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Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications

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Abstract. We prove existence and uniqueness of renormalized solutions of some transport equations with a vector field that is not $W^{1,1}$ with respect to all variables but is of a particular form. Two specific applications of this new result are then treated, based upon the equivalence between transport equations and ordinary differential equations. The first one consists of a result about the dependance upon initial conditions for solutions of ODEs. The second one is related to some stochastic differential equations arising in the modelling of polymeric fluid flows.

1. Introduction and motivation

Our purpose in this article is to show some slight extension, together with some new applications, of the theory of renormalized solutions of the linear transport equations introduced by Di Perna and the second author in [1,2]. This extension aims to consider cases when some coordinates b_i of the vector field b appearing in the transport equation,

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0 \text{ in } (0, \infty) \times \mathbb{R}^N, \tag{1.1}$$

are not $W^{1,1}$ with respect to some space variables x_j. In that case, the Di Perna–Lions theory does not apply straightforwardly. For our result to apply, such coordinates b_i should of course not be of any form, but have to be of some specific form, that we shall make precise below. Typically, we have in mind the situations when

$$b(x_1, x_2) = (b_1(x_1), b_2(x_1, x_2)) \text{ with } b_1 \in W^{1,1}_{x_1}, \text{ and } b_2 \in L^1_{x_1}\left(W^{1,1}_{x_2}\right), \tag{1.2}$$

together with more technical assumptions that will be detailed when needed. But before turning to the heart of the matter, let us say a word on the applications that require such results.

Our motivation stems from two specific applications that we want to briefly describe now.

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The first application aims at raising the state of the art of the theory of solutions of ordinary differential equations with coefficients in Sobolev spaces to the level of that of the Cauchy–Lipschitz theory for those with regular coefficients. We consider the ordinary differential equation

$$\dot{Y}(t, y) = c\,(t, Y(t, y)), \tag{1.3}$$

complemented with the initial condition $Y(0, y) = y$. For simplicity in this introductory section, the space variable y is assumed to be one-dimensional. It is well known in the Cauchy–Lipschitz theory that, once the existence and uniqueness of a solution of (1.3) are proved, which essentially requires the Lipschitz regularity (with respect to y) of the field $c(t, y)$ appearing in (1.3), then, under the additional assumption that the first derivative of c with respect to the space variable y exists and is continuous, one may prove that the solution $Y(t, y)$ to (1.3) is differentiable with respect to its initial datum y, and that its first derivative satifies the linearized ODE

$$\frac{\partial}{\partial t}\frac{\partial Y}{\partial y}(t, y) = \frac{\partial c}{\partial Y}(t, Y(t, y))\,\frac{\partial Y}{\partial y}. \tag{1.4}$$

The Cauchy problem for this ODE is in turn well posed in view of the Cauchy–Lipschitz theorem. On the other hand, the theory of renormalized solutions of linear transport equations (1.1) has been allowed to obtain (see [2]) existence and uniqueness results for the ODEs of type (1.3) with only $W^{1,1}$ coefficients c (with a bounded divergence, say). The observation that allows us to do so is the fact that a generalized almost everywhere flow $Y(t, y)$ associated to (1.3) is indeed related in a unique way to a renormalized group solution $u_0(Y(t, y))$ to a transport equation of the type (1.1), u_0 being the initial condition complementing (1.1) (see [4]). It has already been pointed out in [2, Remark 2, page 536], that it would be interesting to give more than a formal sense to equation (1.4) in this non-regular setting. We shall see in Section 4 that it is indeed possible. Exactly like in the framework of the Cauchy–Lipschitz theory, the number of derivatives that is allowed on Y is equal to that allowed on c. Of course, all this is loosely stated, and will be made precise in Section 4. It is sufficient to say for the time being that the pair $\left(Y(t, y), \dfrac{\partial Y}{\partial y}(t, y)\right)$ is a solution of the system (1.3)–(1.4), which is indeed of the form

$$\begin{cases} \dot{X}_1(t, x) = b_1(t, X_1(t, x)), \\ \dot{X}_2(t, x) = b_2(t, X_1(t, x), X_2(t, x)), \end{cases} \tag{1.5}$$

with $b_1(\cdot) = c\,(\cdot)$ and $b_2(x_1, x_2) = \dfrac{\partial c}{\partial x}(t, x_1)x_2$ satisfying (1.2) as soon as $c \in W^{1,1}$.

The second situation we want to address is close to numerical simulations. It is related to the so-called micro-macro simulations of polymeric fluids. For a pedagogic introduction to these models along with a comprehensive list of relevant references, we refer the reader to [3]. We shall only briefly indicate here the main lines of the modelling.

Micro-macro simulations of non-newtonian flows are now commonly used in computational fluid dynamics. The approach consists of coupling the macroscopic conservation equations for the velocity of the fluid c together with a kinetic theory model – a Fokker–Planck equation – modelling the evolution of the microstructure of the fluid, instead of using a constitutive equation to evaluate the non-newtonian contribution to the stress. Next, this kinetic model can be computed numerically by solving the stochastic differential equation underlying the Fokker–Planck equation. Let us consider in this introduction only the simplest case when the microstructure of the fluid is described by the stochastic evolution of a single Hookean dumbell: the polymeric chain has only one length and the force it experiences is purely entropic. In that case, the equations under consideration are the classic macroscopic continuity equations for an incompressible fluid

$$\begin{cases} \dfrac{\partial c}{\partial t} - \Delta_y c + \nabla_y p = \operatorname{div}_y \tau \\ \operatorname{div}_y c = 0 \end{cases} \tag{1.6}$$

set for $t > 0$, $y \in \mathbb{R}^N$, together with

$$\tau(y) = E(R_t(y) \otimes R_t(y)), \tag{1.7}$$

and

$$dR_t(y) + c \cdot \nabla_y R_t(y)\, dt = (\nabla c \cdot R_t(y) - R_t(y))\, dt + dW_t, \tag{1.8}$$

where $R_t(y)$ denotes the N-dimensional stochastic process defining the evolution of the microstructure at the macroscopic point y (E is the expectation value, W_t is a N-dimensional Wiener process).

Giving a sense to (1.8), which formally is a stochastic partial differential equation, is not straightforward. When c is sufficiently regular, one can consider the Lagrangian form of this equation and treat the stochastic process *along the characteristics*. This amounts to setting

$$\widetilde{R}_t(y) = R_t(Y(t)), \tag{1.9}$$

where $Y(t)$ denotes the flow of c, and then rewriting (1.8) as

$$d\widetilde{R}_t = (\nabla c \cdot \widetilde{R}_t - \widetilde{R}_t)\, dt + dW_t. \tag{1.10}$$

One then ends up with a stochastic differential equation (not a *partial* one), that has a proper mathematical meaning when c is sufficiently regular (and bounded). On the contrary, when c is only a weak solution of the macroscopic equation (1.6), the previous argument has to be adapted. Formally, one may consider the almost everywhere flow $Y(t)$ associated to c in the sense of Di Perna–Lions, and then give a sense to (1.10), at least almost everywhere in y. We shall attack the problem in a slightly different way, by treating it globally in the variables (y, r). As such, the existence of an almost everywhere flow in (y, r) will be a corollary of our general result for transport equations of the form (1.1)–(1.2). One may measure the difficulty simply by considering the solution of this question in the fully deterministic case (which embodies, in fact, most of the difficulties of the stochastic one).

Up to a change of function (replace R by $e^t R$), the question is to give a rigorous mathematical sense to

$$\frac{dR}{dt} + c \cdot \nabla_y R - \nabla c\, R = 0, \tag{1.11}$$

which is the Eulerian form of the Lagrangian dynamics

$$\begin{cases} \dot{Y}(t) & = c\,(Y(t)), \\ \dot{R}(t) & = \nabla_y c\,(Y(t))\, R(t), \\ Y(0) = y\,, & R(0) = r. \end{cases} \tag{1.12}$$

It is straightforward to see that in (1.12), we are back again to vector fields of the form (1.2). The stochastic character of (1.8) only is a slight additional difficulty, that turns system (1.12) into

$$\begin{cases} \dot{Y}(t) & = c\,(Y(t)), \\ dR_t & = \nabla_y c\,(Y(t))\, R_t\, dt + dW_t, \\ Y(0) = y\,, & R_{t=0} = R_0, \end{cases} \tag{1.13}$$

and that will be easily treated in a second step. Section 5 will be devoted to the treatment of this second application.

It is now time to outline our mathematical strategy for providing a mathematical foundation to these types of equations.

Our main task consists of proving an existence and uniqueness theorem for the solution of (1.1) when the vector field is of the form (1.2). This is the purpose of Theorem 2.1 of Section 2. In fact, the uniqueness part is the crucial step (Lemma 2.2) and the proof of it is a consequence of a regularization lemma (Lemma 2.1). From this result we deduce, in the same manner as in [2] an existence, uniqueness and stability result, contained in Theorem 3.1, for renormalized solutions of (1.1). This second result will not be as detailed as the first one as it is really close to the standard case developed in [2].

In Section 4, we turn to the first application mentioned above. We first recall the link between solutions of transport equations and solutions of ordinary differential equations, and next explain how our previous results of Section 2 and Section 3 allow us to prove a result (Theorem 4.1) on perturbations on initial conditions for solutions of the ODE.

Finally, Section 5 is devoted to our second application. We begin by considering in Subsection 5.1 a given trajectory of the stochastic differential equation, which allows us to define the stochastic flow in Subsection 5.2. Next, in Subsections 5.3 and 5.4, we examine the Fokker–Planck equation associated to the SDE. Finally in Subsection 5.5 we take an alternative viewpoint provided by the stochastic transport equation associated to the SDE.

2. Main result: Uniqueness of solutions of some transport equations

This section is devoted to the statement and proof of our main existence and uniqueness result for the solutions of (1.1). All other results will be drawn from this result.

Let us now make precise the mathematical setting and introduce some notation.

As announced in the introduction, we consider the linear transport equation (1.1),

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0 \text{ in } (0, \infty) \times \mathbb{R}^N.$$

The space variable x is partitioned into $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{N_1}, x_2 \in \mathbb{R}^{N_2}, N_1 + N_2 = N$. Accordingly, the vector field b is written $b = (b_1, b_2)$ with $b_i : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_i}$, and the differential operators gradient and divergence are decomposed along $\nabla = (\nabla_{x_1}, \nabla_{x_2})$, $\operatorname{div}_x = \operatorname{div}_{x_1} + \operatorname{div}_{x_2}$.

We make the following assumptions on the vector field :

$$\begin{aligned}
&(H1) && b_1 = b_1(x_1) \in W^{1,1}_{x_1,loc} \ \textbf{(it does not depend on } x_2\textbf{)}\\
&(H2) && \frac{b_1}{1+|x_1|} \in L^1_{x_1}(\mathbb{R}^{N_1}) + L^\infty_{x_1}(\mathbb{R}^{N_1}),\\
&(H3) && \operatorname{div}_{x_1} b_1 = 0,\\
&(H4) && b_2 = b_2(x_1, x_2) \in L^1_{x_1,loc}\big(\mathbb{R}^{N_1},\ W^{1,1}_{x_2,loc}\big),\\
&(H5) && \frac{b_2}{1+|x_2|} \in L^1_{x_1,loc}\big(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2}) + L^\infty_{x_2}(\mathbb{R}^{N_2})\big),\\
&(H6) && \operatorname{div}_{x_2} b_2 = 0.
\end{aligned}$$

In view of (H1), we rewrite equation (1.1) in a particular form :

$$\frac{\partial u}{\partial t} + b_1(x_1) \cdot \nabla_{x_1} u + b_2(x_1, x_2) \cdot \nabla_{x_2} u = 0 \text{ in } (0, \infty) \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \quad (2.1)$$

Assumptions (H1) to (H6) deserve some comments. The crucial assumption is (H1), which justifies our study. It allows us to treat the variables x_1 and x_2 in a different way. Assumptions (H3) to (H6) must be compared to the standard situation which is dealt with in [2]. We need here to control *both* the operators div_{x_1} and div_{x_2} and *not only* their sum div_x. Once (H1), (H3) and (H6) are satisfied, this allows one to avoid any assumption of $W^{1,1}$ regularity of b_2 with respect to x_1. Note indeed that in (H4), only the $L^1_{x_1}$ regularity is required! It should be compared to the standard case of [2] when $b_2 \in W^{1,1}_{x_1,x_2}$ is required.

Some other comments about possible generalizations are in order.

