.

Case 2 When $p > 3$, we get from (56) that

$$|I_{1,2}| \leq C L_2 M_2^{p-1} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \|\mathbf{G} - \mathbf{G}_0\| \, \mathrm{d}\mathbf{x}$$
$$+ C L_2 M_2^p \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-3}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \|\mathbf{A}\| \, \mathrm{d}\mathbf{x}$$
$$+ C L_2 M_2^{p-1} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(\|\nabla \mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{1}{2}} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \|\mathbf{A}\|^{p-3} \|\mathbf{G} - \mathbf{G}_0\| \, \mathrm{d}\mathbf{x}$$
$$+ C L_2 M_2^p \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{A}\|^{p-2} \|\nabla \mathbf{w} - \mathbf{F}\|^2 \, \mathrm{d}\mathbf{x}.$$

Recalling (49) and using Young's inequality yields

$$\frac{|I_{1,2}|}{L_2 M_2^p} \le C\left(M_2^{p-4}\mu_0 R^\beta + \varepsilon\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla\mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\left(\frac{1}{\varepsilon^{p-3}} + 1\right)\left(\sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2\right)^{\frac{p-2}{2}} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x}.$$

As in Case 1, we have that $\sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2 \le \varepsilon^2\left(\|\mathbf{F}\mathbf{G}_0\|^2 + \lambda^2\right)$, so

$$\frac{|I_{1,2}|}{L_2 M_2^p} \le C\left(M_2^{p-4}\mu_0 R^\beta + \varepsilon\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(a^2 + \|\nabla\mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\left(\varepsilon + \varepsilon^{p-2}\right)\left(\|\mathbf{F}\mathbf{G}_0\|^2 + \lambda^2\right)^{\frac{p-2}{2}} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x}.$$

Young's inequality, Lemma 2 and the definition of λ imply

$$|I_{1,2}| \le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} \tag{59}$$
$$+ C\left(\varepsilon + \varepsilon^{p-2}\right)\lambda^p |\mathcal{B}_R|$$
$$\le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \tag{60}$$

Collecting our estimates for each of the cases, (58) and (60), we obtain

$$|I_{1,2}| \le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \tag{61}$$

We now conclude our analysis of I_1 by estimating $I_{1,3}$. For this integral, we first note that $\mathbf{G}\mathbf{G}^{\mathrm{T}} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$; indeed, by (49)

$$\sup_{\mathbf{x}\in\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x}) - \mathbf{G}_0\mathbf{G}_0^{\mathrm{T}}\| \le 2M_2\mu_0 R^\beta.$$

This, (41) and Lemma 2 imply

$$|I_{1,3}| \le C L_2 \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(1 + \|\nabla\mathbf{w}\mathbf{G}\|^2 + \|\mathbf{F}\mathbf{G}\|^2 + \|\mathbf{A}\mathbf{G}\|^2\right)^{\frac{p-2}{2}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \|\mathbf{G}_0\mathbf{G}_0^{\mathrm{T}} - \mathbf{G}\mathbf{G}^{\mathrm{T}}\| \,\mathrm{d}\mathbf{x}$$
$$\le C L_2 M_2^{2p-3}\mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(1 + \|\nabla\mathbf{w}\|^2 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\nabla\mathbf{w} - \mathbf{F}\|^2 \,\mathrm{d}\mathbf{x}$$
$$\le C c_1 L_2 M_2^{2p-3}\mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \tag{62}$$

Combining, in (52), our estimates for $|I_{1,1}|$, $|I_{1,2}|$ and $|I_{1,3}|$ from (55), (61) to (62), we finally conclude that, under assumption (45),

$$|I_1| \leq \widetilde{c}_1 C L_2 M_2^{3p} \left\{\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right\} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}, \tag{63}$$

with C depending only on p.

Our analysis of I_2 is similar to that of I_1. In exact analogy with (52), we find

$$\begin{aligned} I_2 =& \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} f([t\nabla \mathbf{v} + (1-t)\mathbf{F}]\mathbf{G}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{v} + (1-t)\mathbf{F}]\mathbf{G}_0)\right] \\ &\qquad :: ([\nabla \mathbf{v} - \mathbf{F}]\mathbf{G}_0) \otimes ([\nabla \mathbf{v} - \mathbf{F}]\mathbf{G}_0)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} \\ &+ \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \left[\frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{v} + (1-t)\mathbf{F}]\mathbf{G}_0) - \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{v} + (1-t)\mathbf{F} + \mathbf{A}]\mathbf{G})\right] \\ &\qquad :: ([\nabla \mathbf{v} - \mathbf{F}]\mathbf{G}_0) \otimes ([\nabla \mathbf{v} - \mathbf{F}]\mathbf{G}_0)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} \\ &+ \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \int\limits_0^1 \frac{\partial^2}{\partial \mathbf{F}^2} g([t\nabla \mathbf{v} + (1-t)\mathbf{F} + \mathbf{A}]\mathbf{G}) \\ &\qquad :: ([\nabla \mathbf{v} - \mathbf{F}]) \otimes ([\nabla \mathbf{v} - \mathbf{F}](\mathbf{G}_0\mathbf{G}_0^{\mathrm{T}} - \mathbf{G}\mathbf{G}^{\mathrm{T}}))\,\mathrm{d}t\,\mathrm{d}\mathbf{x} \\ =& I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

For $I_{2,1}$, in analogy with (54), we find that

$$\begin{aligned} |I_{2,1}| \leq& C\varepsilon \widetilde{c}_1 M_2^p \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} \\ &+ \varepsilon C M_2^{p-2} a^{p-2} \left\{\lambda^p |\mathcal{B}_R| + \lambda^2 |\mathcal{B}_R| (1 + \|\mathbf{F}\|^2)^{\frac{p-2}{2}}\right\}. \end{aligned}$$

From this we may deduce that

$$\begin{aligned} |I_{2,1}| \leq& \varepsilon C \widetilde{c}_1 M_2^p \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x} \\ &+ \varepsilon C \widetilde{c}_1 M_2^p \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} \end{aligned} \tag{64}$$

Turning to $I_{2,2}$, we have 2 cases, as we did for $I_{1,2}$.

Case 1 If $2 < p \leq 3$, then the same argument that provided (57) leads us to

$$\frac{|I_{2,2}|}{L_2 M_2^p} \leq C\widetilde{c}_1 \left\{\mu_0 R^\beta + M_2^{p-2} \varepsilon^{p-2}\right\} \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} + \varepsilon^{p-2} C \lambda^p |\mathcal{B}_R|,$$

and thus

$$|I_{2,2}| \le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} \qquad (65)$$

Case 2 When $p > 3$, the analogue for (59) is

$$\frac{|I_{2,2}|}{L_2 M_2^{2p-2}} \le C\widetilde{c}_1\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x} + C\left(\varepsilon + \varepsilon^{p-2}\right)\lambda^p |\mathcal{B}_R|.$$

From which we conclude that

$$|I_{2,2}| \le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \qquad (66)$$

Comparing estimates (65) and (66), we see

$$|I_{2,2}| \le C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\widetilde{c}_1 L_2 M_2^{2p-2}\left(\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right)\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \qquad (67)$$

holds in each case. Finally for $I_{2,3}$, we have

$$|I_{2,3}| \le C c_1 L_2 M_2^{2p-3} \mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C c_1 L_2 M_2^{2p-3} \mu_0 R^\beta \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}. \qquad (68)$$

Collecting (64), (67), (68) together gives our final estimate for $|I_2|$:

$$|I_2| \le C\widetilde{c}_1 L_2 M_2^{3p}\left\{\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right\}\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,\mathrm{d}\mathbf{x}$$
$$+ C\widetilde{c}_1 L_2 M_2^{3p}\left\{\mu_0 R^\beta + \varepsilon + \varepsilon^{p-2}\right\}\int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \,\mathrm{d}\mathbf{x}, \qquad (69)$$

since $L_2 \ge L$ and $\widetilde{c}_1 \ge c_1$.

For the integral I_3, we first note that $\mathbf{AG} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+}; \mathbb{R}^{N\times n})$ and by (49) that

$$\sup_{\mathbf{x}\in\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{A}(\mathbf{x})\mathbf{G}(\mathbf{x}) - \mathbf{A}_0\mathbf{G}_0\| \le 2M_2\mu_0 R^\beta.$$

We begin with the estimate

$$|I_3| \le \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left\| \frac{\partial}{\partial \mathbf{F}} g([\mathbf{F}+\mathbf{A}_0]\mathbf{G}_0) \right\| \|\mathbf{G}_0 - \mathbf{G}\| \|\nabla\mathbf{w} - \nabla\mathbf{v}\| \,\mathrm{d}\mathbf{x}$$
$$+ \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left\| \frac{\partial}{\partial \mathbf{F}} g([\mathbf{F}+\mathbf{A}]\mathbf{G}) - \frac{\partial}{\partial \mathbf{F}} g([\mathbf{F}+\mathbf{A}_0]\mathbf{G}_0) \right\| \|\mathbf{G}\| \|\nabla\mathbf{w} - \nabla\mathbf{v}\| \,\mathrm{d}\mathbf{x}$$

Recalling the bounds (41) and (42), we have

$$|I_3| \le L_2 \left(1+2M_2^4+2M_2^2\|\mathbf{F}\|^2\right)^{\frac{p-1}{2}} \mu_0 R^\beta \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla\mathbf{w}-\nabla\mathbf{v}\| \,\mathrm{d}\mathbf{x}$$
$$+ L_2 M_2 \left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} (2M_2+\|\mathbf{F}\|)\mu_0 R^\beta \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla\mathbf{w}-\nabla\mathbf{v}\| \,\mathrm{d}\mathbf{x}$$
$$\le 4L_2 M_2^2 \left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p-1}{2}} \mu_0 R^\beta \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\nabla\mathbf{w}-\nabla\mathbf{v}\| \,\mathrm{d}\mathbf{x}.$$

Applying Young's inequality and Lemma 2, we obtain

$$|I_3| \le 4M_2^2 L_2 \mu_0 R^\beta \left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p}{2}} |\mathcal{B}^+_{\mathbf{x}_0,R}|$$
$$+ CM_2^2 L_2 \mu_0 R^\beta \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\nabla\mathbf{v})\|^2 \,\mathrm{d}\mathbf{x}. \tag{70}$$

We consider two cases to estimate the first term in (70).

