Found. Comput. Math. 81–95 (2008)

DOI: 10.1007/s10208-008-9033-1

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Errata for Computing the Top Betti Numbers of Semialgebraic Sets Defined by Quadratic Inequalities in Polynomial Time (DOI: 10.1007/s10208-005-0208-8)

Saugata Basu

Corollary 1.3 has been amended. The text is reproduced here. The original Section 4.3 has been deleted. Substantial changes have been made to Sections 6 and 7, and the text is reproduced here on the following pages.

Corollary 1.3. *Let* R *be a real closed field and let* $S \subset \mathrm{R}^k$ *be defined by*

$$\bigwedge_{P\in\mathcal{P}_1} P = 0 \bigwedge_{P\in\mathcal{P}_2} P > 0 \bigwedge_{P\in\mathcal{P}_3} P < 0$$

with $\deg(P) \leq 2$ *for each* $P \in \bigcup_{i=0,1,2} \mathcal{P}_i$, *and* $\# \bigcup_{i=0,1,2} \mathcal{P}_i = s$.
Then, *for all* $\ell \geq 0$,

$$b_{k-\ell}(S) \leq \binom{s}{\ell} k^{O(\ell)}.$$

In the following proof, as well as later in the paper, we will extend the ground field R by infinitesimal elements. We denote by $\mathrm{R}\langle\zeta\rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in R (see [**9**] for more details). The sign of a Puiseux series in $\mathrm{R}\langle\zeta\rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ. This induces a unique order on $\mathrm{R}\langle\zeta\rangle$ which makes ζ infinitesimal: ζ is positive and smaller than any positive element of R. When $a \in \mathrm{R}\langle\zeta\rangle$ is bounded from above and below by some elements of R, $\lim_\zeta(a)$ is the constant term of a, obtained by substituting 0 for ζ in a. Given a semialgebraic set S in R^k, the *extension* of S to R', denoted $\mathrm{Ext}(S, \mathrm{R}')$, is the semialgebraic subset of R'^k defined by the same quantifier-free formula that defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is

Online publication: April 23, 2008.

well defined (i.e., it only depends on the set S and not on the quantifier-free formula chosen to describe it). This is an easy consequence of the transfer principle (see, for instance, [**9**]).

Proof of Corollary 1.3. Let $0 < \delta \ll \varepsilon \ll 1$ be infinitesimals. We first replace the set S by the set $S' \subset \mathrm{R}\langle\varepsilon\rangle^k$ defined by $S' = \mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle) \cap \bar{B}_k(0, 1/\varepsilon)$, where $\bar{B}_k(0, r)$ denotes the closed ball of radius r centered at the origin. It follows from Hardt's triviality theorem for semialgebraic mappings [**19**] that $b_i(S) = b_i(S')$ for all $i \geq 0$. We then replace S' by the set $S'' \subset \mathrm{R}\langle\varepsilon, \delta\rangle^k$ defined by

$$\bigwedge_{P\in\mathcal{P}_1} P \ \leq\ 0 \wedge -P \leq 0 \bigwedge_{P\in\mathcal{P}_2} -P + \delta \leq 0 \bigwedge_{P\in\mathcal{P}_3} -P - \delta \\ \leq\ 0 \bigwedge \varepsilon^2(X_1^2 + \cdots + X_k^2) - 1 \leq 0.$$

It follows from Hardt's triviality again that $b_i(S') = b_i(S'')$ for all $i \geq 0$. Now apply Theorem 1.1. $\square$

6. Computing the Cohomology Groups of a Basic Semi-Algebraic Set Defined by Homogeneous Quadratic Inequalities

In this section, we will show how to effectively compute the spectral sequence described in the previous section.

Let $\mathcal{P} = (P_1, \ldots, P_s) \subset \mathrm{R}[X_0, \ldots, X_k]$ be a s-tuple of quadratic forms. For any subset $\mathcal{Q} \subset \mathcal{P}$, we denote by $T_{\mathcal{Q}} \subset \mathbf{S}^k$, the semi-algebraic set,

$$T_{\mathcal{Q}} = \bigcup_{P \in \mathcal{Q}} \{x \in \mathbf{S}^k \mid P(x) \leq 0\},$$

and let

$$S = \bigcap_{P \in \mathcal{P}} \{x \in \mathbf{S}^k \mid P(x) \leq 0\}.$$

We denote by $C^\bullet(\mathcal{H}(T_{\mathcal{Q}}))$ the co-chain complex of a cellular subdivision, $\mathcal{H}(T_{\mathcal{Q}})$ of $T_{\mathcal{Q}}$, which is to be chosen sufficiently fine (to be specified later).