Remark 2.1. For the sake of simplicity, we have chosen to present our results when the vector field b is assumed *not* to depend on the time variable, although our results also hold *mutatis mutandis* in the time-dependent case $b = b\,(t, x)$ when we allow an L^1 dependence with respect to time. Then, assumption (H1) needs to be replaced by

$$(H1') \qquad b_1 = b_1(t, x_1) \in L^1\big([0,T],\ W^{1,1}_{x_1,loc}\big),$$

while (H3) and (H6) hold for almost any time t, and (H2), (H4) and (H5) are replaced by

$$(H2') \qquad \frac{b_1}{1+|x_1|} \in L^1\Big([0,T], \big(L^1_{x_1}(\mathbb{R}^{N_1}) + L^\infty_{x_1}(\mathbb{R}^{N_1})\big)\Big),$$

$$(H4') \qquad b_2 = b_2(t, x_1, x_2) \in L^1\Big([0,T],\ L^1_{x_1,loc}\big(\mathbb{R}^{N_1},\ W^{1,1}_{x_2,loc}\big)\Big),$$

$$(H5') \qquad \frac{b_2}{1+|x_2|} \in L^1\Big([0,T],\ L^1_{x_1,loc}\big(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2}) + L^\infty_{x_2}(\mathbb{R}^{N_2})\big)\Big).$$

Remark 2.2. Similarly, we have chosen to present the case of divergence-free velocities. Again, our results admit straightforward extensions to situations when the divergence is controlled in the L^∞ norm

$$(H3'') \qquad \mathrm{div}_{x_1}\, b_1 \in L^\infty_{x_1}(\mathbb{R}^{N_1}),$$

$$(H6'') \qquad \mathrm{div}_{x_2}\, b_2 \in L^\infty_{x}(\mathbb{R}^{N}).$$

Remark 2.3. Of course, the above two remarks may be combined with one another to allow the hypotheses

$$(H3''') \qquad \mathrm{div}_{x_1}\, b_1 \in L^1\big([0,T],\ L^\infty_{x_1}(\mathbb{R}^{N_1})\big),$$

$$(H6''') \qquad \mathrm{div}_{x_2}\, b_2 \in L^1\big([0,T], L^\infty_{x}(\mathbb{R}^{N})\big).$$

We are now in a position to state our main result:

Theorem 2.1. *We assume* (H1) *to* (H6). *Let*

$$u_0 \in \big(L^1 \cap L^\infty(\mathbb{R}^N)\big) \cap L^\infty_{x_1}\big(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})\big). \tag{2.2}$$

Then there exists one and only one solution,

$$\begin{aligned} u(t,x) \in\ & L^\infty\big([0,T],\ L^1_x \cap L^\infty_x(\mathbb{R}^N)\big) \\ & \cap L^\infty\Big([0,T],\ L^\infty_{x_1}\big(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})\big)\Big), \end{aligned} \tag{2.3}$$

to (2.1) *satisfying the initial condition* $u(t=0,\cdot) = u_0$.

The proof of this theorem is divided into three steps. The uniqueness being the central issue, we begin by proving it. It is the consequence of the following two lemmas, the first one dealing with regularization, the second one stating the uniqueness *per se*. Finally, we show the existence part.

Lemma 2.1. *We assume* (H1)–(H4) *(or, in the time-dependent case,* (H1′)–(H4′)*). Let $f \in L^\infty([0,T],\ L^1_x \cap L^\infty_x(\mathbb{R}^N))$ be a solution of* (2.1)*. Let ρ_{α_1} and ρ_{α_2} be two regularization kernels, respectively, in the variable x_1 and x_2 ($\rho_{\alpha_i} = \frac{1}{{\alpha_i}^{N_i}}\rho_i(\frac{\cdot}{\alpha_i})$, $\rho_i \in \mathcal{D}_+(\mathbb{R}^{N_i})$, $\int_{\mathbb{R}^{N_i}} \rho_i = 1$, for $i = 1, 2$). Then $f_{\alpha_1,\alpha_2} = (f * \rho_{\alpha_1}) * \rho_{\alpha_2}$ is a smooth (in x) solution of*

$$\frac{\partial f_{\alpha_1,\alpha_2}}{\partial t} + b \cdot \nabla f_{\alpha_1,\alpha_2} = \epsilon_{\alpha_1,\alpha_2}, \tag{2.4}$$

with

$$\lim_{\alpha_2 \longrightarrow 0}\ \lim_{\alpha_1 \longrightarrow 0} \epsilon_{\alpha_1,\alpha_2} = 0 \ \text{ in } L^\infty\big([0,T],\ L^1_{x,loc} \cap L^\infty_{x,loc}(\mathbb{R}^N)\big). \tag{2.5}$$

Lemma 2.2. *We assume* (H1) *to* (H6) *(or the analogous hypotheses in the other cases). Let $f(t,x) \in L^\infty([0,T],\ L^1_x \cap L^\infty_x(\mathbf{R}^N)) \cap L^\infty([0,T], L^\infty_{x_1}(\mathbb{R}^{N_1}, L^1_{x_2}(\mathbb{R}^{N_2})))$ be a* non-negative *solution of* (2.1) *with the vanishing initial value $f_0 = 0$. Then $f = 0$ for all times.*

Remark 2.4. Note that we need the additional assumption that $f \in L^\infty([0,T],$ $L^\infty_{x_1}(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})))$ in comparison with the standard case when only $f(t,x) \in L^\infty([0,T],\ L^1_x \cap L^\infty_x(\mathbf{R}^N))$ is needed. As it will be seen later on, this regularity is indeed propagated by the equation itself as soon as the initial condition satisfies the second part of (2.2).

We now prove successively these two lemmas, and next complete the proof of Theorem 2.1.

Proof of Lemma 2.1. We make the proof under the time-independent assumptions (H1)–(H4). All the functional spaces used here are *local*, this is clearly enough for such a regularization result. However, in order to lighten the notations, we skip the subscript *loc*.

We first regularize in the x_2 variable by convoluting (2.1) with ρ_{α_2}

$$\frac{\partial(f * \rho_{\alpha_2})}{\partial t} + b_1 \cdot \nabla_{x_1}(f * \rho_{\alpha_2}) + (b_2 \cdot \nabla_{x_2} f) * \rho_{\alpha_2} = 0,$$

using the fact that, in view of (H1), b_1 does not depend on x_2. Denoting by

$$[b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}]\,(f) = b_2 \cdot \nabla_{x_2}(f * \rho_{\alpha_2}) - \rho_{\alpha_2} * (b_2 \cdot \nabla_{x_2} f), \tag{2.6}$$

this can be written

$$\frac{\partial(f * \rho_{\alpha_2})}{\partial t} + b_1 \cdot \nabla_{x_1}(f * \rho_{\alpha_2}) + b_2 \cdot \nabla_{x_2}(f * \rho_{\alpha_2}) = [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}]\,(f).$$

It is a standard fact (see [2]) that

$$\varepsilon_{\alpha_2} = [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}]\,(f) \overset{\alpha_2 \longrightarrow 0}{\longrightarrow} 0 \text{ in } L^1_x. \tag{2.7}$$

Indeed, it is clear for smooth b_2 and f, while, as in [2], the general case follows by density through the estimate

$$\|[b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}](f)\|_{L^1_x} \leq C\,\|b_2\|_{L^1_{x_1}\left(W^{1,1}_{x_2}\right)}\,\|f\|_{L^\infty_{x_1,x_2}}, \tag{2.8}$$

which can in turn be obtained by integrating in x_1 the standard estimate

$$\|[b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}](f)\|_{L^1_{x_2}} \leq C\,\|b_2\|_{W^{1,1}_{x_2}}\,\|f\|_{L^\infty_{x_2}}. \tag{2.9}$$

Of course, this is here where (H4) plays a role. At this stage, we have obtained for $f_{\alpha_2} = f * \rho_{\alpha_2}$,

$$\frac{\partial f_{\alpha_2}}{\partial t} + b_1 \cdot \nabla_{x_1} f_{\alpha_2} + b_2 \cdot \nabla_{x_2} f_{\alpha_2} = \varepsilon_{\alpha_2}, \tag{2.10}$$

with ε_{α_2} satisfying (2.7).

Next, we regularize in the x_1 variable by convoluting (2.10) with ρ_{α_1}

$$\begin{aligned}\frac{\partial(f_{\alpha_2} * \rho_{\alpha_1})}{\partial t} &+ b_1 \cdot \nabla_{x_1}(f_{\alpha_2} * \rho_{\alpha_1}) + b_2 \cdot \nabla_{x_2}(f_{\alpha_2} * \rho_{\alpha_1}) = \\ &[b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](f_{\alpha_2}) + [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_1}](f_{\alpha_2}) + \varepsilon_{\alpha_2} * \rho_{\alpha_1}.\end{aligned} \tag{2.11}$$

This is exactly (2.4) with

$$\begin{aligned}\epsilon_{\alpha_1,\alpha_2} = {} & [b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](f_{\alpha_2}) + [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_1}](f_{\alpha_2}) \\ & + ([b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}](f)) * \rho_{\alpha_1}.\end{aligned} \tag{2.12}$$

We now successively treat each of the three terms of $\epsilon_{\alpha_1,\alpha_2}$ in order to show the convergence (2.5) claimed in the lemma.

The first term is the standard error term for the regularization in the variable x_1 of the function $f_{\alpha_2} \in L^\infty_{x_1,x_2}$. For almost each x_2, it is controlled in the following way:

$$\|[b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](f_{\alpha_2})\|_{L^1_{x_1}} \leq C\,\|b_1\|_{W^{1,1}_{x_1}}\,\|f_{\alpha_2}\|_{L^\infty_{x_1}},$$

and thus, as b_1 does not depend on x_2,

$$\|[b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](f_{\alpha_2})\|_{L^\infty_{x_2}(L^1_{x_1})} \leq C\,\|b_1\|_{W^{1,1}_{x_1}}\,\|f_{\alpha_2}\|_{L^\infty_{x_1,x_2}},$$

where on the right-hand side, $\|f_{\alpha_2}\|_{L^\infty_{x_1,x_2}}$ may be replaced by $\|f\|_{L^\infty_{x_1,x_2}}$ itself in view of the regularity assumed on f. Considering the above estimate and approximating b_1 and f_{α_2} by density, we therefore obtain, for the first term of (2.12),

$$\lim_{\alpha_1 \longrightarrow 0} [b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](f_{\alpha_2}) = 0, \tag{2.13}$$

in $L^1_{x_1,x_2}$, when α_2 is fixed. Let us now turn to the second term of (2.12). We have

$$\begin{aligned}[b_2 \cdot \nabla_{x_2}, \rho_{\alpha_1}](f_{\alpha_2}) &= b_2 \cdot \nabla_{x_2}(\rho_{\alpha_1} * f_{\alpha_2}) - \rho_{\alpha_1} * (b_2 \cdot \nabla_{x_2} f_{\alpha_2}) \\ &= b_2 \cdot ((\nabla_{x_2} f_{\alpha_2}) * \rho_{\alpha_1}) - \rho_{\alpha_1} * (b_2 \cdot \nabla_{x_2} f_{\alpha_2}) \\ &= [b_2, \rho_{\alpha_1}](\nabla_{x_2} f_{\alpha_2}).\end{aligned} \tag{2.14}$$

The latter bracket may be controlled as follows:

$$\|[b_2, \rho_{\alpha_1}](\nabla_{x_2} f_{\alpha_2})\|_{L^1_{x_1,x_2}} \leq C\|b_2\|_{L^1_{x_1,x_2}} \|\nabla_{x_2} f_{\alpha_2}\|_{L^\infty_{x_1,x_2}},$$

where it should be noted that no norm of derivatives of b_2 with respect to x_1 is needed, and where the last norm depends on α_2. Arguing by density as above, we therefore have

$$\lim_{\alpha_1 \longrightarrow 0} [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_1}]\,(f_{\alpha_2}) = 0, \tag{2.15}$$

in $L^1_{x_1,x_2}$, as α_1 goes to zero and α_2 is kept fixed.

There remains to treat the third term of (2.12) which is the easiest one. Indeed, α_2 being fixed, it is clear that

$$\lim_{\alpha_1 \longrightarrow 0} \varepsilon_2 * \rho_{\alpha_1} = \varepsilon_2, \tag{2.16}$$

in L^1.