Case 1 If $\|\mathbf{F}\| < M_2$, then

$$\left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p}{2}} < CM_2^{2p}. \tag{71}$$

Case 2 If $M_2 \le \|\mathbf{F}\|$, then

$$\left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p}{2}} \le CM_2^p \left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}} \le CM_2^p \left(1+\|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\mathbf{F}\|^2$$
$$= CM_2^p \|\mathbf{V}(\mathbf{F})\|^2.$$

Since $\mathbf{V}(\mathbf{F}) = (\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,R}$, Jensen's inequality implies that

$$\left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p}{2}} \le CM_2^p \fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w})\|^2 \,\mathrm{d}\mathbf{x} \le CM_2^p \fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(1+\|\nabla\mathbf{w}\|^2\right)^{\frac{p}{2}} \,\mathrm{d}\mathbf{x}$$
$$\le CM_2^{2p} \fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} \left(1+\|\nabla\mathbf{w}\mathbf{G}_0\|^2\right)^{\frac{p}{2}} \,\mathrm{d}\mathbf{x}.$$

Recalling $(\mathrm{U})_{\mathrm{ii}}$, we obtain

$$\left(1+4M_2^4+4M_2^2\|\mathbf{F}\|^2\right)^{\frac{p}{2}} \le C\frac{M_2^{2p}}{\Lambda_*}\fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} f(\nabla\mathbf{w}\mathbf{G}_0)\,\mathrm{d}\mathbf{x}. \tag{72}$$

Returning to (70), our estimates in (71) and (72) give us

$$\begin{aligned}|I_3| \le{}& CM_2^{2p+2}L_2\mu_0R^{n+\beta}+C\frac{M_2^{p+2}L_2}{\Lambda_*}\mu_0R^{\beta}J^+_{\mathbf{x}_0,R}[\mathbf{w}] \\ &+CM_2^2L_2\mu_0R^{\beta}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\nabla\mathbf{v})\|^2\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{73}$$

We now turn to the last integral in (51). Plugging in the definition of $\omega(R)$ and using Lemma 2, we get

$$\begin{aligned}|I_4| \le{}& \omega_0R^{n+\beta}|\mathcal{B}|+M_2^p\omega_0R^{\beta}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\left(1+\|\nabla\mathbf{w}\mathbf{G}_0\|^2\right)^{\frac{p}{2}}\,\mathrm{d}\mathbf{x} \\ &+Cc_1\omega_0R^{\beta}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\nabla\mathbf{v})\|^2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Property $(\mathrm{H}')_{\mathrm{i}}$ implies

$$\begin{aligned}|I_4| \le{}& C\omega_0R^{n+\beta}+\frac{M_2^p}{\Lambda_*}\omega_0R^{\beta}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} f(\nabla\mathbf{w}\mathbf{G}_0)\,\mathrm{d}\mathbf{x} \\ &+Cc_1\omega_0R^{\beta}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\nabla\mathbf{v})\|^2\,\mathrm{d}\mathbf{x}\end{aligned} \tag{74}$$

Putting into (51) our estimates for $|I_1|,\ldots,|I_4|$, from (63), (69), (73), (74), we finally obtain

$$\begin{aligned}&\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{v})-\mathbf{V}(\nabla\mathbf{w})\|^2\,\mathrm{d}\mathbf{x} \\ &\le C\frac{\widetilde{c}_1M_2^{3p}L_2}{\Lambda_*}\left\{\mu_0R^{\beta}+\omega_0R^{\beta}+\varepsilon+\varepsilon^{p-2}\right\}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{v})-\mathbf{V}(\nabla\mathbf{w})\|^2\,\mathrm{d}\mathbf{x} \\ &\quad+C\frac{\widetilde{c}_1M_2^{3p}L_2}{\Lambda_*}\left\{\mu_0R^{\beta}+\varepsilon+\varepsilon^{p-2}\right\}\int_{\mathcal{B}^+_{\mathbf{x}_0,R}}\|\mathbf{V}(\nabla\mathbf{w})-\mathbf{V}(\mathbf{F})\|^2\,\mathrm{d}\mathbf{x} \\ &\quad+C\frac{M_2^{2p+2}L_2}{\Lambda_*}(\mu_0+\omega_0)R^{\beta}\left\{R^n+\frac{1}{\Lambda_*}J^+_{\mathbf{x}_0,R}[\mathbf{w}]\right\},\end{aligned}$$

where C depends only on p. Define

$$R_\alpha := \min\left\{1 - \|\mathbf{x}_0\|, \left(\frac{\alpha\Lambda_*}{4C\widetilde{c}_1 M_2^{3p} L_2(1+\mu_0+\omega_0)}\right)^{\frac{1}{\beta}}\right\}.$$

Since, for each $p > 2$, the function $\varepsilon \mapsto \varepsilon + \varepsilon^{p-2}$ is a strictly increasing function that equals 0 at 0, we may choose ε_α to be the unique positive solution to the equation

$$\varepsilon + \varepsilon^{p-2} = \frac{\alpha\Lambda_*}{4\widetilde{c}_1 C M_2^{3p} L_2}.$$

So defined, for each $0 < \varepsilon \leq \varepsilon_\alpha$ and $0 < R \leq R_\alpha$, we have

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{v}) - \mathbf{V}(\nabla\mathbf{w})\|^2 \mathrm{d}\mathbf{x}$$
$$\leq \alpha \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2 \mathrm{d}\mathbf{x} + C\frac{M_2^{2p+2}L_2}{\Lambda_*}(\mu_0+\omega_0)R^\beta\left\{R^n + \frac{1}{\Lambda_*}J^+_{\mathbf{x}_0,R}[\mathbf{w}]\right\},$$

where we again point out that C depends only on p. □

By exactly the same argument, we also have the following analogue of Lemma 9 for the interior of a ball.

Lemma 10 *Suppose that* $\{f_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *which has a uniform p-Uhlenbeck structure. Suppose also that* $\{g_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}} \subset \mathcal{C}^3(\mathbb{R}^{N\times n})$ *possesses the growth properties* (G′) *and is uniformly asymptotically related to* $\{f_\mathbf{y}\}_{\mathbf{y}\in\mathcal{B}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *Let the mappings* $\mathbf{A} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{n\times n})$*, with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{n\times n})$*, be given. For each* $0 < \alpha < 1$*, there exist* $R_\alpha, \varepsilon_\alpha > 0$ *with the following property: for every* $0 < \varepsilon \leq \varepsilon_\alpha$ *and* $\mathcal{B}_{\mathbf{x}_0,R} \subset \mathcal{B}$*, with* $0 < R \leq R_\alpha$*, if* $\mathbf{v} \in W^{1,p}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$ *is a minimizer for*

$$J_{\mathbf{x}_0,R}[\mathbf{v}] := \int_{\mathcal{B}_{\mathbf{x}_0,R}} f_{\mathbf{x}_0}(\nabla\mathbf{v}(\mathbf{x})\mathbf{G}(\mathbf{x}_0))\,\mathrm{d}\mathbf{x}$$

and $\mathbf{w} \in W^{1,1}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$ *is a* $(K_{\mathbf{x}_0,R}, \omega)$*-minimizer for*

$$K_{\mathbf{x}_0,R}[\mathbf{w}] := \int_{\mathcal{B}_{\mathbf{x}_0,R}} g_{\mathbf{x}_0}([\nabla\mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

satisfying $[\mathbf{w} - \mathbf{v}] \in W^{1,1}_0(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$*, then*

$$\|\mathbf{F}_{\mathbf{x}_0,R}\mathbf{G}(\mathbf{x}_0)\|^2 + \lambda_{\mathbf{x}_0,R}(\mathbf{F}_{\mathbf{x}_0,R})^2 > Q_\varepsilon^2 + \frac{1}{\varepsilon^2}\sup_{\mathbf{x}\in\mathcal{B}^+}\|\mathbf{A}(\mathbf{x})\|^2 \tag{75}$$

implies

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{v}) - \mathbf{V}(\nabla\mathbf{w})\|^2\,\mathrm{d}\mathbf{x} \leq \alpha \int_{\mathcal{B}_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\mathbf{F})\|^2\,\mathrm{d}\mathbf{x}$$
$$+ \frac{c_8 L_2}{\Lambda_*(\mathbf{x}_0)}R^\beta\left(R^n + \frac{1}{\Lambda_*(\mathbf{x}_0)}J_{\mathbf{x}_0,R}[\mathbf{w}]\right), \tag{76}$$

with c_8 *depending only on* $\mathbf{A}$, $\mathbf{G}$, $\mathbf{G}^{-1}$ *and the structural constants for* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}}$ *and* $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}}$. *In particular, the constant* c_8 *is independent of* α *and* R.

Here, the function $\lambda_{\mathbf{x}_0,r} : \mathbb{R}^{N\times n} \to \mathbb{R}$ *is the same as that defined in* (47) *and*

$$\mathbf{F}_{\mathbf{x}_0,r} := \mathbf{V}^{-1}((\mathbf{V}(\nabla\mathbf{w}))_{\mathbf{x}_0,r}). \tag{77}$$

We now prove a theorem, which provides bounds up to the set $\mathcal{D}_{\mathbf{0},1}$ for the gradient of a minimizer for an asymptotically convex functional satisfying homogeneous Dirichlet boundary conditions on $\mathcal{D}_{\mathbf{0},1}$.

Theorem 9 *Suppose that the family of functions* $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^3(\mathbb{R}^{N\times n})$ *possesses property* (G′) *and is uniformly asymptotically related to the family of functions* $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *which has a uniform p-Uhlenbeck structure. Let* $\mathbf{A} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+};\mathbb{R}^{N\times n})$ *and* $\mathbf{G} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$, *with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$, *be given. For each* $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$, *define the functional*

$$K^+_{\mathbf{y}}[\mathbf{w}] := \int\limits_{\mathcal{B}^+} g_{\mathbf{y}}([\nabla\mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

If $\mathbf{w} \in W^{1,1}(\mathcal{B}^+;\mathbb{R}^N)$ *is a* $(K^+_{\mathbf{y}},\omega)$*-minimizer at each* $\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}$ *satisfying* $\mathbf{w} = \mathbf{0}$ *on* $\mathcal{D}_{\mathbf{0},1}$, *in the sense of traces, then* $\mathbf{w} \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1};\mathbb{R}^N)$.

Proof With Lemma 9 above, the proof is along the same lines as the one provided for Theorem 3.1 in [23]. We define

$$\Sigma_1 := \left\{ \mathbf{x}_0 \in \mathcal{B}^+ \;\middle|\; \liminf_{r\to 0^+} \fint\limits_{\mathcal{B}^+_{\mathbf{x}_0,r}} \|\nabla\mathbf{w} - (\nabla\mathbf{w})_{\mathbf{x}_0,r}\|^p > 0 \right\}$$

and

$$\Sigma_2 := \left\{ \mathbf{x}_0 \in \mathcal{B}^+ \;\middle|\; \sup_{r>0} \|(\nabla\mathbf{w})_{\mathbf{x}_0,r}\| = +\infty \right\}.$$

Set $\widetilde{\Omega} := \mathcal{B}^+ \setminus (\Sigma_1 \cup \Sigma_2)$. For each $\mathbf{x}_0 \in \widetilde{\Omega}$, we have that

$$\lim_{r\to 0^+} (\nabla\mathbf{w})_{\mathbf{x}_0,r} = \nabla\mathbf{w}(\mathbf{x}_0)$$

and

$$\|\nabla\mathbf{w}(\mathbf{x}_0)\|^p \le \|\mathbf{V}(\nabla\mathbf{w}(\mathbf{x}_0))\|^2 = \lim_{r\to 0^+} \|\mathbf{V}(\mathbf{F}_{\mathbf{x}_0,r})\|^2,$$

where

$$\mathbf{F}_{\mathbf{x}_0,r} := \mathbf{V}^{-1}((\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,r}).$$

It is sufficient to show that there is an $M < +\infty$ such that $\|\nabla\mathbf{w}(\mathbf{x}_0)\|^p \le M$ for every $\mathbf{x}_0 \in \widetilde{\Omega}$.