We first describe for each subset $\mathcal{Q} \subset \mathcal{P}$ with $\#\mathcal{Q} = \ell < k$, a complex, $\mathcal{M}^\bullet_{\mathcal{Q}}$, and natural homomorphisms,

$$\psi_{\mathcal{Q}} : C^\bullet(\mathcal{H}(T_{\mathcal{Q}})) \to \mathcal{M}^\bullet_{\mathcal{Q}},$$

which induce isomorphisms,

$$\psi^*_{\mathcal{Q}} : H^*(C^\bullet(\mathcal{H}(T_{\mathcal{Q}}))) \to H^*(\mathcal{M}^\bullet_{\mathcal{Q}}).$$

Moreover, for $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}$ with $\#\mathcal{A} = \#\mathcal{B} + 1 < k$, we construct a homomorphism of complexes,

$$\varphi_{\mathcal{A},\mathcal{B}} : \mathcal{M}^\bullet_{\mathcal{A}} \to \mathcal{M}^\bullet_{\mathcal{B}},$$

such that the following diagram commutes,

$$\begin{array}{ccc} H^*(\mathcal{M}^\bullet_{\mathcal{A}}) & \xrightarrow{\varphi^*_{\mathcal{A},\mathcal{B}}} & H^*(\mathcal{M}^\bullet_{\mathcal{B}}) \\ \psi^*_{\mathcal{A}} \uparrow & & \uparrow \psi^*_{\mathcal{B}} \\ H^*(C^\bullet(\mathcal{H}(T_{\mathcal{A}}))) & \xrightarrow{r^*} & H^*(C^\bullet(\mathcal{H}(T_{\mathcal{B}}))) \end{array} \quad (6.1)$$

where $\varphi^*_{\mathcal{A},\mathcal{B}}$ and r^* are the induced homomorphisms of $\varphi_{\mathcal{A},\mathcal{B}}$ and the restriction homomorphism r respectively.

Now, consider a fixed subset $\mathcal{Q} \subset \mathcal{P}$, which without loss of generality we take to be $\{P_1, \ldots, P_\ell\}$. Let

$$P = (P_1, \ldots, P_\ell) : \mathrm{R}^{k+1} \to \mathrm{R}^\ell$$

denote the corresponding quadratic map.

As in the previous section, let $\mathrm{R}^{\mathcal{Q}} = \mathrm{R}^{\ell}$, and

$$\Omega_{\mathcal{Q}} = \{\omega \in \mathrm{R}^{\ell} \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq \ell\}.$$

Let $B_{\mathcal{Q}} \subset \Omega_{\mathcal{Q}} \times \mathbf{S}^k$ be the set defined by,

$$B_{\mathcal{Q}} = \{(\omega, x) \mid \omega \in \Omega_{\mathcal{Q}}, x \in \mathbf{S}^k \text{ and } \omega P(x) \geq 0\},$$

and we denote by $\varphi_{1,\mathcal{Q}} : B_{\mathcal{Q}} \to \Omega_{\mathcal{Q}}$ and $\varphi_{2,\mathcal{Q}} : B_{\mathcal{Q}} \to \mathbf{S}^k$ the two projection maps.

For each subset $\mathcal{Q}' \subset \mathcal{Q}$ we have a natural inclusion $\Omega_{\mathcal{Q}'} \hookrightarrow \Omega_{\mathcal{Q}}$. Let

$$h_{\mathcal{Q}} : \Delta_{\mathcal{Q}} \to \Omega_{\mathcal{Q}}$$

be a semi-algebraic triangulation of $\Omega_{\mathcal{Q}}$, which is compatible with the subsets $\Omega_{\mathcal{Q}'}$ for every $\mathcal{Q}' \subset \mathcal{Q}$, and such that for any simplex σ of $\Delta_{\mathcal{Q}}$, $\mathrm{index}(\omega P)$, as well as the multiplicities of the eigenvalues of ωP, stay invariant as ω varies over $h_{\mathcal{Q}}(\sigma)$. Note that by a simplex of certain dimension we mean an open simplex of that dimension. The following proposition relates the homotopy type of $\varphi_{1,\mathcal{Q}}^{-1}(h_{\mathcal{Q}}(\sigma))$ to that of a single fiber.