We are now able to complete our argument. We fix α_2. In view of the convergences (2.13) and (2.15), the first two terms of (2.12) go to zero in $L^1_{x_1,x_2}$, while the third one behaves according to (2.16). It follows that

$$\lim_{\alpha_1 \longrightarrow 0} \varepsilon_{12} = \varepsilon_2$$

in L^1, and finally we let α_2 go to zero and use (2.7) to get the convergence (2.5). □

Proof of Lemma 2.2. Let f be a non-negative solution as claimed in the lemma. We introduce two cut-off functions, respectively, with respect to each variable x_1 and x_2. For $m, n \in \mathbf{N}$, we denote them by $\varphi_m = \varphi(\frac{x_1}{m})$ and $\psi_n(x_2) = \psi(\frac{x_2}{n})$, where $\varphi \in \mathcal{D}(\mathbb{R}^{N_1})$, $\varphi \geq 0$, $\varphi \equiv 1$ for $|x_1| \leq 1$ and $\varphi \equiv 0$ for $|x_1| \geq 2$. The analogous properties are required on ψ with respect to the variable x_2. We first multiply

$$\frac{\partial f}{\partial t} + b_1 \cdot \nabla_{x_1} f + b_2 \cdot \nabla_{x_2} f = 0$$

by ψ_n and integrate over the x_2 space to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^{N_2}} f\, \psi_n \, dx_2 + b_1 \cdot \nabla_{x_1} \int_{\mathbb{R}^{N_2}} f\, \psi_n \, dx_2 & \\ + \int_{\mathbb{R}^{N_2}} (b_2 \cdot \nabla_{x_2} f)\; \psi_n \, dx_2 = 0. & \end{aligned} \tag{2.17}$$

We treat the last term as follows:

$$\begin{aligned} \int_{\mathbf{R}^{N_2}} (b_2 \cdot \nabla_{x_2} f)\; \psi_n \, dx_2 = & - \int_{\mathbf{R}^{N_2}} f\, \psi_n \, (\operatorname{div}_{x_2} b_2)\; \psi_n \, dx_2 \\ & - \int_{\mathbf{R}^{N_2}} f\, b_2 \cdot \nabla_{x_2} \psi_n \, dx_2 \\ = & - \int_{\mathbf{R}^{N_2}} f\, \frac{1+|x_2|}{n}\, \frac{b_2}{1+|x_2|} \cdot (\nabla_{x_2} \psi)\Big(\frac{x_2}{n}\Big)\, dx_2, \end{aligned} \tag{2.18}$$

where we have used the fact that b_2 is divergence free (the modifications in the other cases are rather straightforward and will be skipped here).

Let us now multiply (2.17) by ψ_m and integrate over x_1:

$$\frac{d}{dt}\int_{\mathbf{R}^N} f\,\psi_n\,\varphi_m\,dx_1\,dx_2 + \int_{\mathbf{R}^{N_1}} \varphi_m\,b_1\,\cdot\,\nabla_{x_1}\int_{\mathbf{R}^{N_2}} f\,\psi_n\,dx_2\,dx_1 \\ - \int_{\mathbb{R}^{N_1}} \varphi_m \int_{\mathbf{R}^{N_2}} f\,\frac{1+|x_2|}{n}\,\frac{b_2}{1+|x_2|}\,\cdot\,(\nabla_{x_2}\psi)\Big(\frac{x_2}{n}\Big)\,dx_2\,dx_1 = 0.$$

that is to say, by a simple integration by parts of the second term (we use this time assumption (H3) to get rid of $\mathrm{div}_{x_1} b_1$),

$$\frac{d}{dt}\int_{\mathbf{R}^N} f\,\psi_n\,\varphi_m\,dx_1\,dx_2 - \int_{\mathbf{R}^{N_1}} (b_1\,\cdot\,\nabla_{x_1}\varphi_m)\int_{\mathbf{R}^{N_2}} f\,\psi_n\,dx_2\,dx_1 \\ - \int_{\mathbb{R}^{N_1}} \varphi_m \int_{\mathbf{R}^{N_2}} f\,\frac{1+|x_2|}{n}\,\frac{b_2}{1+|x_2|}\,\cdot\,(\nabla_{x_2}\psi)\Big(\frac{x_2}{n}\Big)\,dx_2\,dx_1 = 0. \tag{2.19}$$

We now remark that, *uniformly with respect to n*, the second term of (2.19) goes to zero as m goes to infinity. Indeed, writing $\dfrac{|b_1|}{1+|x_1|} = c_1 + c_\infty$ with $c_1 \in L^1_{x_1}$, $c_\infty \in L^\infty_{x_1}$, thanks to (H2), we have

$$\int_{\mathbf{R}^{N_1}} |b_1\,\cdot\,\nabla_{x_1}\varphi_m| \int_{\mathbf{R}^{N_2}} f\,\psi_n\,dx_2\,dx_1 \\ = \int_{\mathbf{R}^{N_1}} \frac{1+|x_1|}{m}\,\frac{|b_1|}{1+|x_1|}\,\Big|(\nabla_{x_1}\varphi)\Big(\frac{x_1}{m}\Big)\Big| \int_{\mathbf{R}^{N_2}} f\,\psi_n\,dx_2\,dx_1 \\ \leq c\|\nabla\varphi\|_{L^\infty_{x_1}} \int_{m\leq|x_1|\leq 2m} |c_1(x_1)|\,dx_1\ \Big\|\int_{\mathbf{R}^{N_2}} f\,dx_2\Big\|_{L^\infty_{x_1}} \\ + c\|c_\infty\|_{L^\infty_{x_1}}\|\nabla\varphi\|_{L^\infty_{x_1}}\Big\|\int_{\{m\leq|x_1|\leq 2m\}\times\mathbf{R}^{N_2}} f\,dx_1\ dx_2,$$

using the fact that $\nabla\varphi$ is L^∞ and supported in the annular $\{1 \leq |x_1| \leq 2\}$ and the assumption $f \in L^1_{x_1}(L^1_{x_2}) \cap L^\infty_{x_1}(L^1_{x_2})$.

In addition, m this time being fixed, we claim that the third term of (2.19) goes to zero as n goes to infinity by Lebesgue's dominated convergence theorem.

Indeed, as $\nabla\psi$ is L^∞ and supported in the annular $\{1 \leq |x_1| \leq 2\}$, and $\dfrac{b_2}{1+|x_2|}$ may be decomposed, in view of (H4), into $d_1 + d_\infty$ with $d_1 \in L^1_{x_1,x_2}$ and $d_\infty \in L^1_{x_1}(L^\infty_{x_2})$, we have for almost all $x_1 \in \mathbb{R}^{N_1}$,

$$|\varphi_m(x_1)| \int_{\mathbf{R}^{N_2}} f\,\frac{1+|x_2|}{n}\,\frac{|b_2|}{1+|x_2|}\,|\nabla_{x_2}\psi|\Big(\frac{x_2}{n}\Big)\,dx_2 \\ \leq c|\varphi_m(x_1)|\|f(x_1,\cdot)\|_{L^\infty_{x_2}} \int_{n\leq|x_2|\leq 2n} |d_1(x_1,\cdot)|\,dx_2 \\ + c|\varphi_m(x_1)|\|d_\infty(x_1,\cdot)\|_{L^\infty_{x_2}} \int_{n\leq|x_2|\leq 2n} f\,dx_2 \\ \longrightarrow 0$$

as n goes to infinity. In addition,

$$
\begin{aligned}
|\varphi_m(x_1)| \int_{\mathbf{R}^{N_2}} f \, \frac{1+|x_2|}{n} \, \frac{|b_2|}{1+|x_2|} \, |\nabla_{x_2} \psi| \left(\frac{x_2}{n} \right) dx_2 & \\
\leq c |\varphi_m(x_1)| \, \| f(x_1, \cdot) \|_{L^\infty_{x_2}} \| d_1(x_1, \cdot) \|_{L^1_{x_2}} & \\
+ c |\varphi_m(x_1)| \, \| d_\infty(x_1, \cdot) \|_{L^\infty_{x_2}} \| f \|_{L^1_{x_2}} &
\end{aligned}
$$

and the right-hand side is in $L^1_{x_1}$, since $f \in L^\infty_{x_1}(L^1 \cap L^\infty_{x_2})$, $d_1 \in L^1_{x_1,x_2}$, $d_\infty \in L^1_{x_1}(L^\infty_{x_2})$, $\varphi_m \in L^\infty_{x_1}$. Therefore Lebesgue's theorem applies and we get the convergence of the third term of (2.19) to zero as n goes to infinity, m being kept fixed.

Collecting the behaviours of the last two terms, we obtain with (2.19), as n, and next m, go to infinity,

$$
\frac{d}{dt} \int_{\mathbf{R}^N} f = 0.
$$

As $f_0 = 0$, this yields $f = 0$ for all t since $f \geq 0$ and this concludes the proof. $\square$

Remark 2.5. Let us emphasize the fact that local integrability in x_1 is enough for assumption (H5), as we indeed argue on Supp(φ_m), m being fixed (large enough).

Having proved Lemma 2.1 and Lemma 2.2, we now complete the proof of Theorem 2.1:

Proof of Theorem 2.1. Assume for the time being that we have at hand two solutions u_1 and u_2 to (2.1) satisfying the regularity stated in Theorem 2.1, and sharing the same initial value. By virtue of Lemma 2.1, the difference $f = u_1 - u_2$ satisfies

$$
\frac{\partial f_{\alpha_1,\alpha_2}}{\partial t} + b \cdot \nabla f_{\alpha_1,\alpha_2} = \epsilon_{\alpha_1,\alpha_2},
$$

with obvious notation. As we are in a regular setting here, we may multiply the above equation by $\beta'(f_{\alpha_1,\alpha_2})$ for some function $\beta \in C^1(\mathbb{R})$, β' bounded, and obtain

$$
\frac{\partial \beta(f_{\alpha_1,\alpha_2})}{\partial t} + b \cdot \nabla \beta(f_{\alpha_1,\alpha_2}) = \epsilon_{\alpha_1,\alpha_2} \beta'(f_{\alpha_1,\alpha_2}).
$$

By letting α_2, and next α_1, go to zero, we obtain the equation

$$
\frac{\partial \beta(f)}{\partial t} + b \cdot \nabla \beta(f) = 0,
$$

for such functions β. Now, letting β approximate the absolute value, we end up with

$$
\frac{\partial |f|}{\partial t} + b \cdot \nabla |f| = 0.
$$

Therefore we have a non-negative solution $|f|$ to (2.1), which vanishes at initial time and belongs to the right functional space. Applying Lemma 2.2, we get $u_1 = u_2$. There remains now to prove the existence part.

Existence in the functional space $L^\infty([0,T],\ L^1_x \cap L^\infty_x(\mathbb{R}^N))$ is given in a straightforward way by an application of Proposition II.1 of [2]. For the sake of consistency, let us only mention here that it is a simple matter of regularization of the vector field b appearing in (2.1). One introduces the solution u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} + b_\varepsilon \cdot \nabla u_\varepsilon = 0 \text{ in } (0,\infty) \times \mathbb{R}^N, \tag{2.20}$$

where $b_\varepsilon = \rho_\varepsilon * b \in L^1([0,T], C^1_b(\mathbb{R}^N))$ converges to b, next shows the right estimates on u_ε and finally passes to the limit. The only non-standard thing we have to prove here is the fact that such a solution necessarily belongs to $L^\infty([0,T],\ L^\infty_{x_1}(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})))$. This is actually a consequence of the specific form of the transport equation and of the regularization work we have already done. Indeed, formally we deduce by integration of

$$\frac{\partial u}{\partial t} + b_1 \cdot \nabla_{x_1} u + b_2 \cdot \nabla_{x_2} u = 0,$$

over the x_2-space, that

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^{N_2}} u\, dx_2 + b_1 \cdot \nabla_{x_1} \int_{\mathbb{R}^{N_2}} u\, dx_2 + \int_{\mathbb{R}^{N_2}} b_2 \cdot \nabla_{x_2} u\, dx_2 = 0,$$

that is to say, the third term vanishes as b_2 has zero divergence with respect to x_2,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^{N_2}} u\, dx_2 + b_1 \cdot \nabla_{x_1} \int_{\mathbb{R}^{N_2}} u\, dx_2 = 0.$$

Formally, this yields the conservation over time of

$$\left\| \int_{\mathbb{R}^{N_2}} u\ dx_2 \right\|_{L^\infty_{x_1}},$$

and therefore the $L^\infty([0,T],\ L^\infty_{x_1}(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})))$ regularity. In order to make this estimate rigorous, it suffices to go back to the regular form (2.20). For this purpose, we need to specify the regularization kernel ρ_ε used to define b_ε in (2.20). We choose a product of two regularization kernels $\rho_{1,\varepsilon}(x_1)$ and $\rho_{2,\varepsilon}(x_2)$. This is useful to keep both parts of b_ε, namely $b_{1,\varepsilon}$ and $b_{2,\varepsilon}$, divergence free. Then all the computations made formally above take a rigorous sense, and we obtain the conservation of $\| \int_{\mathbb{R}^{N_2}} u_\varepsilon\ dx_2 \|_{L^\infty_{x_1}}$ and thus, letting ε go to zero, that of the same norm for u. This completes the proof of Theorem 2.1. $\square$

Remark 2.6. When assumptions (H3)–(H6) are replaced e.g. by (H3″)–(H6″), the same technique allows one to control both divergences in the right functional spaces, and consequently the claimed regularity also holds, by a straightforward adaptation of our proof.