Put $R_{\mathbf{x}_0} := 1 - \|\mathbf{x}_0\|$, and let $\mathbf{x}_0 \in \widetilde{\Omega}$ and $0 < R < R_{\mathbf{x}_0}$ be given. Let the mapping $\mathbf{v} \in W^{1,p}(\mathcal{B}^+_{\mathbf{x}_0,R};\mathbb{R}^N)$ be the minimizer for

$$J^+_{\mathbf{x}_0,R}[\mathbf{v}] := \int\limits_{\mathcal{B}^+_{\mathbf{x}_0,R}} f_{\mathbf{x}_0}(\nabla\mathbf{v}(\mathbf{x})\mathbf{G}_0)\,\mathrm{d}\mathbf{x}$$

satisfying $[\nabla \mathbf{w} - \nabla \mathbf{v}] \in W_0^{1,p}(\mathcal{B}^+_{\mathbf{x}_0,R}; \mathbb{R}^N)$. For each $0 < \rho < R$, we have

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,\rho})\|^2 \,d\mathbf{x}$$
$$\leq 4\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\nabla \mathbf{v})\|^2 \,d\mathbf{x} + 4\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{v}) - (\mathbf{V}(\nabla \mathbf{v}))^+_{\mathbf{x}_0,\rho}\|^2 \,d\mathbf{x}$$
$$+ 4\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \| (\mathbf{V}(\nabla \mathbf{v}))^+_{\mathbf{x}_0,\rho} - (\mathbf{V}(\nabla \mathbf{w}))^+_{\mathbf{x}_0,\rho}\|^2 \,d\mathbf{x}.$$

We invoke Theorem 5 and Jensen's inequality to get

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,\rho})\|^2 \,d\mathbf{x} \leq 2^5 c_0 \left(\frac{\rho}{R}\right)^{n+2\sigma_0} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,R})\|^2 d\mathbf{x}$$
$$+ 2^5 c_0 \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\nabla \mathbf{v})\|^2 \,d\mathbf{x}, \tag{78}$$

for every $\sigma_0 \in (0, 1)$.

Fix $\sigma_0 \in (0, 1)$, and let $0 < \alpha < 1$ be given. Lemma 9 provides $R_\alpha, \varepsilon_\alpha > 0$ with the following property: If $0 < R \leq R_\alpha$, then for every $0 < \varepsilon \leq \varepsilon_\alpha$

$$\|\mathbf{F}_{\mathbf{x}_0,R}\|^2 + \lambda^2_{\mathbf{x}_0,R} \left(1 + \|\mathbf{F}_{\mathbf{x}_0,R}\|^2\right)^{\frac{p-2}{2}} > Q^2_\varepsilon + \frac{1}{\varepsilon^2} \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2 \tag{79}$$

implies

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,d\mathbf{x} \leq \alpha \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,R})\|^2 \,d\mathbf{x}$$
$$+ \frac{c_8 L_2}{\Lambda_*} R^\beta \left(R^n + \frac{\Lambda^*}{\Lambda_*} J^+_{\mathbf{x}_0,R}[\mathbf{w}]\right).$$

Since $\mathbf{A} \in L^\infty(\mathcal{B}^+; \mathbb{R}^{N\times n})$, for each $0 < R \leq R_\alpha$ we get from Theorem 6, in particular (40), that

$$J^+_{\mathbf{x}_0,R}[\mathbf{w}] \leq \Lambda^* \left[c_5 \left(\frac{1}{R_{\mathbf{x}_0}}\right)^{n+2\nu-\beta} \int_{\mathcal{B}^+_{\mathbf{x}_0,R_{\mathbf{x}_0}}} H(\nabla \mathbf{w} \mathbf{G}_0) \,d\mathbf{x} + c_6 \right] R^{n+2\nu-\beta},$$

for each $0 \leq \nu < \frac{1}{2}\beta$. Hence, whenever (79) is satisfied, we have that

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{w})\|^2 \,d\mathbf{x} \leq \alpha \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,R})\|^2 \,d\mathbf{x}$$
$$+ \frac{L_2}{\Lambda_*} \left(c'_9 + \left(\frac{\Lambda^*}{\Lambda_*}\right)^2 \frac{c'_{10}}{R^{n-\beta}_{\mathbf{x}_0}} \right) R^{n+2\nu},$$

with

$$c_9' := (1+c_6)c_8 \quad \text{and} \quad c_{10}' := c_5 \int_{\mathcal{B}^+} f_{\mathbf{x}_0}(\nabla \mathbf{w} \mathbf{G}_0)\, d\mathbf{x}.$$

Inserting this into (78) yields

$$\int_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,\rho})\|^2\, d\mathbf{x} \leq 2^5(1+c_0)\left[\left(\frac{\rho}{R}\right)^{n+2\sigma_0} + \alpha\right]$$
$$\times \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - \mathbf{V}(\mathbf{F}_{\mathbf{x}_0,R})\|^2 + \frac{2^5 L_2 c_0}{\Lambda_*}\left(c_9' + \left(\frac{\Lambda^*}{\Lambda_*}\right)^2 \frac{c_{10}'}{R_{\mathbf{x}_0}^{n-\beta}}\right) R^{n+2\nu}. \quad (80)$$

Fix ν and γ, so that $0 < 2\nu < \beta < 2\gamma < 2\sigma_0$, and fix $\alpha = \left(\frac{1}{2^6(1+c_0)}\right)^{\frac{n+2\sigma_0}{2(\sigma_0-\gamma)}}$, so $\varepsilon_{\alpha>0}$ is also fixed. Define

$$M_{\varepsilon_\alpha}^2 := Q_{\varepsilon_\alpha}^2 + \frac{1}{\varepsilon_\alpha^2} \sup_{\mathbf{x}\in\mathcal{B}^+} \|\mathbf{A}\|^2$$

For each $R \in (0, R_\alpha]$, the inequality in(80) must hold whenever

$$\|\mathbf{F}_{\mathbf{x}_0,R}\|^2 + \lambda_{\mathbf{x}_0,R}^2 \left(1 + \|\mathbf{F}_{\mathbf{x}_0,R}\|^2\right)^{\frac{p-2}{2}} > M_{\varepsilon_\alpha}^2 \quad (81)$$

As in [23], we consider three cases.

Case 1 If there is a sequence $\{R_j\}_{j=1}^{\infty} \subset (0, 1 - \|\mathbf{x}_0\|)$ such that $\lim_{j\to+\infty} R_j = 0$ and

$$\|\mathbf{F}_{\mathbf{x}_0,R_j}\|^2 + \lambda_{\mathbf{x}_0,R_j}^2 \left(1 + \|\mathbf{F}_{\mathbf{x}_0,R_j}\|^2\right)^{\frac{p-2}{2}} \leq M_{\varepsilon_\alpha}^2,$$

then

$$\|\nabla \mathbf{w}(\mathbf{x}_0)\|^p \leq \lim_{j\to\infty} \|\mathbf{V}(\mathbf{F}_{\mathbf{x}_0,R_j})\|^2 \leq C \lim_{j\to\infty} \left(\|\mathbf{F}_{\mathbf{x}_0,R_j}\|^2 + \|\mathbf{F}_{\mathbf{x}_0,R_j}\|^p\right)$$
$$\leq C\left(M_{\varepsilon_\alpha}^2 + M_{\varepsilon_\alpha}^p\right), \quad (82)$$

with C depending only on p.

Case 2 There is an $\widetilde{R} \in (0, R_\alpha]$ such that for each $R \in (0, \widetilde{R})$ the inequality in (81) holds, but

$$\|\mathbf{F}_{\mathbf{x}_0,\widetilde{R}}\|^2 + \lambda_{\mathbf{x}_0,\widetilde{R}}^2 (\mathbf{F}_{\mathbf{x}_0,\widetilde{R}})^2 \leq M_{\varepsilon_\alpha}^2. \quad (83)$$

In this case, we may apply Lemma 1 to obtain

$$\fint_{\mathcal{B}^+_{\mathbf{x}_0,\rho}} \|\mathbf{V}(\nabla \mathbf{w}) - (\mathbf{V}(\nabla \mathbf{w}))^+_{\mathbf{x}_0,\rho}\|^2\, d\mathbf{x}$$
$$\leq \frac{2}{\alpha}\left(\frac{\rho}{R}\right)^{2\gamma} \fint_{\mathcal{B}^+_{\mathbf{x}_0,R}} \|\mathbf{V}(\nabla \mathbf{w}) - (\mathbf{V}(\nabla \mathbf{w}))^+_{\mathbf{x}_0,R}\|^2\, d\mathbf{x} + c_9 \rho^{2\nu}, \quad (84)$$

for each $0 < \rho \le R \le \widetilde{R}$. Here, we have put

$$c_9 := \frac{2^5 L_2 c_0}{\Lambda_*}\left(c_9' + \left(\frac{\Lambda^*}{\Lambda_*}\right)^2 \frac{c_{10}'}{R_{\mathbf{x}_0}^{n-\beta}}\right)\left(\frac{1}{\alpha}\right)^{\frac{n+2\nu}{n+2\sigma_0}}\left[\frac{1}{\alpha^{\frac{n+2\nu}{n+2\sigma_0}} - \alpha^{\frac{n+2\gamma}{n+2\sigma_0}}} + 1\right].$$

Since the uniform p-Uhlenbeck structure for the family $\{f_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{B}^+\cup\mathcal{D}_{\mathbf{0},1}}$ implies that $\frac{L_2}{\Lambda_*}, \frac{\Lambda^*}{\Lambda_*} \in L^\infty(\mathcal{B}^+;\mathbb{R})$, we conclude that $c_9 < +\infty$ at each $\mathbf{x}_0 \in \mathcal{B}^+$. Define $\{r_j\}_{j=1}^\infty \subset [0, \widetilde{R})$ by $r_j := 2^{-j}\widetilde{R}$, for each $j = 1, 2, \ldots$ From (84), we have

$$\begin{aligned}
&\fint_{\mathcal{B}^+_{\mathbf{x}_0,r_j}} \|\mathbf{V}(\nabla\mathbf{w}) - (\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,r_j}\|^2\,\mathrm{d}\mathbf{x} \\
&\quad\le \left(\frac{2}{\alpha}\right) 2^{-2\gamma j}\fint_{\mathcal{B}^+_{\mathbf{x}_0,\widetilde{R}}} \|\mathbf{V}(\nabla\mathbf{w}) - (\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,R_0}\|^2\,\mathrm{d}\mathbf{x} + c_9 2^{-2\nu j}\widetilde{R}^{2\nu}.
\end{aligned}$$

Since $\mathbf{x}_0 \in \widetilde{\Omega}$, it follows that

$$\begin{aligned}
\|\mathbf{V}(\nabla\mathbf{w}(\mathbf{x}_0))\| &\le C\sum_{j=1}^{\infty}\left(\fint_{\mathcal{B}^+_{\mathbf{x}_0,r_j}}\|\mathbf{V}(\nabla\mathbf{w}) - (\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,r_j}\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} + \|\mathbf{V}(\mathbf{F}_{\mathbf{x}_0,\widetilde{R}})\| \\
&\le \frac{C}{\alpha^{\frac{1}{2}}}\left(\fint_{\mathcal{B}^+_{\mathbf{x}_0,\widetilde{R}}}\|\mathbf{V}(\nabla\mathbf{w}) - (\mathbf{V}(\nabla\mathbf{w}))^+_{\mathbf{x}_0,\widetilde{R}}\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\sum_{j=1}^{\infty}2^{-\gamma j} \\
&\quad + c_9^{\frac{1}{2}}\widetilde{R}^{\nu}\sum_{j=1}^{\infty}2^{-\nu j}.
\end{aligned}$$

Using the definition of $\lambda_{\mathbf{x}_0,\widetilde{R}}(\mathbf{F}_{\mathbf{x}_0,\widetilde{R}})$, which we will denote by λ, yields

$$\begin{aligned}
\|\mathbf{V}(\nabla\mathbf{w}(\mathbf{x}_0))\| \le{}& \frac{C}{\alpha^{\frac{1}{2}}(2^{\gamma}-1)}\left\{\lambda^p + \lambda^2(1+\|\mathbf{F}_{\mathbf{x}_0,\widetilde{R}}\|^2)^{\frac{p-2}{2}}\right\}^{\frac{1}{2}} \\
&+ \frac{c_9^{\frac{1}{2}}\widetilde{R}}{2^{\nu}-1} + \|\mathbf{V}(\mathbf{F}_{\mathbf{x}_0,\widetilde{R}})\|
\end{aligned}$$

Now (83) implies

$$\|\nabla\mathbf{w}(\mathbf{x}_0)\|^p \le \|\mathbf{V}(\nabla\mathbf{w}(\mathbf{x}_0))\|^2 \le C\left\{\frac{1}{\alpha(2^{\gamma}-1)^2} + \left(M^p_{\varepsilon_\alpha} + M^2_{\varepsilon_\alpha}\right) + c_9\right\}, \tag{85}$$

with C depending only on n and p.

Case 3 For this case, the inequality in (81) holds for every $R \in (0, R_\alpha]$, so we may use (84) for any ρ and R satisfying $0 < \rho \leq R \leq R_\alpha$, and arguing as we did in Case 2, we arrive at

$$\begin{aligned}\|\mathbf{V}(\nabla \mathbf{w}(\mathbf{x}_0))\| &\leq \frac{C}{\alpha^{\frac{1}{2}}} \left(\fint_{\mathcal{B}^+_{\mathbf{x}_0, R_\alpha}} \|\mathbf{V}(\nabla \mathbf{w}) - (\mathbf{V}(\nabla \mathbf{w}))^+_{\mathbf{x}_0, R_\alpha}\|^2 \, d\mathbf{x} \right)^{\frac{1}{2}} \sum_{j=1}^{\infty} 2^{-\gamma j} \\ &\quad + c_9^{\frac{1}{2}} R_\alpha^\nu \sum_{j=1}^{\infty} 2^{-\nu j} + \|\mathbf{V}(\mathbf{F}_{\mathbf{x}_0, R_\alpha})\| \\ &\leq \frac{C}{\alpha^{\frac{1}{2}} R_\alpha^{\frac{n}{2}} (2^\gamma - 1)} \|\mathbf{V}(\nabla \mathbf{w})\|_{L^2(\mathcal{B}^+_{\mathbf{x}_0, R_\alpha})} + \frac{c_9^{\frac{1}{2}}}{2^\nu - 1}.\end{aligned}$$

Hence, by Lemma 7,

$$\|\nabla \mathbf{w}(\mathbf{x}_0)\|^p \leq C \left\{ \frac{1}{\alpha R_\alpha (2^\gamma - 1)^2} + \frac{1}{\Lambda_*} \left(K^+[\mathbf{w}] + c_4 \right) + c_9 \right\} \tag{86}$$

In each case, we have $\|\nabla \mathbf{w}(\mathbf{x}_0)\|^p \leq M$, where M depends only on $K^+[\mathbf{w}]$, the structural constants for $\{f_\mathbf{y}\}_{\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}}$ and $\{g_\mathbf{y}\}_{\mathbf{y} \in \mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}}$, the mappings $\mathbf{A}$, $\mathbf{G}$, and $\mathbf{G}^{-1}$ and $\text{dist}(\mathbf{x}_0, \partial \mathcal{B})$. In particular, the bound is independent of $\text{dist}(\mathbf{x}_0, \mathcal{D}_{\mathbf{0},1})$. It follows that $\mathbf{w} \in W^{1,\infty}_{\text{loc}}(\mathcal{B}^+ \cup \mathcal{D}_{\mathbf{0},1}; \mathbb{R}^N)$. □

Using Lemma 10 and essentially the same argument provided for Theorem 9, we obtain

Theorem 10 *Suppose that the family* $\{g_\mathbf{y}\}_{\mathbf{y} \in \mathcal{B}} \subset \mathcal{C}^3(\mathbb{R}^{N \times n})$ *possesses property* (G′) *and is uniformly asymptotically related to* $\{f_\mathbf{y}\}_{\mathbf{y} \in \mathcal{B}} \subset \mathcal{C}^2(\mathbb{R}^{N \times n})$*, which has a uniform p-Uhlenbeck structure. Let* $\mathbf{A} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{N \times n})$ *and* $\mathbf{G} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{n \times n})$*, with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^{0,\beta}(\overline{\mathcal{B}}; \mathbb{R}^{n \times n})$*, be given. For each* $\mathbf{y} \in \mathcal{B}$*, define the functional*

$$K_\mathbf{y}[\mathbf{w}] := \int_{\mathcal{B}} g_\mathbf{y}([\nabla \mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x})) \, d\mathbf{x}.$$

If $\mathbf{w} \in W^{1,1}(\mathcal{B}; \mathbb{R}^N)$ *is a* $(K_\mathbf{y}, \omega)$*-minimizer at each* $\mathbf{y} \in \mathcal{B}$*, then* $\mathbf{w} \in W^{1,\infty}_{\text{loc}}(\mathcal{B}; \mathbb{R}^N)$*.*

With Theorems 9 and 10 available, the main result for this section may be proved in the same manner as Theorem 8.

Theorem 11 *Suppose that* $\Omega \subset \mathbb{R}^n$ *and that* Γ *is a* $\mathcal{C}^{1,\beta}$ *portion of* $\partial\Omega$*. Suppose also that* $\{g_\mathbf{y}\}_{\mathbf{y} \in \Omega \cup \Gamma} \subset \mathcal{C}^3(\mathbb{R}^{N \times n})$ *possesses the growth properties in* (G′) *and is uniformly asymptotically related to* $\{f_\mathbf{y}\}_{\mathbf{y} \in \Omega \cup \Gamma} \subset \mathcal{C}^2(\mathbb{R}^{N \times n})$*, which has a uniform p-Uhlenbeck structure. Let* $\mathbf{A} \in \mathcal{C}^{0,\beta}_{\text{loc}}(\Omega \cup \Gamma; \mathbb{R}^{N \times n})$ *and* $\mathbf{G} \in \mathcal{C}^{0,\beta}_{\text{loc}}(\Omega \cup \Gamma; \mathbb{R}^{n \times n})$*, with a point-wise matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}^{0,\beta}_{\text{loc}}(\Omega \cup \Gamma; \mathbb{R}^{n \times n})$*, be given. For each* $\mathbf{y} \in \Omega \cup \Gamma$*, define the functional*

$$K_\mathbf{y} := \int_{\Omega} g_\mathbf{y}([\nabla \mathbf{w}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]\mathbf{G}(\mathbf{x})) \, d\mathbf{x}$$

If $\mathbf{w} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *satisfies* $\mathbf{w} = \mathbf{0}$ *on* Γ*, in the sense of traces, and* $\mathbf{w}$ *is a* $(K_\mathbf{y}, \omega)$*-minimizer at each* $\mathbf{y} \in \Omega \cup \Gamma$*, then* $\mathbf{w} \in W^{1,\infty}_{\text{loc}}(\Omega \cup \Gamma; \mathbb{R}^N)$*.*

We conclude this section with the following

Corollary 2 *Suppose that* $\Omega \subset \mathbb{R}^n$ *has a* $\mathcal{C}^{1,\beta}$ *boundary, for some* $0 < \beta \leq 1$. *Let* $\overline{\mathbf{u}} \in \mathcal{C}^{1,\beta}(\overline{\Omega}; \mathbb{R}^N)$ *be given. If* $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *is a* (K, ω)*-minimizer for the functional*

$$K[\mathbf{u}] := \int_{\Omega} g(\nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,1}(\Omega; \mathbb{R}^N)$, *then* $\mathbf{u} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$.

8 Global regularity for nonhomogeneous functionals

In this section, we apply the results from the previous sections to some variational problems with integrands that are nonhomogeneous. The basic idea is to show that a minimizer for such problems is a $(K, \omega, \{\nu_\varepsilon\})$-minimizer with a suitable choice of ω and family $\{\nu_\varepsilon\}$. We will not state the most general possible applications. In particular, we will only state the results for almost minimizers with full Dirichlet conditions.

Fix $0 \leq \kappa < n$ and $0 < \beta \leq 1$. Recall the Sobolev conjugate exponent defined by

$$p^* := \begin{cases} \frac{np}{n-p}, & 1 \leq p < n; \\ +\infty, & p = n. \end{cases}$$

For our first application, we shall prove global Morrey regularity for almost minimizers for functionals with integrands of the form $g(\mathbf{x}, \mathbf{F})$. With an aim to establish regularity for minimizers of functionals that depend on the mapping itself, as a well as its gradient, we allow g to possess some discontinuous behavior with respect to the $\mathbf{x}$ argument.

Theorem 12 *Let* $\Omega \subset \mathbb{R}^n$, *with a* $\mathcal{C}^{1,0}$ *boundary, and* $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ *be given. Let* $\zeta \in \mathscr{L}^{q,\kappa}(\Omega)$, *for some* $q \geq 1$, *be given. Suppose that* $\delta,\ \eta \in \mathcal{C}^0([0,\infty))$ *are non-decreasing functions satisfying* $\delta(0) = \eta(0) = 0$ *and* $g : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$ *possesses the following properties*

(i) *for each* $\mathbf{x} \in \overline{\Omega}$, *we have* $g(\mathbf{x}, \cdot) \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, *and there exists a function* $b \in L^{p,\kappa}(\Omega)$ *such that*

$$b(\mathbf{x}) \geq |g(\mathbf{x}, \mathbf{0})|^{\frac{1}{p}} + \left\| \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x}, \mathbf{0}) \right\|^{\frac{1}{p-1}},$$

for all $\mathbf{x} \in \overline{\Omega}$;

(ii) *there is a family* $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\overline{\Omega}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ *with a uniform p-Uhlenbeck structure such that for each* $\varepsilon > 0$, *there exists a* $\sigma_\varepsilon \in L^{p,\kappa}(\Omega)$ *such that for every* $\mathbf{x} \in \overline{\Omega}$

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x}, \mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f_{\mathbf{x}}(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}} \varepsilon \|\mathbf{F}\|^{p-2}$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon(\mathbf{x})$; *and there is an* $a \in L^{p,\kappa}(\Omega)$ *satisfying*

$$a(\mathbf{x}) \geq 1 + \max\left\{ \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x}, \mathbf{F}) \right\|^{\frac{1}{p-2}} : \|\mathbf{F}\| \leq \sigma_{\Lambda^*(\mathbf{x})}(\mathbf{x}) \right\},$$

for all $\mathbf{x} \in \overline{\Omega}$;

(iii) *for each* $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$, *we have*

$$|g(\mathbf{x}, \mathbf{F}) - g(\mathbf{y}, \mathbf{F})| \leq \delta(\|\mathbf{x} - \mathbf{y}\|) \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} + |\zeta(\mathbf{x})|^q + |\zeta(\mathbf{y})|^q + (|\zeta(\mathbf{x})| + |\zeta(\mathbf{y})|) \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p(q-1)}{2q}},$$

for every $\mathbf{F} \in \mathbb{R}^{N \times n}$.