3. Renormalized solutions of transport equations (2.1)

We devote this short section to the extension of the previous result to less regular initial data through the notion of renormalized solutions, as introduced in [2]. As almost all the arguments and statements are identical to those of [2], here we sketch only the proofs and give the main lines of the results.

As in [2], we consider the set L^0 of all measurable functions u on $\mathbb{R}^N$ with values in $\overline{\mathbb{R}}$ such that $meas\{|u| > \lambda\} < \infty$, for all $\lambda > 0$. For any $\beta \in C(\mathbf{R})$, bounded and vanishing near zero, we thus have $\beta(u) \in L^1 \cap L^\infty(\mathbb{R}^N)$ for any $u \in L^0$. As in [2], we shall say that a sequence u_n is bounded (respectively, converges) in L^0 whenever $\beta(u_n)$ is bounded (respectively, converges) in L^1 for any such β.

Here we need some additional assumptions on our initial datum, and that is why we consider the subset L^{00} of L^0 consisting of functions u satisfying

$$\forall\delta > 0, \quad meas\{x_2 \,/\, |u(x_1, x_2)| > \delta\} < c_\delta(x_1) \in L^\infty_{x_1}(\mathbb{R}^{N_1}). \tag{3.1}$$

This subset is equipped with the topology induced by that of L^0. For any $u \in L^{00}$, we have $\beta(u) \in L^\infty_{x_1}(\mathbf{R}^{N_1}, L^1_{x_2}(\mathbb{R}^{N_2}))$. Indeed, for δ small enough such that β vanishes on $[0, \delta]$, we have

$$\begin{aligned}\int_{\mathbf{R}^{N_2}} \big|\beta(u(x_1, x_2))\big|\, dx_2 &= \int_{\{x_2/\,|u(x_1,x_2)|>\delta\}} \big|\beta(u(x_1, x_2))\big|\, dx_2 \\ &\quad + \int_{\{x_2/\,|u(x_1,x_2)|<\delta\}} \big|\beta(u(x_1, x_2))\big|\, dx_2 \\ &\le \|\beta\|_{L^\infty}\, c_\delta(x_1) + 0,\end{aligned}$$

which belongs to $L^\infty_{x_1}$.

It follows that if we choose u_0 in L^{00}, then $\beta(u_0)$ is a convenient initial condition for the transport equation considered in the previous section.

We therefore say that u is a renormalized solution of (2.1) complemented by an initial condition $u_0 \in L^{00}$ whenever $\beta(u)$ is a solution of (2.1) in the sense of Section 2 with initial condition $\beta(u_0)$.

The stability result for renormalized solutions of (2.1) that we shall need in the rest of this article is a straightforward consequence of that of [2, Theorem II.4]. We therefore reproduce the main result of it here for convenience. Only slight modifications are done with respect to that of [2].

Theorem 3.1. *Let us consider a sequence $b_n = (b_{1,n}(x_1), b_{2,n}(x_1, x_2))$ of vector fields satisfying* (H1) *to* (H6). *We assume that $b_{1,n}$ converges to b_1 in $L^1_{x_1,loc}$, that $b_{2,n}$ converges to b_2 in $L^1_{x_1,x_2,loc}$. We also assume that $b = (b_1(x_1), b_2(x_1, x_2))$ satisfies* (H1) *to* (H6).

Let u_n be a bounded sequence in $L^\infty([0, T], L^{00})$ of renormalized solutions of (2.1) *with vector field b_n, and initial condition $u_{n,0} \in L^{00}$. We assume that $u_{n,0}$ converges in L^{00}_{loc} to some u_0 in L^{00}.*

Then u_n converges in $C([0, T], L^{00}_{loc})$ to the renormalized solution of (2.1) *associated to the initial condition u_0.*

Let us emphasize that we have only mentioned above the basic conclusion of stability. Many extensions (under alternatives and/or additional assumptions) are available in the spirit of [2]. In particular, the following remark contains a noteworthy extension, as far as time-dependent vector fields b are concerned.

Remark 3.1. We consider time-dependent vector fields b (satisfying (H1′), (H2′), (H3), (H4′), (H5′), (H6)). Then, for Theorem 3.1 to hold, one may allow for a convergence of $b_n = (b_{1,n}(x_1), b_{2,n}(x_1, x_2))$ to $b = (b_1(x_1), b_2(x_1, x_2))$ either in the strong topology of $L^1([0, T], L^1_{x_1,loc} \times L^1_{x_1,x_2,loc})$, or also for only a convergence in the weak topology of the same functional space provided the additional condition of "regularity" in x

$$\sup_n \|b_n(t, x + h) - b_n(t, x)\|_{L^1\left([0,T],L^1_{x_1,loc} \times L^1_{x_1,x_2,loc}\right)} \overset{h\longrightarrow 0}{\longrightarrow} 0$$

is fulfilled. The fact that the above Theorem 3.1 can be extended to this latter situation is a straightforward consequence of arguments already used to prove Theorem II.7 of [2].

Likewise, the convergence of u_n towards u can be improved (local convergence for better topologies, and global convergence) under additional assumptions. All these are consequences of Theorem II.4 of [2] in the present setting, and are skipped here.

We also skip the proof as it is the same as that of [2]. It is sufficent to say that the heart of the matter is the uniqueness theorem for the renormalized solution, basically to obtain that

$$\beta(\lim u_n) = \lim(\beta(u_n)).$$

4. Application 1: Dependance upon initial data for solutions of ODEs

We turn here to our first application, that deals with ordinary differential equations.

We consider the ordinary differential equation (1.3) that again for the sake of simplicity we rewrite here in the case for a vector field, denoted here by c, which does not depend on time

$$\begin{cases} \dot{Y}(t, y) &= c\,(Y(t, y)), \\ Y(t = 0, y) &= y. \end{cases} \tag{4.1}$$

Let us assume that c satifies the following properties:

$$(P1) \qquad c \in W^{1,1}_{y,loc},$$

$$(P2) \qquad \frac{c}{1 + |y|} \in L^1 + L^\infty(\mathbb{R}^N),$$

$$(P3) \qquad \operatorname{div} c = 0,$$

which are the standard assumptions that allow us to define an almost everywhere flow for (4.1) in the sense of [2,1]). As in [2], assumption (P2) will possibly be strenghtened into

$$(Q2) \qquad c \in L^p + (1+|y|)\, L^\infty(\mathbb{R}^N), \text{ for some } p \in [1,\infty],$$

in order to obtain the L^1_{loc} integrability of Y with respect to the y variable. Otherwise, almost everywhere in t, $Y(t, y)$ will only belong to the space L_y of almost everywhere finite measurable functions of y, the space being equipped with the topology of the convergence in measure on arbitrary balls.

Remark 4.1. Let us at once mention that, like in [2], our whole study can be carried through when the vector field c depends on time, the properties (P1)–(P2) being changed into

$$(P1') \qquad c = c\,(t, y) \in L^1\big([0, T],\ W^{1,1}_{y,loc}\big),$$

$$(P2') \qquad \frac{c}{1+|y|} \in L^1([0, T],\ L^1 + L^\infty(\mathbb{R}^N)).$$

Likewise, we may allow, like in [2], a divergence of c that is not zero, provided it is controlled in L^∞_y, or respectively in the time-dependent case in $L^1_t(L^\infty_y)$.

As mentioned in the introduction, we aim at differentiating the flow Y with respect to its initial condition. Let us fix some $r \in \mathbb{R}^N$, and differentiate (formally for the moment) Y with respect to the initial condition y along the direction r. We obtain

$$\frac{\partial}{\partial t}(r \cdot \nabla_y Y)(t, y) = \nabla_y\, c\,(Y(t, y))\,(r \cdot \nabla_y Y)(t, y).$$

Grouping the two equations together, we may write

$$\begin{cases} \dot{Y}(t, y) &= c\,(Y(t, y)), \\ \dot{R}(t, y, r) &= \nabla_y\, c\,(Y(t, y))\, R(t, y, r), \\ Y(t=0, y) &= y, \\ R(t=0, y, r) &= r, \end{cases} \tag{4.2}$$

where we have denoted by $R(t, y, r) = (r \cdot \nabla_y Y)(t, y)$.

Our aim is to give a rigorous sense to this system. Clearly, if it has a sense, it should be the limit in some sense of the system obtained by a small perturbation of the initial condition of (4.1). For $\varepsilon > 0$ small, we indeed may consider

$$\begin{cases} \dot{Y}(t, y+\varepsilon r) &= c\,(Y(t, y+\varepsilon r)), \\ Y(t=0, y+\varepsilon r) &= y+\varepsilon r, \end{cases} \tag{4.3}$$

and by comparing with (4.1) we deduce the system

$$\begin{cases} \dot{Y}(t,y) & = c\,(Y(t,y)), \\ \dfrac{Y(t,y+\varepsilon r) - Y(t,y)}{\varepsilon}^{\boldsymbol{\cdot}} & = \dfrac{c\,(Y(t,y+\varepsilon r)) - c\,(Y(t,y))}{\varepsilon}, \\ Y(t=0,y) & = y, \\ \dfrac{Y(t=0,y+\varepsilon r) - Y(t=0,y)}{\varepsilon} & = r. \end{cases} \tag{4.4}$$

We shall see that, indeed, the limit as ε goes to zero "(4.4) giving to (4.2)" can be made rigorous.

For this purpose, our first task is to remark that both systems (4.2) and (4.4) can be formally recast as systems of the form

$$\begin{cases} \dot{X}_1(t,x) & = b_1(X_1(t,x)), \\ \dot{X}_2(t,x) & = b_2(X_1(t,x), X_2(t,x)), \\ X_1(t=0,x_1) & = x_1, \\ X_2(t=0,x_2) & = x_2, \end{cases} \tag{4.5}$$

for $(x_1, x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. It suffices that:

- for system (4.2), to set $x_1 = y$, $x_2 = r$, $N_1 = N_2 = N$, $X_1 = Y$, $X_2 = \dfrac{\partial Y}{\partial y}$, $b_1 = c$, $b_2 = \nabla_y c\,(y)\,r$, (note that this latter notation means $(b_2)_i = \sum_j (\partial_j c_i)\,r_j$);
- for system (4.4), it is natural to introduce a subscript ε, and we set $x_1 = y$, $x_2 = r$, $N_1 = N_2 = N$, $X_1 = Y$, $X_{2\varepsilon} = \dfrac{Y(t,y+\varepsilon r) - Y(t,y)}{\varepsilon}$, $b_1 = c$, $b_{2\varepsilon} = \dfrac{c\,(y+\varepsilon r) - c\,(y)}{\varepsilon}$.

The programme is now clear. First, we recall the equivalence between systems of ODEs and linear transport equations. Secondly, check that both systems satisfy assumptions (H1) to (H6). Theorem 2.1 will thus imply the existence and uniqueness of the solution of the associated transport equation, and therefore this will allow us to obtain the existence and uniqueness of the almost everywhere flow for both systems. Thirdly, we check by the stability Theorem 3.1 that the solution of the transport equation associated to (4.4) converges, as ε goes to zero, to the solution of that associated to (4.2). Consequently, we shall obtain the convergence of the flows.

System (4.5) will also be written in the more compact form

$$\begin{cases} \dot{X}(t,x) & = b\,(X(t,x)), \\ X(t=0,x) & = x, \end{cases} \tag{4.6}$$

but the reader should keep in mind that the field b has here the special form appearing in (4.5).