If there are $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}$ *and* $\{T_\varepsilon\}_{\varepsilon>0} \subseteq [0, \infty)$ *such that* $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *is a* $(K, \eta, \{\gamma_\varepsilon, T_\varepsilon\})$*-minimizer for the functional*

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,1}(\Omega; \mathbb{R}^N)$, *then* $\nabla \mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N \times n})$. *In addition, we find that* $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^N)$.

Proof We only need to show that $\mathbf{u}$ is an almost minimizer for a family of functionals with integrands satisfying the hypotheses of Theorem 8. For each $\mathbf{y} \in \overline{\Omega}$, define the function $g_\mathbf{y} \in C^2(\mathbb{R}^{N \times n})$ by

$$g_\mathbf{y}(\mathbf{F}) := g(\mathbf{y}, \mathbf{F}),$$

and define the functional $K_\mathbf{y} : W^{1,1}(\Omega; \mathbb{R}^N) \to \overline{\mathbb{R}}$ by

$$K_\mathbf{y}[\mathbf{u}] := \int_\Omega g_\mathbf{y}(\nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}.$$

Clearly the family $\left\{g_\mathbf{y}\right\}_{\mathbf{y}\in\overline{\Omega}}$ is $L^{p,\kappa}$-asymptotically related to the family $\left\{f_\mathbf{y}\right\}_{\mathbf{y}\in\overline{\Omega}}$ and satisfies the growth conditions (G) in Sect. 6.

We now show that there is a family $\{\nu_\varepsilon\}_{\varepsilon>0} \subset \mathscr{L}^{p,\kappa}(\Omega)$ such that $\mathbf{u}$ is a $(K_\mathbf{y}, \omega, \{\nu_\varepsilon, T_\varepsilon\})$-minimizer at each $\mathbf{y} \in \overline{\Omega}$ for an appropriate ω. To this end, let $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega_{\mathbf{y},\rho}; \mathbb{R}^N)$ and $\varepsilon > 0$ be given. Since $\mathbf{u}$ is a $(K, \eta, \{\gamma_\varepsilon, T_\varepsilon\})$-minimizer, for some $t > 1$ we have

$$\begin{aligned}
K_\mathbf{y}[\mathbf{u}] &\leq K_\mathbf{y}[\mathbf{u}+\boldsymbol{\varphi}] + K[\mathbf{u}+\boldsymbol{\varphi}] - K[\mathbf{u}] + K_\mathbf{y}[\mathbf{u}] - K_\mathbf{y}[\mathbf{u}+\boldsymbol{\varphi}] \\
&\quad + \{\eta(\rho) + \varepsilon\} \int_{\mathrm{supp}(\varphi)} \left(1 + \|\nabla \boldsymbol{\varphi}\|^p + \|\nabla \mathbf{u}\|^p\right) \mathrm{d}\mathbf{x} \\
&\quad + T_\varepsilon \left(\int_{\mathrm{supp}(\varphi)} \|\nabla \boldsymbol{\varphi}\|^p\, \mathrm{d}\mathbf{x} \right)^t + \int_{\mathrm{supp}(\varphi)} |\gamma_\varepsilon|\, \mathrm{d}\mathbf{x} + \rho^n |\gamma_\varepsilon(\mathbf{y})| \\
&\leq K_\mathbf{y}[\mathbf{u}+\boldsymbol{\varphi}] + \int_{\mathrm{supp}(\varphi)} |g(\mathbf{x}, \nabla\mathbf{u}+\nabla\boldsymbol{\varphi}) - g(\mathbf{y}, \nabla\mathbf{u}+\nabla\boldsymbol{\varphi})|\, \mathrm{d}\mathbf{x} \\
&\quad + \int_{\mathrm{supp}(\varphi)} |g(\mathbf{x}, \nabla\mathbf{u}) - g(\mathbf{y}, \nabla\mathbf{u})|\, \mathrm{d}\mathbf{x} + \{\eta(\rho)+\varepsilon\} \int_{\mathrm{supp}(\varphi)} \left(1 + \|\nabla \boldsymbol{\varphi}\|^p + \|\nabla \mathbf{u}\|^p\right) \mathrm{d}\mathbf{x} \\
&\quad + T_\varepsilon \left(\int_{\mathrm{supp}(\varphi)} \|\nabla \boldsymbol{\varphi}\|^p\, \mathrm{d}\mathbf{x} \right)^t + \int_{\mathrm{supp}(\varphi)} |\gamma_\varepsilon|\, \mathrm{d}\mathbf{x} + \rho^n |\gamma_\varepsilon(\mathbf{y})|.
\end{aligned}$$

Using the regularity of g with respect to its first argument in property (iii), we may write

$$K_{\mathbf{y}}[\mathbf{u}] \le K_{\mathbf{y}}[\mathbf{u}+\boldsymbol{\varphi}] + C\{\eta(\rho)+\delta(\rho)+\varepsilon\}\int\limits_{\mathrm{supp}(\varphi)}\left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\mathrm{d}\mathbf{x}$$
$$+C\int\limits_{\mathrm{supp}(\varphi)}(|\zeta(\mathbf{x})|+|\zeta(\mathbf{y})|)\left(1+\|\nabla\boldsymbol{\varphi}\|^2+\|\nabla\mathbf{u}\|^2\right)^{\frac{p(q-r)}{2q}}\mathrm{d}\mathbf{x}$$
$$+C\int\limits_{\mathrm{supp}(\varphi)}\left(|\zeta(\mathbf{x})|^q+|\zeta(\mathbf{y})|^q\right)\mathrm{d}\mathbf{x}+T_\varepsilon\left(\int\limits_{\mathrm{supp}(\varphi)}\|\nabla\boldsymbol{\varphi}\|^p\,\mathrm{d}\mathbf{x}\right)^t$$
$$+\int\limits_{\mathrm{supp}(\varphi)}|\gamma_\varepsilon|\,\mathrm{d}\mathbf{x}+\rho^n|\gamma_\varepsilon(\mathbf{y})|.$$

Next, we apply Young's inequality to conclude that for each $\varepsilon > 0$

$$K_{\mathbf{y}}[\mathbf{u}] \le K_{\mathbf{y}}[\mathbf{u}+\boldsymbol{\varphi}] + \{C(n(\rho)+\delta(\rho))+2\varepsilon\}\int\limits_{\mathrm{supp}(\varphi)}\left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\mathrm{d}\mathbf{x}$$
$$+T_\varepsilon\left(\int\limits_{\mathrm{supp}(\varphi)}\|\nabla\boldsymbol{\varphi}\|^p\,\mathrm{d}\mathbf{x}\right)^t+C\left(\frac{1}{\varepsilon^{q-1}}+1\right)\int\limits_{\mathrm{supp}(\varphi)}\left(|\zeta(\mathbf{x})|^q+|\zeta(\mathbf{y})|^q\right)\mathrm{d}\mathbf{x}$$
$$+\int\limits_{\mathrm{supp}(\varphi)}|\gamma_\varepsilon|\,\mathrm{d}\mathbf{x}+\rho^n|\gamma_\varepsilon(\mathbf{y})|$$
$$\le K_{\mathbf{y}}[\mathbf{u}+\boldsymbol{\varphi}]+\{\omega(\rho)+2\varepsilon\}\int\limits_{\mathrm{supp}(\varphi)}\left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\mathrm{d}\mathbf{x}$$
$$+T_\varepsilon\left(\int\limits_{\mathrm{supp}(\varphi)}\|\nabla\boldsymbol{\varphi}\|^p\,\mathrm{d}\mathbf{x}\right)^t+\int\limits_{\mathrm{supp}(\varphi)}|\nu_\varepsilon(\mathbf{x})|\mathrm{d}\mathbf{x}+\rho^n|\nu_\varepsilon(\mathbf{y})|,$$

with $\omega(\rho) := C\{\eta(\rho)+\delta(\rho)\}$ for each $\rho \ge 0$ and $\nu_\varepsilon := C\left(\frac{1}{\varepsilon^{q-1}}+1\right)|\zeta|^q+|\gamma_\varepsilon|$. The constant C depends only on p and q. The continuity of ω follows from the hypotheses on δ and η. We also see that $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$. We deduce from Theorem 8 that $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega;\mathbb{R}^N)$. □

It is worth noting that if $p+\kappa$ is large enough, then we obtain uniform continuity of the almost minimizer $\mathbf{u}$.

Corollary 3 *Suppose that all of the hypotheses of Theorem 12 are satisfied with $p+\kappa > n$. Then $\mathbf{u} \in \mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\overline{\Omega};\mathbb{R}^N)$.*

Example 3 Let

(a) $y \in \mathcal{C}^0(\overline{\Omega};(0,\infty))$;
(b) $z \in L^{q,\kappa}(\Omega;\mathbb{R})$, for some $q \ge 1$

be given. The function $g : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ defined by

$$g(\mathbf{x},\mathbf{F}) := y(\mathbf{x})\|\mathbf{F}\|^p + z(\mathbf{x})\left(1+\|\mathbf{F}\|^2\right)^{\frac{p(q-1)}{2q}}$$

satisfies the hypotheses of Theorem 12. Indeed, hypotheses (i) and (iii) are easily verified. For hypothesis (ii), we may use $\sigma_\varepsilon(\mathbf{x}) := 1 + C\left|\frac{z(\mathbf{x})}{\varepsilon}\right|^{\frac{q}{p}}$, for each $\varepsilon > 0$, and $a(\mathbf{x}) := 1 + C\left(\frac{1}{y(\mathbf{x})^{\frac{1}{p-2}}} + y(\mathbf{x})^{\frac{1}{p-2}}\right)\left|\frac{z(\mathbf{x})}{y(\mathbf{x})}\right|^{\frac{q}{p}}$, where C is a constant that depends only on p and q. Thus, if $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ and $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^{N\times n})$ is a minimizer for the functional

$$\mathbf{u} \mapsto \int_\Omega \left\{ y(\mathbf{x})\|\nabla\mathbf{u}(\mathbf{x})\|^p + z(\mathbf{x})\left(1+\|\nabla\mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p(q-1)}{2q}}\right\} \mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u}-\overline{\mathbf{u}}] \in W_0^{1,1}(\Omega;\mathbb{R}^N)$, then we may use Theorem 12 to conclude that $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega;\mathbb{R}^{N\times n})$.