Since $b \in L^1_{loc}$ (both b_i are, in particular, in $L^1_{x_1,x_2,loc}$), it is a consequence of [4] that we have an equivalence between the ordinary differential equation (4.6) and the transport equation (2.1). More precisely, as announced in the introduction, we know from Proposition 1 of [4] that if X is an almost everywhere flow solution of (4.6), which means that X satisfies (4.6), for almost all x, in the sense of distributions together with the following three properties:

$$\begin{cases} (i) \quad X \in C(\mathbb{R}, L^1), \\ (ii) \quad \displaystyle\int \varphi(X(t,x))\,dx = \int \varphi(x)\,dx, \ \forall \varphi \in \mathcal{C}^\infty, \ \forall t \in \mathbb{R}, \\ (iii) \ X(t+s,x) = X(t, X(s,x)), \ \forall\, t, s \in \mathbb{R}, \ \forall\, a.e.\, x, \end{cases} \tag{4.7}$$

then $[S(t)u_0](x) = u_0(X(t,x))$ is a renormalized group solution of (2.1) (here and in the rest of this section, $-b$ replaces b in (2.1)), that is

$$\begin{cases} (i) \quad S(t)u_0 \in C(\mathbb{R}, L^1), \\ (ii) \quad S(t)\beta(u_0) = \beta(S(t)u_0), \ \forall \beta \in C_0^\infty, \ \forall u_0 \in L^\infty, \ \forall t \in \mathbb{R}, \\ (iii) \ S(t) \text{ is linear}, \ \forall t \in \mathbb{R}, \\ (iv) \quad S(t+s) = S(t) \circ S(s), \ \forall s, t \in \mathbb{R}, \\ (v) \quad u(t,x) = (S(t)u_0)(x) \text{ is solution of } (2.1), \ \forall u_0 \in L^\infty. \end{cases} \tag{4.8}$$

Conversely, we define an almost everywhere flow $X(t,x)$ for (4.1) from a renormalized group $S(t)$ solution of (2.1) simply by setting $(X(t,x))_i = (S(t)x_i)(x)$, $1 \le i \le N$ (coordinate by coordinate).

It is to be remarked that the equivalence we have recalled above holds when the equation is set on the torus. A necessary and sufficient modification when one works on the whole space is to impose a convenient behaviour at infinity for the vector field b. In our case, assumption (P2) will play this role. However, the L^1_{loc} integrability in (i) of (4.7) is not immediate (see [2]). We shall see below (in Remark 4.2) that in our case the flow is not L^1 with respect to some variables. Nevertheless, we shall call our flow an almost everywhere flow in the sense of [4] with a slight abuse of language.

We define the notion of the solution of (4.6) as follows. We shall say that X is a solution of (4.6) whenever, for all $\beta \in C_0^\infty$, we have

$$\begin{cases} \dfrac{\partial}{\partial t}\beta(X) \qquad\quad = D\beta(X(t,y)) \cdot b\,(X(t,y)), \\ \beta(X)(t=0,y) = \beta(y), \end{cases} \tag{4.9}$$

in the sense of distributions.

4.1. *Main result: first-order derivatives*

We are now in position to state and prove our main result of this section.

Theorem 4.1. *We assume* (P1) *to* (P3). *Then, there exists a unique almost everywhere flow* (Y, R), *such that:*

- *Y is continuous from $[0,T]$ with values in $L_{y,loc}$;*
- *R is continuous from $[0,T]$ to the set of functions of (y,r) that, for almost all y, are $L^1_{r,loc}$ and, for almost all r, $L_{y,loc}$, i.e. almost everywhere finite measurable functions of y;*
- *$X=(Y,R)$ satisfies the conservation property (4.7)-(ii) of the Lebesgue measure in (y,r), and the semigroup property (4.7)-(iii) (again in (y,r));*
- *(Y,R) satisfies (4.2) in the sense of (4.9).*

In addition, almost everywhere in (y,r), $X=(Y,R)$ satisfies (4.2) in the sense of distributions in time.

If we assume, in addition, (Q2), then Y is continuous from $[0,T]$ with values in $L^1_{y,loc}$, R is continuous from $[0,T]$ to $L^1_{y,r,loc}$.

As ε goes to zero, the flow (Y,R) is, in $C^0([0,T],L_{y,loc}\times(L_{r,loc}\cap L_{y,loc}))$, and almost everywhere in (y,r), the limit of the unique almost everywhere flow

$$\left(Y,\frac{Y(t,y+\varepsilon r)-Y(t,y)}{\varepsilon}\right)$$

associated to (4.4) in the same sense, and enjoying the same measurability and regularity properties as those of (Y,R) above.

Proof of Theorem 4.1. Let us first check that for both systems (4.2) and (4.4), that the assumptions (H1) to (H6), which allow to use the theory developed in Section 2, are fulfilled.

For both systems, assumptions (H1) to (H3) on b_1 exactly are assumptions (P1) to (P3) on c. Next, for system (4.2), since $b_2=(\nabla_{x_1}c(x_1))\,x_2$, we have $b_2\in L^1_{x_1,x_2,loc}$. Next, $\nabla_{x_2}b_2=\nabla_{x_1}c(x_1)\in L^1_{x_1,x_2,loc}$. Therefore (H4) is satisfied. Assumption (H5) is clear since

$$\frac{|b_2|}{1+|x_2|}=|\nabla_{x_1}c(x_1)|\in L^1_{x_1,loc}(L^\infty_{x_2}).$$

Finally, (H6) comes from a simple algebraic calculation showing that $\mathrm{div}_{x_2}\,b_2(x_1,x_2)=\mathrm{div}_x\,c(x)$ and (P3).

For system (4.4), assumption (H4) is clear since we even have $b_{2\varepsilon}\in L^\infty_{x_1,loc}(W^{1,1}_{x_2,loc})$. For (H5), we remark that

$$\frac{c(x_1+\varepsilon x_2)}{1+|x_2|}\in L^\infty_{x_1,loc}\big(L^1_{x_2}+L^\infty_{x_2}\big),$$

because of (P2), and

$$\frac{c(x_1)}{1+|x_2|}\in L^1_{x_1,loc}\big(L^\infty_{x_2}\big),$$

because $c\in L^1_{x_1,loc}$.

Finally (H6) follows from a simple calculation.

Having checked that assumptions (H1) to (H6) are verified by both systems (4.2) and (4.4), allows us to apply the results of Section 2 for the corresponding transport equations (2.1). There exist unique solutions (in the sense of Theorem 2.1) to both transport equations. Next using the equivalence stated in [2,4] between the transport equations and their respective ODEs, we therefore may claim that there exist flows for both systems. These flows are solution of (4.2) and (4.4), respectively, in the sense of (4.9).

Moreover, now using the stability result of Section 3, we may say that as ε goes to zero, the unique renormalized solution of the transport equation associated to (4.4) converges to the unique renormalized solution of the transport equation associated to (4.2). Consequently, the flows also converge, almost everywhere in x. All the properties of measurability and integrability of Y and R stated in Theorem 4.1 are straightforward consequences of the arguments of [2]. In particular, the L^1_{loc} integrability of R with respect to the r variable comes from the fact that the associated flow, namely $\nabla_y\, c(y)\, r$, fulfills the condition

$$\nabla_y\, c(y)\, r \;\in |r|\, L^\infty(\mathbb{R}^N) \;\subset\; L^1 + (1+|r|)\, L^\infty(\mathbb{R}^N),$$

which is a condition of type (Q2) in the r variable.

We have therefore proven the theorem. $\square$

Remark 4.2. It should be noted that R is not in L^1_{loc} in the variables y, r (for each time t). In order to illustrate the obstruction, we can formally integrate

$$\dot{R} = \nabla c\,(Y)R.$$

Then, we remark that ∇c is not exponentially integrable with respect to the variable y. Therefore R is only measurable in y, and not integrable.

Remark 4.3. The result contained in Theorem 4.1 of existence of a stable almost everywhere flow for ODEs (4.6) with a vector field satisfying (H1) to (H6) was announced in a somewhat vague way in Remarque 4 - alinea (iii) of [4]. In particular the case when b_1 in (H1) is constant was explicitly mentioned there.

4.2. *Second-order derivatives*

It is next a natural idea to try and differentiate the flow Y of (4.1) with respect to its initial condition at a higher order than one. It turns out that when assuming some $W^{m,1}$ regularity of c with respect to y, it is indeed possible to differentiate m times. We therefore recover a result of the same nature as that of the classical Cauchy–Lipschitz theory. In order to illustrate this fact, we now briefly describe the case $m = 2$. The cases $m \geq 3$ basically follow the same line, without any additional difficulty, and we leave such easy extensions to the reader.

Let us come back to the system (4.2), namely

$$\begin{cases} \dot{Y}(t,y) &= c\,(Y(t,y)), \\ \dot{R}(t,y,r) &= \nabla_y\, c\,(Y(t,y))\, R(t,y,r), \\ Y(t=0,y) &= y, \\ R(t=0,y,r) &= r. \end{cases}$$

Let us fix $(y', r') \in \mathbb{R}^{2N}$, and differentiate Y and R with respect to (y, r) along the direction (y', r'). In other words, we look for which equations the functions

$$Y' = (y', r') \cdot \nabla_{y,r} Y = y' \cdot \nabla_y Y \tag{4.10}$$

and

$$R' = (y', r') \cdot \nabla_{y,r} R \tag{4.11}$$

are solutions of. It is a simple calculation to show that they are solutions of:

$$\begin{cases} \dot{Y}'(t, y, y') & = \nabla_y c\,(Y(t, y))\, Y'(t, y, y'), \\ \dot{R}'(t, y, r, y', r') & = \nabla_y c\,(Y(t, y))\, R'(t, y, r, y', r') \\ & \quad + \nabla^2_{yy} c\,(Y(t, y)) \cdot (R(t, y, r), Y'(t, y, y')), \\ Y'(t = 0, y, y') & = y', \\ R'(t = 0, y, r, y', r') & = r'. \end{cases} \tag{4.12}$$

For the sake of clarity, let us make precise that the above equations explicitly read, componentwise, and using the convention of summation over repeated indices,

$$\frac{d}{dt} Y'_i(t, y, y') = \frac{\partial}{\partial y_j} c_i\,(Y(t, y)) Y'_j(t, y, y')$$

and

$$\begin{aligned} \frac{d}{dt} R'_i(t, y, r, y', r') = & \frac{\partial}{\partial y_j} c_i\,(Y(t, y)) R'_j(t, y, r, y', r') \\ & + \frac{\partial^2}{\partial y_j \partial y_k} c_i\,(Y(t, y)) Y'_j(t, y, y') R'_k(t, y, r, y', r'). \end{aligned}$$

Collecting (4.12) with (4.2), we obtain the closed system:

$$\begin{cases} \dot{Y}(t, y) & = c\,(Y(t, y)), \\ \dot{R}(t, y, r) & = \nabla_y c\,(Y(t, y))\, R(t, y, r), \\ \dot{Y}'(t, y, y') & = \nabla_y c\,(Y(t, y))\, Y'(t, y, y'), \\ \dot{R}'(t, y, r, y', r') & = \nabla_y c\,(Y(t, y))\, R'(t, y, r, y', r') \\ & \quad + \nabla^2_{yy} c\,(Y(t, y)) \cdot (R(t, y, r), Y'(t, y, y')), \\ Y(t = 0, y) & = y, \\ R(t = 0, y, r) & = r, \\ Y'(t = 0, y, y') & = y', \\ R'(t = 0, y, r, y', r') & = r'. \end{cases} \tag{4.13}$$

It is easy to see that this system of ODEs is associated to the following transport equation:

$$\begin{aligned} & \frac{\partial}{\partial t} u \;-\; c(y) \,\cdot\, \nabla_y u \;-\; \nabla_y c(y)\, r\, \nabla_r u \\ & \quad - \nabla_y c(y)\, y'\, \nabla_{y'} u \;-\; \big(\nabla_y c(y)\, r' \;-\; \nabla^2_{yy} c(y) \;\cdot (r, y')\big)\, \nabla_{r'} u = 0 \end{aligned} \tag{4.14}$$

set on a function $u = u(t, y, r, y', r')$.

Proving that an almost everywhere flow (in the sense of Theorem 4.1) exists for system (4.13) amounts to showing the existence and uniqueness of renormalized solutions of (4.14). It is easily seen that properties (H1) to (H6) are fulfilled by the vector field

$$\begin{aligned} b\,(y, r, y', r') = \\ - \,(c(y), \nabla_y\, c(y)\, r, \nabla_y\, c(y)\, y', \nabla_y\, c(y)\, r', \nabla^2_{yy}\, c(y)\; \cdot (r, y')). \end{aligned} \tag{4.15}$$

In particular, the $W^{2,1}$ regularity allows (H1) and (H4) to be fulfilled, while the crucial assumption (H5) is satisfied because

$$\frac{\nabla_y\, c(y)\, r' \;+\; \nabla^2_{yy}\, c(y)\; \cdot (r, y')}{1 + |r'|} \in L^1_{y,r,y',loc}\big(L^1_{r'} + L^\infty_{r'}\big).$$

5. Application 2: Micro-macro models for polymeric flows

We concentrate here on a stochastic differential equation of the type

$$dX_t = b(X_t)dt + dW_t, \tag{5.1}$$

complemented with an initial condition at time $t = 0$

$$X_t\,|_{t=0} = X_0.$$

As mentioned in the introduction, our interest in such equations stems from our aim to give a sound mathematical meaning to systems of the type (1.13).