Under stronger hypotheses, we get the Lipschitz continuity of an almost minimizer **u**. The proof of the following Theorem is the same as that given for Theorem 12, except that we invoke Thereom 11 instead of Theorem 8.

Theorem 13 *Let $\Omega \subset \mathbb{R}^n$, with a $\mathcal{C}^{1,\beta}$ boundary, and $\overline{\mathbf{u}} \in C^{1,\beta}(\overline{\Omega};\mathbb{R}^N)$ be given. Suppose that the functions $\delta, \eta \in \mathcal{C}^{0,\beta}([0,\infty))$ are non-decreasing and satisfy $\delta(0) = \eta(0) = 0$. Suppose also that $g : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses the following properties*

(i) *for each $\mathbf{x} \in \overline{\Omega}$, we have $g(\mathbf{x},\cdot) \in \mathcal{C}^3(\mathbb{R}^{N\times n})$ and there exist $a, b, L < +\infty$ such that*

$$\left\|\frac{\partial^3}{\partial\mathbf{F}^3} g(\mathbf{x},\mathbf{F})\right\| \le L\left(a + \|\mathbf{F}\|^2\right)^{\frac{p-3}{2}}$$

and

$$b \ge |g(\mathbf{x},\mathbf{0})| + \left\|\frac{\partial}{\partial\mathbf{F}} g(\mathbf{x},\mathbf{0})\right\| + \left\|\frac{\partial^2}{\partial\mathbf{F}^2} g(\mathbf{x},\mathbf{0})\right\|$$

for every $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{F} \in \mathbb{R}^{N\times n}$;

(ii) *there is a family $\{f_\mathbf{x}\}_{\mathbf{x}\in\overline{\Omega}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a uniform p-Uhlenbeck structure such that for each $\varepsilon > 0$, there exists a $\sigma_\varepsilon < +\infty$ such that for every $\mathbf{x} \in \overline{\Omega}$*

$$\left\|\frac{\partial^2}{\partial\mathbf{F}^2} g(\mathbf{x},\mathbf{F}) - \frac{\partial^2}{\partial\mathbf{F}^2} f_\mathbf{x}(\mathbf{F})\right\| < \frac{1}{2^{\frac{p}{2}}}\varepsilon\|\mathbf{F}\|^{p-2}$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon$;

(iii) *for each $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$, we have*

$$|g(\mathbf{x},\mathbf{F}) - g(\mathbf{y},\mathbf{F})| \le \delta(\|\mathbf{x}-\mathbf{y}\|)\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}},$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$.

If $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a (K,η)-minimizer for the functional

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x},\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u}-\overline{\mathbf{u}}] \in W_0^{1,1}(\Omega;\mathbb{R}^N)$, then $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^N)$.

We next apply our results to the case where the functionals depend on the mapping itself.

Theorem 14 *Suppose that $p \leq n$ and $\Omega \subset \mathbb{R}^n$ has a $C^{1,0}$ boundary, and let the mapping $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ be given. Let $\delta, \eta \in C^0([0,\infty))$ be non-decreasing functions satisfying $\delta(0) = \eta(0) = 0$. Let $0 \leq s < p^*$ be given. Suppose that the function $g : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses the following properties:*

(i) *for each $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{u} \in \mathbb{R}^N$, we have $g(\mathbf{x}, \mathbf{u}, \cdot) \in \mathcal{C}^2(\mathbb{R}^{N\times n})$, and there exists a function $b \in L^{p,\kappa}(\Omega)$ and a number $B < +\infty$ such that*

$$B\|\mathbf{u}\|^{\frac{s}{p}} + b(\mathbf{x}) \geq |g(\mathbf{x}, \mathbf{u}, \mathbf{0})|^{\frac{1}{p}} + \left\| \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x}, \mathbf{u}, \mathbf{0}) \right\|^{\frac{1}{p-1}},$$

for all $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{u} \in \mathbb{R}^N$;

(ii) *there is a family $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\overline{\Omega}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a uniform p-Uhlenbeck structure such that for each $\varepsilon > 0$, there exists a $\sigma_\varepsilon \in L^{p,\kappa}(\Omega)$ and a $\Sigma_\varepsilon < +\infty$ such that for every $\mathbf{x} \in \overline{\Omega}$*

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x}, \mathbf{u}, \mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f_{\mathbf{x}}(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}} \varepsilon \|\mathbf{F}\|^{p-2}$$

whenever $\|\mathbf{F}\| > \Sigma_\varepsilon \|\mathbf{u}\|^{\frac{s}{p}} + \sigma_\varepsilon(\mathbf{x})$; and that there is an $a \in L^{p,\kappa}(\Omega)$ and an $A < +\infty$ satisfying

$$\begin{aligned} &A\|\mathbf{u}\|^{\frac{s}{p}} + a(\mathbf{x}) \\ &\geq 1 + \max\left\{ \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x}, \mathbf{u}, \mathbf{F}) \right\|^{\frac{1}{p-2}} \,\middle|\, \|\mathbf{F}\| \leq \Sigma_{\Lambda^*(\mathbf{x})} \|\mathbf{u}\|^{\frac{s}{p}} + \sigma_{\Lambda^*(\mathbf{x})}(\mathbf{x}) \right\}, \end{aligned}$$

for all $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{u} \in \mathbb{R}^N$;

(iii) *there is an $r \in (0, s)$ and an $M < +\infty$ such that for each $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ and each $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, we have*

$$\begin{aligned} |g(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g(\mathbf{y}, \mathbf{v}, \mathbf{F})| &\leq \delta(\|\mathbf{x} - \mathbf{y}\|) \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} + M\left(\|\mathbf{u}\|^s + \|\mathbf{v}\|^s\right) \\ &\quad + M\left(1 + \|\mathbf{u}\|^r + \|\mathbf{v}\|^r\right)\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p(s-r)}{2s}}, \end{aligned}$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$.

If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a (K, η)-minimizer for the functional

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,p}(\Omega; \mathbb{R}^N)$, then we find that $\nabla \mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N\times n})$ and that $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^N)$.

Proof By applying Young's inequality in property (iii), without loss of generality we may assume that $s > p$. First, we show that we may apply Theorem 12. Put $\gamma_1 := n\left(1 - \frac{s}{p^*}\right)$. Since $s < p^*$, we see that $\gamma_1 > 0$. Since by hypothesis $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$, Sobolev's imbedding theorem implies $\mathbf{u} \in L^{p^*}(\Omega; \mathbb{R}^N)$. Therefore $\|\mathbf{u}\|^s \in L^{1,\gamma_1}(\Omega)$ and $\left[A\|\mathbf{u}\|^{\frac{s}{p}} + a\right], \left[B\|\mathbf{u}\|^{\frac{s}{p}} + b\right] \in L^{p,\gamma_1}(\Omega)$. For each $\mathbf{y} \in \overline{\Omega}$, define $\widetilde{g} : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$ by

$$\widetilde{g}(\mathbf{y}, \mathbf{F}) := g(\mathbf{y}, \mathbf{u}(\mathbf{y}), \mathbf{F})$$

and the functional $\widetilde{K} : W^{1,1}(\Omega;\mathbb{R}^N) \to \overline{\mathbb{R}}$ by

$$\widetilde{K}[\mathbf{w}] := \int_{\Omega} \widetilde{g}(\mathbf{x}, \nabla\mathbf{w}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

Let $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega_{\mathbf{y},\rho};\mathbb{R}^N)$ and $\varepsilon > 0$ be given. Since the inequality in Definition 1 is trivial if $\nabla\boldsymbol{\varphi} \notin L^p(\Omega_{\mathbf{y},\rho};\mathbb{R}^{N\times n})$, we assume that $\boldsymbol{\varphi} \in W_0^{1,p}(\Omega_{\mathbf{y},\rho};\mathbb{R}^N)$. We have

$$\begin{aligned}\widetilde{K}[\mathbf{u}] &\leq \widetilde{K}[\mathbf{u}+\boldsymbol{\varphi}] + K[\mathbf{u}+\boldsymbol{\varphi}] - K[\mathbf{u}] + \widetilde{K}[\mathbf{u}] - \widetilde{K}[\mathbf{u}+\boldsymbol{\varphi}] \\ &\quad + \eta(\rho)\int_{\operatorname{supp}(\varphi)} \left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\mathrm{d}\mathbf{x} \\ &\leq \widetilde{K}[\mathbf{u}+\boldsymbol{\varphi}] + \int_{\operatorname{supp}(\varphi)} |g(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})) - g(\mathbf{y},\mathbf{u}(\mathbf{y}),\nabla\mathbf{u}(\mathbf{x}))|\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\operatorname{supp}(\varphi)} |g(\mathbf{x},\mathbf{u}(\mathbf{x})+\boldsymbol{\varphi}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})+\nabla\boldsymbol{\varphi}(\mathbf{x})) - g(\mathbf{y},\mathbf{u}(\mathbf{y}),\nabla\mathbf{u}(\mathbf{x})+\nabla\boldsymbol{\varphi}(\mathbf{x}))|\,\mathrm{d}\mathbf{x} \\ &\quad + \eta(\rho)\int_{\operatorname{supp}(\varphi)} \left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\mathrm{d}\mathbf{x}.\end{aligned}$$

Using property (iii) and Young's inequality, we continue with

$$\begin{aligned}\widetilde{K}[\mathbf{u}] &\leq \widetilde{K}[\mathbf{u}+\boldsymbol{\varphi}] + C(M+1)\{\delta(\rho)+\eta(\rho)+\varepsilon\}\int_{\operatorname{supp}(\varphi)} \left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\,\mathrm{d}\mathbf{x} \\ &\quad + C(M+1)\left(1+\frac{1}{\varepsilon^{\frac{s-r}{r}}}\right)\int_{\operatorname{supp}(\varphi)} \left(\|\mathbf{u}\|^s+\|\boldsymbol{\varphi}\|^s\right)\,\mathrm{d}\mathbf{x} + C(M+1)\rho^n\|\mathbf{u}(\mathbf{y})\|^s.\end{aligned}$$

Now Sobolev's inequality implies that

$$\begin{aligned}\widetilde{K}[\mathbf{u}] &\leq \widetilde{K}[\mathbf{u}+\boldsymbol{\varphi}] + C(M+1)\{\delta(\rho)+\eta(\rho)+\varepsilon\}\int_{\operatorname{supp}(\varphi)} \left(1+\|\nabla\boldsymbol{\varphi}\|^p+\|\nabla\mathbf{u}\|^p\right)\,\mathrm{d}\mathbf{x} \\ &\quad + C(M+1)\left(1+\frac{1}{\varepsilon^{\frac{s-r}{r}}}\right)\left(\int_{\operatorname{supp}(\varphi)} \|\nabla\boldsymbol{\varphi}\|^p\,\mathrm{d}\mathbf{x}\right)^{\frac{s}{p}} \\ &\quad + C(M+1)\left(1+\frac{1}{\varepsilon^{\frac{s-r}{r}}}\right)\left(\int_{\operatorname{supp}(\varphi)} \|\mathbf{u}\|^s\,\mathrm{d}\mathbf{x} + \rho^n\|\mathbf{u}(\mathbf{y})\|^s\right).\end{aligned}$$