Our purpose is to give a meaning to the stochastic flow of such an equation when the field b does not fulfill the usual assumptions of regularity needed in the theory of stochastic flows (namely b Lipschitz for strong solutions or b continuous bounded, say, for the corresponding martingale problem, see, e.g., [5]). We shall show that conditions of weak differentiability analogous to that used for the ordinary differential equations suffice to give a generalized sense to flows in this stochastic context. The notion of the Fokker–Planck equation and of a stochastic transport equation will also be extended to the present framework.

For clarity, and with a view to concentrate here on the additional difficulties created by the non-standard term dW_t in (5.1), we shall mainly deal in this section with equations of the form (5.1) when the vector field b satisfies the classical assumptions of the theory of ODEs treated in [2], namely assumptions (P1), (P2), (P3) of Section 4 (assumption (P2) being possibly strenghtened into (Q2)). It is however to be borne in mind that our arguments can be slightly modified in order to extend our results to the other cases:

- (α) b is a vector field of the particular form (1.2) with b_1, b_2 satisfying assumptions (H1) to (H6), with possibly (H3″), (H6″) replacing (H3), (H6), respectively; in this case, $M = 2N$ where N is the space dimension for x_1 and x_2, it exactly corresponds to the case when system (1.13) is dealt with;
- (β) b is not divergence free, but $\operatorname{div} b$ is controlled in the L^∞_x norm;
- (γ) b is time dependent and satifies (H1′), (H2′), (H3), (H4′), (H5′), (H6);
- (δ) b satifies assumptions that are any combinations of (α), (β), (γ) above.

We mainly skip such extensions, but shall incidentally indicate the modifications of the arguments, in particular when dealing with case (α). Apart from the extensions (α), (β), (γ), (δ), we shall also consider, from Subsection 5.3 on, the case when b has only an L^2 regularity.

Our strategy is first to come back to a deterministic situation. Without entering the details of the theory of diffusion processes (which will be made below, see Subsection 5.2), it suffices to mention here that in order to give a meaning to (5.1), we shall make use of the special form of this SDE to prove it can be given a meaning pathwise. This motivates the introduction of the following problem: giving a meaning to an equation of the form

$$dX(t) = b(X(t))dt + d\omega(t),$$

where ω is a given (deterministic) continuous-in-time function. This will be done in Subsection 5.1 below. Then (essentially by replacing $\omega(t)$ by a trajectory of the Brownian motion W_t) we shall come back to the stochastic framework in Subsection 5.2.

5.1. A deterministic flow

As mentioned above, we concentrate ourselves here on the differential equation

$$dX(t) = b(X(t))dt + d\omega(t), \tag{5.2}$$

complemented with the initial condition

$$X(0) = x, \tag{5.3}$$

where ω is a given (deterministic) function in $C^0([0, +\infty[, \mathbb{R}^M)$. The unknown $X(t)$ takes values in $\mathbb{R}^M$, the vector field b maps $\mathbb{R}^M$ in $\mathbb{R}^M$, and the initial condition x is fixed in $\mathbb{R}^M$. As indicated above, we focus on the “simple” case when the vector field b satisfies assumptions (P1), (P2), (P3) of Section 4 (assumption (P2) being possibly strengthened into (Q2)).

Our main trick is to reduce (5.2) to an ODE of a standard form. Up to a change of variable $Y(t) = X(t) - \omega(t)$, (5.2) is formally equivalent to the ODE $\dot{Y}(t) = b(Y(t) + \omega(t))$ complemented by the initial condition $Y(t)|_{t=0} = x - \omega(0)$. It is therefore natural to introduce the vector field

$$b^\omega(t, y) = b(y + \omega(t)), \tag{5.4}$$

and consider the ordinary differential equation

$$\begin{cases} \dot{Y} & = b^\omega(t, Y), \\ Y(t = s) & = y, \end{cases} \tag{5.5}$$

with its flow $Y^\omega(t, s, y)$ formally defined (for the time being) by

$$\begin{cases} \dfrac{d}{dt} Y^\omega(t, s, y) & = b^\omega(t, Y^\omega(t, s, y)), \\ Y^\omega(t = s, s, y) & = y. \end{cases} \tag{5.6}$$

The formal equivalence above motivates the following definition:

Definition 5.1. *We shall say that $X(t, s, x)$ is an almost everywhere flow for the ordinary differential equation*

$$dX(t) = b(X(t))\,dt + d\omega(t) \qquad \text{for}\, t \geq s \geq 0$$

complemented with the initial condition

$$X(s) = x,$$

when $X(t, s, x)$ reads

$$X(t, s, x) = Y^{\omega}(t, s, x - \omega(s)) + \omega(t), \tag{5.7}$$

for all $t \geq s \geq 0$, where $Y^{\omega}(t, s, y)$ is an almost everywhere flow (in the sense of [2]*) of the ordinary differential equation* (5.6).

Next, let us recall at this point that if we assume that $b(x)$ satisfies assumptions (P1), (P2) and (P3), then the field $b^{\omega}(t, y)$ defined from b by (5.4) satisfies the properties needed, in the time-dependent case to uniquely define an almost everywhere flow Y^{ω} for (5.6). These properties have been written above, namely in (H1′), (H2′) and (H3′). It suffices to remark that all algebraic properties and all norms with respect to the space variable remain unchanged: the change of variables is a simple translation with $\omega(t)$. It is useful, for self consistency, to recall here the properties obtained (in [2]) for the unique flow Y^{ω} of an ODE with a time-dependent right-hand-side such as (5.6), which is defined as a function continuous with respect to (t, s) and valued in the space L of almost everywhere finite measurable functions of y, namely:

- (a) we have the conservation of the Lebesgue measure (in the case when the original vector field b is divergencefree, otherwise the modification of this measure is controlled, see [2]);
- (b) the semigroup property $Y^{\omega}(t_3, t_1, y) = Y^{\omega}(t_3, t_2, Y^{\omega}(t_2, t_1, y))$, for all t_i;
- (c) the renormalized equation for (5.6), namely

  $$\frac{\partial}{\partial t}\beta(Y^{\omega}) = D\beta(Y^{\omega}) \cdot b^{\omega}(t, Y^{\omega}),$$

  with initial condition $\beta(Y^{\omega}) = \beta(y)$ at time $t = s$, holds in the distribution's sense in t, y.

In view of [2], we know that Y^{ω} is uniquely defined by properties (a), (b), (c) above. In addition to other properties for which we refer to [2], we wish to mention that

$$\frac{\partial}{\partial t} Y^{\omega} = b^{\omega}(t, Y^{\omega})$$

holds for all $t \geq 0$, almost all y. When the additional assumption (Q2) on b is fulfilled, then Y^{ω} takes, continuously in time, its values in L^p_{loc} in the y variable.

The above properties of the flow Y^{ω} immediately imply:

Proposition 5.1. *Assume that b satisfies* (P1), (P2) *and* (P3). *Then the almost everywhere flow $X(t,s,x)$ defined by Definition* 5.1 *exists and is uniquely defined by properties (a), (b), (c) above (where $X-\omega$ replaces Y^ω accordingly to relation* (5.7)*). In addition, $X(t,x) = X(t,0,x)$ satisfies the ODE* (5.2)–(5.3) *in the sense that, almost everywhere in x, one has*

$$X(t,x) = x + \int_0^t b(X(s,x))\,ds + \omega(t) - \omega(0).$$

If b satisfies (Q2)*, then $X(t,x)$ is continuous in time with values in L^1_{loc}.*

5.2. The stochastic flow

We now come back to equation

$$dX_t = b(X_t)dt + dW_t, \tag{5.8}$$

complemented with an initial condition at time $t=0$,

$$X_t\,|_{t=0} = X_0. \tag{5.9}$$

In order to set equation (5.8), we fix a Wiener space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, W_t)$ with a standard Brownian motion. Each realization $\omega \in \Omega$ is a pair (ω_1, ω_2), and Ω is equipped with the product measure of that of Ω_1 and Ω_2, respectively. The space Ω_1 is the probability space for the initial condition, here fixed at the random variable X_0, whose properties will be made precise later on. On the other hand, the probability space Ω_2 is that for the trajectories of the M-dimensional Wiener process W_t, valued in $\mathbb{R}^M$. The random variable X_t denotes a stochastic process defined on $\Omega = \Omega_1 \times \Omega_2$, valued in $\mathbb{R}^M$. The drift b, i.e. the vector field in the terminology of the previous sections, will enjoy specific properties which will be the main issue examined in the following. Within this precise mathematical setting, and at least for regular fields b (see above) the meaning of (5.8)–(5.9) can now be made precise:

$$X_t(\omega_1,\omega_2) = X_0(\omega_1) + \int_0^t b(X_s(\omega_1,\omega_2))\,ds + \int_0^t dW_s(\omega_2), \tag{5.10}$$

for all $t \geq 0$, and almost all ω_1, ω_2.

As above, and with a view to now concentrate on the additional difficulties created by the stochastic nature of (5.8), we shall again deal in this section with equations of the form (5.8) when the drift b satisfies assumptions (P1), (P2), (P3) of Section 4 (assumption (P2) being possibly strengthened into (Q2)). Our arguments can be slightly modified in order to extend our results to the other cases (α), (β), (γ), (δ) listed in Subsection 5.1.

As for the stochastic part of the equation is concerned, we have chosen (for clarity again) a simple Wiener process W_t in (5.8), but the following situation (that can also be combined with cases (α), (β), (γ), (δ) above) can be tackled in the same manner as below: dW_t is replaced by $\Lambda(t)dW_t$, where $\Lambda(t)$ is a $M \times M$ matrix with coefficients depending on time t with a $W^{1,1}$ regularity (a particular case is (1.13), where $\Lambda = [\Lambda_{ij}]$ with $\Lambda_{ii} = 0$ when $1 \leq i \leq N$, and $\Lambda_{ii} = 1$ when $N + 1 \leq i \leq M = 2N$).

It is to be emphasized that, on the contrary, when the Wiener process is multiplied by a diffusion coefficient (or matrix), namely when we have

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

the following arguments will generally not apply (at least for general diffusion coefficients σ). It is however possible to extend our approach to such situations but we shall not do so here.

The treatment of (5.8) simply follows from the following observation: for almost all ω_2, we may consider the continuous trajectory $W_t(\omega_2)$ of the Wiener process, next set $\omega(t) = W_t(\omega_2)$ and use the results of Subsection 5.1. More explicitly, we introduce, for almost all ω_2, the vector field

$$b^{\omega_2}(t, y) = b(y + W_t(\omega_2)), \tag{5.11}$$

and consider the ordinary differential equation

$$\begin{cases} \dot{Y} & = b^{\omega_2}(t, Y), \\ Y(t = s) & = y, \end{cases} \tag{5.12}$$

with its flow $Y^{\omega_2}(t, s, y)$ formally defined by

$$\begin{cases} \dfrac{d}{dt} Y^{\omega_2}(t, s, y) & = b^{\omega_2}(t, Y^{\omega_2}(t, s, y)), \\ Y^{\omega_2}(t = s, s, y) = y. & \end{cases} \tag{5.13}$$

Next, in the same spirit as in Definition 5.1, we introduce:

Definition 5.2. *Let us be given a Wiener process W_t on Ω_2. We shall say that $X_t(s, x, \omega_2)$ is an almost everywhere stochastic flow for the stochastic differential equation*

$$dX_t = b(X_t)\, dt + dW_t, \qquad \text{for } t \geq s \geq 0,$$

complemented with the deterministic initial condition

$$X_{t=s}(t, x, \omega_2) = x,$$

when $X_t(s, x, \omega_2)$ reads

$$X_t(s, x, \omega_2) = Y^{\omega_2}(t, s, x - W_s(\omega_2)) + W_t(\omega_2), \tag{5.14}$$

for all $t \geq s \geq 0$ and almost all ω_2, where $Y^{\omega_2}(t, s, y)$ is an almost everywhere flow (in the sense of [2]*) of the ordinary differential equation* (5.12).