Hence $\mathbf{u}$ is a $(\widetilde{K},\omega,\{\nu_\varepsilon,T_\varepsilon\})$-minimizer with $\omega(\rho) = C(M+1)\{\delta(\rho)+\eta(\rho)\}$, $\nu_\varepsilon = C(M+1)\left(1+\frac{1}{\varepsilon^{\frac{s-r}{r}}}\right)\|\mathbf{u}\|^s$, $T_\varepsilon = C(M+1)\left(1+\frac{1}{\varepsilon^{\frac{s-r}{r}}}\right)$ and $t = \frac{s}{p} > 1$. Moreover, by putting $\zeta := M(1+\|\mathbf{u}\|^r)$, we see that $\zeta \in L^{\frac{s}{r},\gamma_1}$ and

$$\begin{aligned}|\widetilde{g}(\mathbf{x},\mathbf{F}) - \widetilde{g}(\mathbf{y},\mathbf{F})| &\leq \delta(\|\mathbf{x}-\mathbf{y}\|)\left(1+\|\mathbf{F}\|^2\right)^{\frac{p}{2}} + \zeta(\mathbf{x})^{\frac{s}{r}} + \zeta(\mathbf{y})^{\frac{s}{r}} \\ &\quad + (\zeta(\mathbf{x})+\zeta(\mathbf{y}))\left(1+\|\mathbf{F}\|^2\right)^{\frac{p(\frac{s}{r}-1)}{\frac{s}{r}}}\end{aligned}$$

If $\gamma_1 \geq \kappa$, then for each $\varepsilon > 0$, we find that $\Sigma_\varepsilon \|\mathbf{u}\|^{\frac{s}{p}} + \sigma_\varepsilon \in L^{p,\kappa}(\Omega)$ and that $\nu_\varepsilon \in L^{1,\kappa}$, so Theorem 12 tells us that $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N\times n})$. Thus Poincaré's inequality implies that $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^N)$, and the Theorem is proved.

If, on the other hand, we find that $\gamma_1 < \kappa$, then $\Sigma_\varepsilon \|\mathbf{u}\|^{\frac{s}{p}} + \sigma_\varepsilon \in L^{p,\gamma_1}(\Omega)$ and $\nu_\varepsilon \in L^{1,\gamma_1}$, and by Theorem 12, we deduce that $\nabla\mathbf{u} \in L^{p,\gamma_1}(\Omega; \mathbb{R}^{N\times n})$, while Poincaré's inequality implies that $\mathbf{u} \in \mathscr{L}^{p^*,\gamma_1}(\Omega; \mathbb{R}^N)$. Since $\gamma_1 < n$ and Ω has no external cusps, there is an isomorphism between $L^{p^*,\gamma_1}(\Omega; \mathbb{R}^N)$ and $\mathscr{L}^{p^*,\gamma_1}(\Omega; \mathbb{R}^N)$ (see [13] or [16]) that we may use to argue that $\mathbf{u} \in L^{p^*,\gamma_1}(\Omega; \mathbb{R}^N)$. Putting $\gamma_2 := \gamma_1\left(1 + \frac{s}{p^*}\right)$, it follows that $\|\mathbf{u}\|^s \in L^{1,\gamma_2}(\Omega)$ and $\left[A\|\mathbf{u}\|^{\frac{s}{p}} + a\right], \left[B\|\mathbf{u}\|^{\frac{s}{p}} + b\right] \in L^{p,\gamma_2}(\Omega; \mathbb{R})$. Another application of Theorem 12, now implies $\nabla\mathbf{u} \in L^{p,\gamma_2}(\Omega; \mathbb{R}^{N\times n})$. We may repeat this argument. Each repetition shows that $\nabla\mathbf{u} \in L^{p,\gamma_j}(\Omega; \mathbb{R}^{N\times n})$, where $\gamma_j := n\left(1 - \left(\frac{s}{p^*}\right)^j\right)$, so long as $\gamma_j < \kappa$. Once $\gamma_j \geq \kappa$, a final application of Theorem 12 tells us that $\nabla\mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N\times n})$, and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^{N\times n})$ follows from Poincaré's inequality. □

We again point out that if $p + \kappa > n$, then we obtain uniform continuity of the almost minimizer $\mathbf{u}$.

Corollary 4 *Suppose that all of the hypotheses of Theorem 14 are satisfied with $p + \kappa > n$. Then $\mathbf{u} \in \mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\overline{\Omega}; \mathbb{R}^N)$.*

Example 4 Suppose that $p \leq n$ and let

(i) $y \in \mathcal{C}^0(\overline{\Omega}; (0,\infty))$;
(ii) $z : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfying, for some $L < +\infty$ and $0 < r < p^*\frac{p-2}{p}$,

$$|z(\mathbf{x},\mathbf{u})| \leq L\left(1 + \|\mathbf{u}\|^r\right),$$

for all $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{u} \in \mathbb{R}^N$

be given. Notice that we require no continuity for the function z. We claim that the function $g : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ given by

$$g(\mathbf{x},\mathbf{u},\mathbf{F}) := y(\mathbf{x})\|\mathbf{F}\|^p + z(\mathbf{x},\mathbf{u})\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p(p^*-r)}{2(p^*+r)}}$$

satisfies the hypotheses of Theorem 14, with $s = \frac{p^*+r}{2}$. Clearly, we have

$$|z(\mathbf{x},\mathbf{u})|^{\frac{1}{p}} \geq |g(\mathbf{x},\mathbf{u},\mathbf{0})|^{\frac{1}{p}} + \left\|\frac{\partial}{\partial\mathbf{F}}g(\mathbf{x},\mathbf{u},\mathbf{0})\right\|^{\frac{1}{p-1}},$$

so hypothesis (i) is satisfied. For hypothesis (ii), we may use $\sigma_\varepsilon(\mathbf{x}) := \frac{C}{\varepsilon^{\frac{p^*+r}{2pr}}}$, for all $\mathbf{x} \in \overline{\Omega}$, and $\Sigma_\varepsilon := \frac{C}{\varepsilon^{\frac{p^*+r}{2pr}}}$, and $a(\mathbf{x})$ and A may also be taken to be constants that depend only on $\sup_{\mathbf{x}\in\overline{\Omega}}\left\{\frac{1}{y(\mathbf{x})} + y(\mathbf{x})\right\}$, r and p. Thus, if $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ and $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^{N\times n})$ is a minimizer for the functional

$$\mathbf{u} \mapsto \int_\Omega \left\{y(\mathbf{x})\|\nabla\mathbf{u}(\mathbf{x})\|^p + z(\mathbf{x},\mathbf{u}(\mathbf{x}))\left(1 + \|\nabla\mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p(p^*-r)}{2(p^*+r)}}\right\} d\mathbf{x}$$

that satisfies $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,p}(\Omega; \mathbb{R}^N)$, then we may apply Theorem 14 and conclude that $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}(\Omega; \mathbb{R}^{N\times n})$. In particular, if $\kappa > n - p$, then $\mathbf{u} \in \mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\overline{\Omega}; \mathbb{R}^N)$.

Again, under some stronger hypotheses, we can obtain Lipschitz regularity of almost minimizers. More precisely, we have the following

Theorem 15 *Let $\Omega \subset \mathbb{R}^n$, with a $\mathcal{C}^{1,\beta}$ boundary, and $\overline{\mathbf{u}} \in C^{1,\beta}(\overline{\Omega};\mathbb{R}^N)$ be given. Suppose that the functions $\delta, \eta \in \mathcal{C}^{0,\beta}([0,\infty))$ are non-decreasing and satisfy $\delta(0) = \eta(0) = 0$. Let $0 \le s < p^*$ be given. Suppose that $g : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses the following properties:*

(i) *for each $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{u} \in \mathbb{R}^N$, we have $g(\mathbf{x},\mathbf{u},\cdot) \in \mathcal{C}^3(\mathbb{R}^{N\times n})$, and there exist $a, L \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and $b < +\infty$ such that*

$$\left\| \frac{\partial^3}{\partial \mathbf{F}^3} g(\mathbf{x},\mathbf{u},\mathbf{F}) \right\| \le L(\mathbf{u}) \left(a(\mathbf{u}) + \|\mathbf{F}\|^2\right)^{\frac{p-3}{2}}$$

and

$$b\left(1 + \|\mathbf{u}\|^{\frac{s}{p}}\right) \ge |g(\mathbf{x},\mathbf{u},\mathbf{0})|^{\frac{1}{p}} + \left\| \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x},\mathbf{u},\mathbf{0}) \right\|^{\frac{1}{p-1}} + \left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x},\mathbf{u},\mathbf{0}) \right\|$$

for every $\mathbf{x} \in \overline{\Omega}$, $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{F} \in \mathbb{R}^{N\times n}$;

(ii) *there is a family $\{f_\mathbf{x}\}_{\mathbf{x}\in\overline{\Omega}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a uniform p-Uhlenbeck structure such that for each $\varepsilon > 0$, there exist $\sigma_\varepsilon, < +\infty$ such that for every $\mathbf{x} \in \overline{\Omega}$*

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x},\mathbf{u},\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} f_\mathbf{x}(\mathbf{F}) \right\| < \frac{1}{2^{\frac{p}{2}}} \varepsilon \|\mathbf{F}\|^{p-2}$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon \left(1 + \|\mathbf{u}\|^{\frac{s}{p}}\right)$;

(iii) *there is an $r \in (0, s-\beta)$ such that for each $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, we have*

$$\begin{aligned} |g(\mathbf{x},\mathbf{u},\mathbf{F}) - g(\mathbf{y},\mathbf{v},\mathbf{F})| &\le \delta(\|\mathbf{x}-\mathbf{y}\|)\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} \\ &+ \delta(\|\mathbf{u}-\mathbf{v}\|)\left(1 + \|\mathbf{u}\|^r + \|\mathbf{v}\|^r\right)\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p(s-r)}{2s}}, \end{aligned}$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$.

If $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a (K,η)-minimizer for the functional

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W^{1,p}_0(\Omega;\mathbb{R}^N)$, then $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^N)$.

Proof First, we use Corollary 4, with $\kappa = n$, to see that $\mathbf{u} \in \mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^N)$ for every $0 < \alpha < 1$. Thus, Theorem 13 can be applied to complete the proof.

9 Application to an open problem

We now apply Theorem 12 to obtain a partial resolution to the open problem discussed in [21, Sect. 4.3]. We establish a continuity result for minimizers of convex functionals with integrands of the form $g(\mathbf{x},\mathbf{F})$ that have a radial structure at infinity and a continuous dependence on $\mathbf{x}$.

Definition 12 Given a measurable set $\mathcal{U} \subset \mathbb{R}^n$, we will say that a family of functions $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ is *uniformly asymptotically radial* if and only if there exists a family $\{\widetilde{f}_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \in \mathcal{C}^2([0,\infty))$ with the following property: for each $\varepsilon > 0$ there exists a $\tau_\varepsilon < +\infty$ such that for every $\mathbf{y} \in \mathcal{U}$, we have

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) - \frac{\partial^2}{\partial \mathbf{F}^2} \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) \right\| < \varepsilon \|\mathbf{F}\|^{p-2}, \tag{87}$$

whenever $\|\mathbf{F}\| > \tau_\varepsilon$.

We will use the following

Lemma 11 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a measurable set. Suppose that the family of functions $\{g_{\mathbf{y}}\} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ is uniformly asymptotically radial and possesses the following property: there exist $\Lambda_* > 0$ and $\Lambda^* < +\infty$ such that for each $\mathbf{y} \in \mathcal{U}$*

$$\Lambda_* \|\mathbf{F}\|^{p-2} \|\boldsymbol{\xi}\|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} g_{\mathbf{y}}(\mathbf{F}) :: \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq \Lambda^* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\boldsymbol{\xi}\|^2,$$

for every $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$. Then there is a family $\{h_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ with a uniform p-Uhlenbeck structure that is also uniformly asymptotically related to $\{g_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}}$.

Proof Put $R_* = 1 + \tau_{\frac{\Lambda_*}{2}}$. First, since $\{g_{\mathbf{y}}\}$ is uniformly asymptotically radial, we deduce from hypothesis (ii) that there is a $\{\widetilde{f}\}_{\mathbf{y}\in\mathcal{U}} \subset \mathcal{C}^2(\mathbb{R}^{N\times n})$ such that $\|\mathbf{F}\| \geq R_*$ implies

$$\frac{\Lambda_*}{2^{\frac{p}{2}}} \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\boldsymbol{\xi}\|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) :: \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq \frac{3\Lambda^*}{2} \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\boldsymbol{\xi}\|^2, \tag{88}$$

for every $\mathbf{y} \in \mathcal{U}$ and $\boldsymbol{\xi} \in \mathbb{R}^{N\times n}$. Since (87) continues to hold if to each $\widetilde{f}_{\mathbf{y}}$ we add a linear function, without loss of generality, we may assume $\widetilde{f}_{\mathbf{y}}(0) = \widetilde{f}'_{\mathbf{y}}(0) = 0$ for all $\mathbf{y} \in \mathcal{U}$. It is straightforward to verify that there is a constant $C > 0$, depending only on n, N and p, such that for every $\mathbf{y} \in \mathcal{U}$ and $\|\mathbf{F}\| \geq R_*$ the following hold:

$$\left\| \frac{\partial^2}{\partial \mathbf{F}^2} \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) \right\| \leq C\Lambda^* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}};$$

$$\left\| \frac{\partial}{\partial \mathbf{F}} \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) \right\| \leq C\Lambda^* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}} \|\mathbf{F}\|;$$

and

$$\frac{\Lambda_*}{C} \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} \leq \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) \leq C\Lambda^* \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}}. \tag{89}$$

Thus for each $\mathbf{y} \in \mathcal{U}$, we see that $\mathbf{F} \mapsto \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2)$ has a p-Uhlenbeck structure outside the ball $\mathcal{B}_{R_*} \subset \mathbb{R}^{N\times n}$. We proceed to extend this function to all of $\mathbb{R}^{N\times n}$ while preserving the p-Uhlenbeck structure.

We make a couple of preliminary observations based on (88). Since

$$\frac{\partial^2}{\partial \mathbf{F}^2} \widetilde{f}_{\mathbf{y}}(\|\mathbf{F}\|^2) :: \boldsymbol{\xi} \otimes \boldsymbol{\xi} = 4\widetilde{f}''_{\mathbf{y}}(\|\mathbf{F}\|^2)(\mathbf{F} : \boldsymbol{\xi})^2 + 2\widetilde{f}'_{\mathbf{y}}(\|\mathbf{F}\|^2)\|\boldsymbol{\xi}\|^2,$$

by selecting $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$ so that $\|\mathbf{F}\| = R_*$, $\boldsymbol{\xi} \neq \mathbf{0}$ and $\boldsymbol{\xi} : \mathbf{F} = 0$, we deduce that

$$0 < \frac{\Lambda_*}{2^{\frac{p+2}{2}}} \left(1 + R_*^2\right)^{\frac{p-2}{2}} \leq \widetilde{f}'_{\mathbf{y}}(R_*^2) \leq \frac{3\Lambda^*}{2} \left(1 + R_*^2\right)^{\frac{p-2}{2}}. \tag{90}$$

Next, using $\mathbf{F} \in \mathbb{R}^{N\times n}$ satisfying $\|\mathbf{F}\| = R_*$ and $\boldsymbol{\xi} = \mathbf{F}$ in (88), we also find that

$$\frac{\Lambda_*\left(1+R_*^2\right)^{\frac{p-2}{2}}}{2^{\frac{p+2}{2}} f_{\mathbf{y}}(R_*^2)} \leq 2\frac{\widetilde{f}_{\mathbf{y}}''(R_*^2)R_*^2}{\widetilde{f}_{\mathbf{y}}'(R_*^2)} + 1 \leq \frac{3\Lambda^*\left(1+R_*^2\right)^{\frac{p-2}{2}}}{4 f_{\mathbf{y}}(R_*^2)}$$
$$\Rightarrow \quad \frac{1}{6}\cdot\frac{1}{2^{\frac{p}{2}}}\cdot\frac{\Lambda_*}{\Lambda^*} - \frac{1}{2} \leq \frac{\widetilde{f}_{\mathbf{y}}''(R_*^2)R_*^2}{\widetilde{f}_{\mathbf{y}}'(R_*^2)} \leq \frac{3}{2}\cdot\frac{1}{2^{\frac{p}{2}}}\cdot\frac{\Lambda^*}{\Lambda_*} - \frac{1}{2}. \tag{91}$$

We now define an appropriate extension of each $\widetilde{f}_{\mathbf{y}}$. Choose $\mu > 0$ so that

$$\left(\frac{\mu + R_*^2}{R_*^2}\right)\left(\frac{1}{6}\cdot\frac{1}{2^{\frac{p}{2}}}\cdot\frac{\Lambda_*}{\Lambda^*} - \frac{1}{2}\right) > \frac{1}{2}. \tag{92}$$

For each $\mathbf{y} \in \mathcal{U}$, define

$$\alpha_{\mathbf{y}} := \frac{\widetilde{f}_{\mathbf{y}}''(R_*^2)\left(\mu + R_*^2\right)}{\widetilde{f}_{\mathbf{y}}'(R_*^2)} \quad \text{and} \quad \beta_{\mathbf{y}} := \frac{\widetilde{f}_{\mathbf{y}}'(R_*^2)\left(\mu + R_*^2\right)^{(1-\alpha_{\mathbf{y}})}}{(1+\alpha_{\mathbf{y}})} - \widetilde{f}_{\mathbf{y}}(R_*^2).$$

Notice that (89), (90), (91) and (92) imply that $\{\alpha_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{U}}$ is contained in a compact subset of $(-\frac{1}{2}, +\infty)$ and that $\beta_{\mathbf{y}}$ is uniformly bounded in $\mathcal{U}$. Define the function $h_{\mathbf{y}} : [0, +\infty) \to [0, +\infty)$ by

$$h_{\mathbf{y}}(t) := \begin{cases} \dfrac{\widetilde{f}_{\mathbf{y}}'(R_*^2)}{\left(1+\alpha_{\mathbf{y}}\right)\left(\mu + R_*^2\right)^{\alpha_{\mathbf{y}}}}\,(\mu + t)^{(1+\alpha_{\mathbf{y}})}, & t < R_*^2 \\ \beta_{\mathbf{y}} + \widetilde{f}_{\mathbf{y}}(t), & t \geq R_*^2. \end{cases}$$

The family $\left\{\mathbf{F} \mapsto h_{\mathbf{y}}(\|\mathbf{F}\|^2)\right\}_{\mathbf{y}\in\mathcal{U}}$ has a uniform p-Uhlenbeck structure. That this family is uniformly asymptotically related to the family $\left\{g_{\mathbf{y}}\right\}_{\mathbf{y}\in\mathcal{U}}$ clearly follows from the hypothesis on $\left\{\widetilde{f}_{\mathbf{y}}\right\}_{\mathbf{y}\in\mathcal{U}}$. □

With Lemma 11 and Theorem 12, we obtain

Theorem 16 *Let $\Omega \subset \mathbb{R}^n$, with a $\mathcal{C}^{1,0}$ boundary, and $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$, for some $\kappa \geq 0$ satisfying $p + \kappa > n$, be given. Suppose that $g : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses the following properties:*

(i) *for each $\mathbf{x} \in \overline{\Omega}$ and each $\mathbf{F} \in \mathbb{R}^{N\times n}$, we have that $g(\mathbf{x}, \cdot) \in \mathcal{C}^2(\mathbb{R}^{N\times n})$ and $g(\cdot, \mathbf{F}) \in \mathcal{C}(\overline{\Omega})$;*

(ii) *there is a $\delta \in \mathcal{C}([0, +\infty))$ that is non-decreasing and satisfies $\delta(0) = 0$ such that for each $\mathbf{x}, \mathbf{y} \in \Omega$, we have*

$$|g(\mathbf{x}, \mathbf{F}) - g(\mathbf{y}, \mathbf{F})| \leq \delta(\|\mathbf{x} - \mathbf{y}\|)\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}},$$

for every $\mathbf{F} \in \mathbb{R}^{N\times n}$;

(iii) *there exist $\Lambda_* > 0$ and $\Lambda^* < +\infty$ such that for each $\mathbf{x} \in \overline{\Omega}$, we have*

$$\Lambda_*\|\mathbf{F}\|^{p-2}\|\boldsymbol{\xi}\|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} g(\mathbf{x}, \mathbf{F}) :: \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq \Lambda^*\left(1 + \|\mathbf{F}\|^2\right)^{\frac{p-2}{2}}\|\boldsymbol{\xi}\|^2,$$

for every $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$;

(iv) *the family $\{g(\mathbf{x}, \cdot)\}_{\mathbf{x}\in\overline{\Omega}}$ is uniformly asymptotically radial.*

If $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *is a* $(K, 0)$*-minimizer for the functional*

$$K[\mathbf{u}] := \int_{\Omega} g(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}$$

satisfying $[\mathbf{u} - \overline{\mathbf{u}}] \in W_0^{1,1}(\Omega; \mathbb{R}^N)$*, then* $\mathbf{u} \in \mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\overline{\Omega}; \mathbb{R}^N) \cap \mathcal{C}^{0,\beta}(\Omega; \mathbb{R}^N)$*, for each* $\beta \in (0, 1)$.

Under the theorem's hypotheses, we actually find $\nabla \mathbf{u} \in L^{p,\kappa}(\Omega; \mathbb{R}^{N \times n})$. We conclude by mentioning that there is a local version of this theorem for local minimizers of the functional K.

Acknowledgments I wish to thank Paolo Marcellini for his interest in this work. I also wish to thank Emilio Acerbi and Giuseppe Mingione for sharing their time and expertise during my visit to Università di Parma. I sincerely appreciate the many useful suggestions provided by an anonymous referee.

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