Definition 5.3. *Let us now fix some absolutely continuous random variable $X_0(\omega_1)$. Then we say that $X_t(\omega_1, \omega_2)$ is an almost everywhere solution of*

$$\begin{cases} dX_t &= b(X_t)\,dt + dW_t, \\ X_{t=0} &= X_0, \end{cases} \tag{5.15}$$

when $X_t(\omega_1, \omega_2)$ satisfies (5.10)*, i.e.,*

$$X_t(\omega_1, \omega_2) = X_0(\omega_1) + \int_0^t b(X_s(\omega_1, \omega_2))\,ds + \int_0^t dW_s(\omega_2),$$

for all $t \geq 0$, and almost all ω_1, ω_2. If $X_t(s, x, \omega_2)$ is a flow in the sense of Definition 5.2 *then*

$$X_t(\omega_1, \omega_2) = X_t(0, X_0(\omega_1), \omega_2) \tag{5.16}$$

is convenient.

Some comments are in order:

- the term "almost everywhere" for the flow may seem redundant in the context of stochastic differential equations; however, we wish to keep it here with a view to have a general definition valid also in the case when equation (5.8) is, e.g., replaced by a system consisting of an ordinary differential equation and a stochastic differential equation, as is the case for (1.13); in such a case, we only recover a notion of *almost everywhere* flow for the ODE part, and not that of a classical solution for *all* initial conditions;
- the notion of flow and solution in the above definitions are "strong" in the sense of diffusion processes theory, since the Wiener process is *a priori* given;
- the assumption on the absolute continuity of the initial value X_0 is needed for consistency, when passing from an "almost everywhere in ω_1" notion (for a stochastic initial condition) to an "almost everywhere in x" notion (for a deterministic initial condition);
- when the drift b is regular enough, we recover by the above definition the usual notions of stochastic flow and solution.

A straightforward application of the results of Subsection 5.1 for almost all ω_2 yields:

Proposition 5.2. *Assume that b satisfies* (P1), (P2) *and* (P3)*. Then the almost everywhere stochastic flow defined by Definition* 5.2 *exists and is uniquely defined by properties (a), (b), (c) above holding almost everywhere in ω_2 (where $X - W$ replaces Y^{ω_2} accordingly to relation* (5.14)*). In addition, $X_t(\omega_1, \omega_2)$ of Definition* 5.3 *satifies the SDE* (5.8)–(5.9) *in the sense that, almost everywhere in ω_1 and ω_2, one has* (5.10)*, i.e.*

$$X_t(\omega_1, \omega_2) = X_0(\omega_1) + \int_0^t b(X_s(\omega_1, \omega_2))\,ds + \int_0^t dW_s(\omega_2).$$

If b satisfies (Q2)*, then X_t is continuous in time with values in L^1_{loc}.*

Remark 5.1. Another way to state the above result is to claim that *almost surely in* ω_2, there exists a flow, uniquely defined by properties (a), (b), (c) (where $X - W$ replaces Y^{ω_2}).

Remark 5.2. We emphasize that, like in the deterministic case, we claim the uniqueness of the *flow* (defined by Definition 5.2) but not that of the *solution* (defined by Definition 5.3) of the SDE for a given initial condition.

5.3. Existence and uniqueness for the Fokker–Planck equation, and consequences

As a counterpart to the viewpoint of trajectories that has been used in the previous subsection to understand mathematically a stochastic differential equation of the form (5.8), one may adopt the viewpoint of densities (or laws). Indeed, the natural issues that can be considered involve the Fokker–Planck (or forward Kolmogorov) equation formally associated to (5.8) when the drift b is again not regular, but enjoys the properties used above. This equation reads

$$\frac{\partial p}{\partial t} + b \cdot \nabla p - \frac{1}{2} \Delta p = 0, \tag{5.17}$$

when b is divergencefree (otherwise, henceforth we use the convention that the term $b \cdot \nabla p$ is to be replaced by $\operatorname{div}(p\, b)$) and is complemented by some given initial condition $p(t = 0, \cdot) = p_0$.

We successively want to address here and in Subsection 5.4 the following questions:

(i) does there exist a solution to (5.17) ? in which functional space ?
(ii) is such a solution unique ?
(iii) suppose the initial condition X_0 of (5.8) admits a density p_0, then does a solution X_t admit a density, p, and is this density a (the ?) solution of (5.17) with initial condition p_0 ?
(iv) conversely, is any solution of (5.17) the density of a solution X_t of (5.8) ?

We shall address these questions under the usual assumptions on the vector field b that we have used above. In particular, we shall consider the case of interest for our application in fluid mechanics: the vector field is then of the form $b = (b_1, b_2)$, satisfies (H1)–(H6) and the Laplacian operator, formally introduced in (5.17), only refers to the x_2 variable and not to the full variable (x_1, x_2). As is natural, we shall see that we must progressively strengthen the assumptions on b to obtain stronger results, going from question (i) to question (iv). We shall not give the details of all proofs, as they are clearly reminiscent of those of [2], but rather concentrate ourselves on some keypoints of the arguments.

As far as the initial condition p_0 of (5.17) is concerned, we shall assume henceforth that it belongs to $L^1 \cap L^\infty(\mathbb{R}^N)$. The L^1 assumption is natural since p_0 plays the role of the density of a random variable, namely X_0. The L^∞ assumption is taken here for simplicity. We actually may weaken these assumptions, in particular the L^∞ assumption which is restrictive from the standpoint of probability theory, and consider much more general initial conditions. Indeed, all the arguments of the following can be adapted through the notion of renormalized solutions, as we did above in Section 3. We leave such easy, but rather tedious, extensions to the reader.

Let us begin with the existence question (i). For simplicity we assume that b is a time-independent, divergencefree vector field that we suppose only belongs to $L^1_{loc}(\mathbb{R}^M)$. Then it is an easy exercise to show (first establishing formal *a priori* estimates, and next giving a rigorous meaning to them by approximation and regularization) that there exists some

$$p \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^M)) \ \cap \ L^2([0,T], H^1(\mathbb{R}^M))$$

solution of (5.17). The H^1 (in x) regularity can be shown to be optimal, while the L^2 (in time) integrability may indeed be improved. The "usual" other cases of vector fields b can be considered likewise. For instance, some b that are time dependent and/or that are not divergence free can be dealt with, provided the assumptions of Section 2 are fulfilled. On the other hand, we may consider the special form of a vector field $b = (b_1, b_2)$ satisfying (H1)–(H6), together with a Laplacian operator with only the x_2 variable. Equation (5.17) then reads

$$\frac{\partial p}{\partial t} + b_1(x_1) \cdot \nabla_{x_1} p + b_2(x_1, x_2) \cdot \nabla_{x_2} p - \frac{1}{2} \Delta_{x_2} p = 0. \tag{5.18}$$

Then, the existence of a solution

$$p \in L^\infty\big([0,T], L^1 \cap L^\infty_{x_1,x_2}(\mathbb{R}^{2N})\big) \ \cap \ L^2_{t,x_1}\big([0,T] \times \mathbb{R}^N, H^1_{x_2}(\mathbb{R}^N)\big)$$

can be shown by the same approximation and regularization techniques. The point is that the H^1 regularity then only holds in the x_2 variable.

Let us now turn to question (ii) about the uniqueness, which obviously is the central issue. Our results are contained in the following:

Proposition 5.3. *The vector field b is assumed to satisfy* (P1), (P2) *and* (P3) *(or any generalization of these in the spirit of Remark* 4.1*). Then there is one and only one solution*

$$p \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^M)) \ \cap \ L^2([0,T], H^1(\mathbb{R}^M))$$

of (5.17) *satisfying the initial condition* $p_0 \in L^1 \cap L^\infty$.

We skip the proof of Proposition 5.3. The proof of the existence statement has been outlined above, while the uniqueness is a standard consequence of the arguments of [2].

Before examining the consequences of Proposition 5.3 on the way questions (iii) and (iv) can be answered, we would like to mention possible extensions of this proposition, namely:

Proposition 5.4. *The vector field b is also assumed to contain a part which is not in* $W^{1,1}_{loc}$ *but in* L^2_{loc}, *namely it satisfies*

$$(\widetilde{P1}) \qquad\qquad b \in L^2_{loc} + W^{1,1}_{loc}(\mathbb{R}^M),$$

together with (P2)–(P3) *(or any generalization of these in the spirit of Remark* 4.1*). Then the existence and uniqueness stated in Proposition* 5.3 *still holds.*

The proof of Proposition 5.4 relies upon the following observation. As the solution p of the Fokker–Planck equation has a better regularity than that of the transport equation, since p automatically enjoys H^1 regularity, one can allow for a somewhat less regular vector field, thus we have the L^2 part in $(\widetilde{P}1)$. As usual, the keypoint in the proof of the uniqueness of the solution is the commutation relation stating (with the notations introduced in (2.6)) that $[b\cdot\nabla, \rho_\varepsilon]\,(p)$ converges to zero in L^1_{loc} as ε goes to zero, thus allowing for a regularization (by convolution with ρ_ε) of the solution. We therefore only detail here the proof of:

Lemma 5.1. *Let $b \in L^2_{loc}+W^{1,1}_{loc}(\mathbb{R}^M)$, $p \in L^\infty \cap H^1(\mathbb{R}^M)$. Let $\rho_\varepsilon = 1/\varepsilon^N\, \rho(\cdot/\varepsilon)$, where ρ is a fixed, non-negative, compactly supported, smooth function. Then*

$$[b\cdot\nabla, \rho_\varepsilon]\,(p) \overset{\varepsilon\longrightarrow 0}{\longrightarrow} 0 \text{ in } L^1_{loc}. \tag{5.19}$$

Proof of Lemma 5.1. To fix the ideas, we assume ρ has compact support in the unit ball and is of total mass one. The convergence (5.19) is clear for smooth b and p, and it is also true when $b \in W^{1,1}_{loc}$ and $p \in L^\infty$ by Lemma II.1 of [2]. Let us now decompose $b = b^1 + b^2$ with $b^1 \in W^{1,1}_{loc}$ and $b^2 \in L^2_{loc}$. We introduce, for each $\delta > 0$, a smooth function b^2_δ that converges to b^2 in L^2_{loc} as δ goes to zero. Likewise, we introduce, for each $\delta > 0$, a smooth function p_δ that converges to p in H^1_{loc} as δ goes to zero. Arguing by linearity and by density as in [2], we remark that in order to prove (5.19) for $b = b^1 + b^2$ it suffices to show

$$\begin{aligned} r_{\varepsilon,\delta}(x) = &\int \nabla p(y)\,(b^2(y) - b^2(x))\,\rho_\varepsilon(x-y)\,dy \\ &- \int \nabla p_\delta(y)\,(b^2_\delta(y) - b^2_\delta(x))\,\rho_\varepsilon(x-y)\,dy \longrightarrow 0, \end{aligned} \tag{5.20}$$

in L^1_{loc}, as δ goes to zero, *uniformly* with respect to ε. We split these two integrals as follows:

$$\begin{aligned} r_{\varepsilon,\delta}(x) = &-b^2(x) \int (\nabla p(y) - \nabla p_\delta(y))\;\rho_\varepsilon(x-y)\,dy \\ &+ \int (\nabla p(y) - \nabla p_\delta(y))\,b^2(y)\,\rho_\varepsilon(x-y)\,dy \\ &+ (b^2_\delta(x) - b^2(x)) \int \nabla p_\delta(y)\,\rho_\varepsilon(x-y)\,dy \\ &+ \int \nabla p_\delta(y)\,(b^2(y) - b^2_\delta(y))\,\rho_\varepsilon(x-y)\,dy. \end{aligned} \tag{5.21}$$

The first term can be shown to go to zero in L^1_{loc} because, for any ball B_R, one has

$$\begin{aligned} &\left| \int_{x\in B_R} b^2(x) \int_y (\nabla p(y) - \nabla p_\delta(y))\;\rho_\varepsilon(x-y)\,dy\,dx \right| \\ &\qquad \le \int_y |\nabla p(y) - \nabla p_\delta(y)| \int_{x\in B_R} |b^2(x)|\;\rho_\varepsilon(x-y)\,dx\,dy, \\ &\qquad \le \|\nabla p - \nabla p_\delta\|_{L^2}\,\|\chi_R\, b^2\|_{L^2}\,\|\rho_\varepsilon\|_{L^1}, \end{aligned} \tag{5.22}$$

using both the Young and Cauchy–Schwarz inequalities, and denoting by χ_R the characteristic function of the ball B_R. The right-hand side vanishes, uniformly with respect to ε. The third term of $r_{\varepsilon,\delta}$ in (5.21) can be treated along the same lines. As for the second term of (5.21), we write

$$\begin{aligned}
&\left|\int_{x\in B_R}\int_y (\nabla p(y)-\nabla p_\delta(y))\, b^2(y)\, \rho_\varepsilon(x-y)\,dy\,dx\right| \\
&\qquad \leq \int_{x\in B_R}\int_{y\in B_\varepsilon(x)} |\nabla p(y)-\nabla p_\delta(y)|\,|b^2(y)|\,\rho_\varepsilon(x-y)\,dy\,dx, \\
&\qquad \leq \int_{y\in B_{R+1}} |\nabla p(y)-\nabla p_\delta(y)|\,|b^2(y)| \int_{x\in B_R} \rho_\varepsilon(x-y)\,dx\,dy, \\
&\qquad\qquad \text{as soon as } \varepsilon \leq 1, \\
&\qquad \leq \|\nabla p-\nabla p_\delta\|_{L^2}\,\|\chi_{R+1}\, b^2\|_{L^2}\,\|\rho_\varepsilon\|_{L^1},
\end{aligned} \tag{5.23}$$

and the right-hand side vanishes uniformly with respect to ε as δ goes to zero. The fourth term of $r_{\varepsilon,\delta}$ in (5.21) can be treated likewise. Collecting the four terms, we complete the proof of Lemma 5.1. □

It is worth mentioning that, beyond Proposition 5.4, another extension of Proposition 5.3 can be considered. It deals with the case when the Fokker–Planck equation reads

$$\begin{aligned}
\frac{\partial p}{\partial t} + b_1(x_1)\cdot\nabla_{x_1} p + b_2(x_1,x_2)\cdot\nabla_{x_2} p - \frac{1}{2}\,\Delta_{x_2} p& \\
= 0 \text{ in } (0,\infty)\times\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}&.
\end{aligned} \tag{5.24}$$

As there is a Laplacian in the x_2 variable, it allows us to treat the case when the component b_2 of the vector field b also has an $L^2_{x_2}$ part. More precisely, under the assumptions (H1)–(H2)–(H3)–(H5)–(H6), together with

$$(\widetilde{H}4) \qquad b_2 = b_2(x_1,x_2) \in L^1_{x_1,loc}\big(\mathbb{R}^{N_1},\ L^2_{x_2,loc} + W^{1,1}_{x_2,loc}\big)$$

instead of (H4), then the existence and uniqueness of the solution of the above Fokker–Planck equation (complemented by an initial condition in $\big(L^1\cap L^\infty(\mathbb{R}^N)\big)\cap L^\infty_{x_1}(\mathbb{R}^{N_1},\ L^1_{x_2}(\mathbb{R}^{N_2})))$ can be proven. Again, the same usual extensions for non-divergence-free and/or time-dependent fields can be considered. We skip the proof of the above assertion, and only mention the fact that the crucial fact is an extension of the commutation Lemma 5.1. The latter can be proven by combining the techniques used in the proof of Lemma 2.1 (see formulae (2.7) and (2.9)) and those of the proof of Lemma 5.1: each term of the approximation error $r_{\varepsilon,\delta}$ is treated by the *ad hoc* technique.

Remark 5.3. We wish to emphasize that all the above results of existence and uniqueness of the solution of the Fokker–Planck equation can be complemented by a result of *stability* of solutions, in the same vein as that of Theorem 3.1. Extensions for the "usual" other cases of vector field b can be proven.

5.4. Correspondence between the SDE and the Fokker–Planck equation

We now consider questions (iii) and (iv) that deal with the relationship linking the solution X_t of the SDE (5.8) with the function p solution of (5.17).

For simplicity, we come back to the case when the vector field satisfies assumptions (P1), (P2) and (P3), i.e. is timeindependent, divergencefree, has $W^{1,1}$ regularity, and satisfies the usual growth condition at infinity. Then we are in a situation when we may both apply the result of existence/uniqueness of a stochastic flow (Proposition 5.2) and that of existence/uniqueness of the solution of the Fokker–Planck equation (Proposition 5.3). It is then easy to answer to questions (iii)–(iv):

Proposition 5.5 (Uniqueness in law). *Assume that b satisfies* (P1), (P2) *and* (P3) *and that* $p_0 \in L^1 \cap L^\infty$. *Then the unique solution* $p(t, \cdot)$ *of the Fokker-Planck equation* (5.17) *with initial condition* p_0 *is the density at time t of the process* $X_t(0, X_0(\omega_1), \omega_2)$ *where* X_0 *has density* p_0 *and* $X_t(s, x, \omega_2)$ *is the unique a.e. stochastic flow (in the sense of Definition* 5.2*) of* (5.8)*. In addition, any solution of* (5.8) *in the sense of Definition* 5.3 *with an initial condition* X_0 *that has density* p_0 *has density* $p(t, \cdot)$ *for all t.*

Remark 5.4. We therefore can also assert that the unique solution $f(t, \cdot)$ of

$$\frac{\partial f}{\partial t} - b \cdot \nabla f - \frac{1}{2} \Delta f = 0$$

with initial condition $f(t = 0, \cdot) = f_0$ writes $f(t, x) = \mathbb{E}_{\omega_2}(f_0(X_t(0, x, \omega_2)))$.

Proof of Proposition 5.5 (sketch). The proof is based upon two arguments: regularization of the vector field, and uniqueness (and stability) for the solution of the Fokker–Planck equation. Let us introduce some field b^ε, which is both a regularization and a truncation of the field b, in such a way that b^ε is a bounded continuous field on $\mathbb{R}^M$ and b^ε converges to b in L^1_x as ε goes to zero. For such a field, the stochastic differential equation

$$dX_t^\varepsilon = b^\varepsilon\big(X_t^\varepsilon\big)dt + dW_t, \tag{5.25}$$

with initial condition $X_{t=0}^\varepsilon = X_0^\varepsilon$ (a regularization of X_0), is well posed by standard arguments: it admits a unique solution X_t^ε. It is also standard that, when X_0^ε admits a density, denoted by p_0^ε, then X_t^ε has a density p^ε, which is the unique solution of

$$\frac{\partial p^\varepsilon}{\partial t} + b^\varepsilon \cdot \nabla p^\varepsilon - \frac{1}{2} \Delta p^\varepsilon = 0 \tag{5.26}$$

associated to the initial condition p_0^ε.

Next, the change of variables (5.14) can be made. More precisely, we may introduce the field

$$b^{\varepsilon,\omega_2}(t, y) = b^\varepsilon(y + W_s(\omega_2)) \tag{5.27}$$

and the flow Y^{ε,ω_2} of the ordinary differential equation

$$\begin{cases} \dfrac{d}{ds} Y^{\varepsilon,\omega_2}(t,s,y) &= b^{\varepsilon,\omega_2}\,(Y^{\varepsilon,\omega_2}(t,s,y)), \\ Y^{\varepsilon,\omega_2}(t=s,s,y) = y. \end{cases} \tag{5.28}$$

Then, the flow $X_t^\varepsilon(s,x,\omega_2)$ and the solution X^ε are related to Y^{ε,ω_2} through

$$\begin{aligned} X_t^\varepsilon(s,x,\omega_2) &= Y^{\varepsilon,\omega_2}(t,s,x-W_s(\omega_2)) + W_t(\omega_2), \\ X_t^\varepsilon(\omega_1,\omega_2) &= Y^{\varepsilon,\omega_2}\big(t,0,X_0^\varepsilon(\omega_1)\big) + W_t(\omega_2). \end{aligned} \tag{5.29}$$

We now want to let ε go to zero, so that b^ε approximates b. By using the result of the stability of the solutions of Fokker–Planck equations that is outlined in Remark 5.3, we may claim that p_ε converges to p the unique solution of

$$\frac{\partial p}{\partial t} + b \cdot \nabla p - \frac{1}{2}\,\Delta p = 0.$$

On the other hand, as ε goes to zero, the generalized flow $Y^{\varepsilon,\omega_2}(t,s,y)$ of (5.28) converges to the generalized flow $Y^{\omega_2}(t,s,y)$ of (5.13), in $C^0_{t,s}(L_y)$, almost surely in ω_2. Therefore the almost everywhere stochastic flow $X_t^\varepsilon(s,x,\omega_2)$ converges to the almost everywhere stochastic flow $X_t(s,x,\omega_2)$ of (5.8). Therefore, $X_t(\omega_1,\omega_2)$ is a solution of (5.8).

Now, we already know that the density p^ε of X_t^ε converges in $L^1_{t,y}$ to p. The two facts imply that $p(t,\cdot)$ is the density of X_t, first for almost all t, and next (in view of the pathwise continuity of X_t^ε and X_t) for any time t. This concludes the proof. □

We now need to make two important comments.

On the one hand, the result of the above proposition clearly also holds when the field b is assumed to satisfy the usual time-dependent and/or non-divergence-free assumptions of Section 2. Likewise, using renormalization techniques, it can be proven to also hold for less regular initial conditions.

On the other hand, however, the case when b has an L^2 part, i.e. satisfies assumption $(\widetilde{P1})$, is unclear. The reason is that the results of Proposition 5.2 are not known to hold in this case. Suppose indeed we consider a regularization b^ε of the vector field, together with the associated flow X_t^ε for the SDE and the density p_ε, solution of the Fokker–Planck equation with regularized field. Then, as ε goes to zero, we can indeed prove, using Proposition 5.4, that p^ε converges to p the unique solution of the Fokker–Planck equation with vector field b. But the convergence of the flow X^ε solution of the regularized SDE is not clear. In fact, the above analysis of the Fokker–Planck equation shows that X^ε converges in law (using the Markov property) and one sees that, through the Fokker–Planck equation, we can only determine uniquely in law a flow solving the SDE, or, equivalently, a solution to the martingale problem “à la Stroock–Varadhan” [5]. However, one needs to define these notions precisely, and we shall not do so here.

5.5. On the stochastic transport equation

We would like to conclude this work by briefly examining another viewpoint, that of the stochastic transport equation associated to (5.8).

We know that a useful rewriting of

$$dX_t = b(X_t)\, dt + dW_t$$

is obtained by setting

$$b^{\omega_2}(t, y) = b(y + W_t(\omega_2))\,,\ Y^{\omega_2}(t, s, x) = X_t(s, x + W_s(\omega_2), \omega_2) - W_t(\omega_2),$$

and reads

$$dY^{\omega_2}(t) = b^{\omega_2}(t, Y^{\omega_2}(t))\, dt.$$

On the other hand, we know from the results of [4], recalled in Section 4 above, that as soon as b is L^1, there is, for almost all ω_2, a one-to-one correspondence between the almost everywhere flow Y^{ω_2} and the (possibly renormalized) group solution f^{ω_2} to the transport equation

$$\frac{\partial f^{\omega_2}}{\partial t} + b^{\omega_2} \cdot \nabla f^{\omega_2} = 0. \tag{5.30}$$

Next, we can formally (for the time being) transform the latter equation into the stochastic transport equation

$$df + (b(x)\, dt + dW_t(\omega_2)) \cdot \nabla f - \left(\frac{1}{2}\Delta f\right) dt = 0, \tag{5.31}$$

simply by considering

$$f(t, x, \omega_2) = f^{\omega_2}(t, x - W_t(\omega_2)), \tag{5.32}$$

and (formally) applying the Ito formula. Equation (5.31) has a unique solution in the distributions sense when the initial condition f_0 is in $L^1 \cap L^\infty$ (and in the sense of renormalized solutions when the initial condition f_0 is only in L^1).

The above formal manipulations leading from (5.30) to (5.31) can be given a rigorous meaning: one gives a distributional sense to (5.30), then applies the change of variable (5.32) and Ito formula on the C^∞ test functions, and obtains (5.31) in the distributional sense.

We therefore have proven the one-to-one correspondence, for almost all ω_2, between the stochastic flow of (5.8) and the solution to the stochastic transport equation (5.31), for any b of regularity L^1 (divergencefree, for simplicity).

In addition, when b enjoys (e.g.) properties (P1), (P2) and (P3), we may prove within this setting the existence and uniqueness of the two objects, the flow X_t on the one hand and the solution f on the other hand, which are uniquely related to one another.

The solution f of the equation

$$\frac{\partial f}{\partial t} + b \cdot \nabla f - \frac{1}{2}\Delta f = 0$$

can then be recovered from f^{ω_2}:

$$f(t, x) = \mathbb{E}_{\omega_2}\big(f(t, x, \omega_2)\big).$$

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