Proposition 4. *For any simplex $\sigma \in \Delta_{\mathcal{Q}}$ and $\omega \in h_{\mathcal{Q}}(\sigma)$, $\varphi_{1,\mathcal{Q}}^{-1}(h_{\mathcal{Q}}(\sigma))$ is homotopy equivalent to $\varphi_{1,\mathcal{Q}}^{-1}(\omega)$, and both these spaces have the homotopy type of the sphere $\mathbf{S}^{k-\mathrm{index}(\omega P)}$.*

Proof. Let $i = \mathrm{index}(\omega P)$. Since $\mathrm{index}(\omega P)$ is invariant as ω varies over $h_{\mathcal{Q}}(\sigma)$, the quadratic forms ωP has exactly i negative eigen-values for each $\omega \in h_{\mathcal{Q}}(\sigma)$. Let $M(\sigma, \omega) \subset \mathrm{R}^{k+1}$ be the orthogonal complement to the linear span of the corresponding eigen-vectors, and let $B(\sigma, \omega) = M(\sigma, \omega) \cap \mathbf{S}^k$. Clearly, $M(\sigma, \omega)$ and $B(\sigma, \omega)$ vary continuously with ω, and $\varphi_{1,\mathcal{Q}}^{-1}(\omega)$ can be retracted to the set $\{\omega\} \times B(\sigma, \omega)$. Finally, since $h_{\mathcal{Q}}(\sigma)$ is contractible to ω, its clear that $\varphi_{1,\mathcal{Q}}^{-1}(h_{\mathcal{Q}}(\sigma))$ retracts to $\{\omega\} \times B(\sigma, \omega)$ and the latter has the homotopy type of $\mathbf{S}^{k-\mathrm{index}(\omega P)}$ by Lemma 5.1. □

Our next goal is to construct a cell complex homotopy equivalent to $B_{\mathcal{Q}}$ obtained by glueing together certain regular cell complexes, $\mathcal{K}(\sigma)$, where $\sigma \in \Delta_{\mathcal{Q}}$.

Let $1 \geq \varepsilon_0 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_s \gg 0$ be infinitesimals. For $\eta \in \Delta_{\mathcal{Q}}$, we denote by C_η the subset of $\bar{\eta}$ defined by,

$$C_\eta = \{x \in \bar{\eta} \mid \text{ and } \mathrm{dist}(x, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \sigma\}.$$

Now, let $\sigma \prec \eta$ be two simplices of $\Delta_{\mathcal{Q}}$. We denote by $C_{\sigma,\eta}$ the subset of $\bar{\eta}$ defined by,

$$C_{\sigma,\eta} = \{x \in \bar{\eta} \mid \mathrm{dist}(x, \sigma) \leq \varepsilon_{\dim(\sigma)}, \text{ and } \mathrm{dist}(x, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \sigma\}.$$

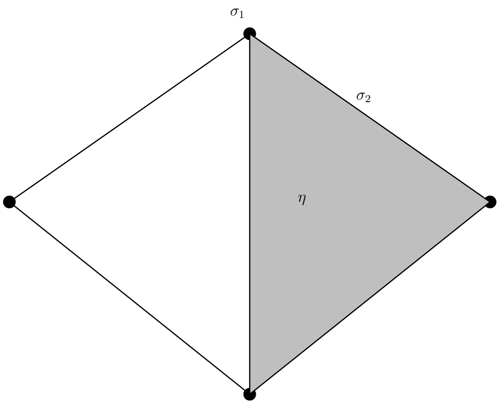

Fig. 1. The complex $\Delta_{\mathcal{Q}}$.

Note that,

$$|\Delta_{\mathcal{Q}}| = \bigcup_{\sigma \in \Delta_{\mathcal{Q}}} C_\sigma \cup \bigcup_{\sigma,\eta \in \Delta_{\mathcal{Q}}, \sigma \prec \eta} C_{\sigma,\eta}.$$

Also, observe that the various C_η's and $C_{\sigma,\eta}$'s are each homeomorphic to a ball, and moreover all non-empty intersections between them also have the same property. Thus, the union of the C_η's and $C_{\sigma,\eta}$'s together with the non-empty intersections between them form a regular cell complex, $\mathcal{C}(\Delta_{\mathcal{Q}})$, whose underlying topological space is $|\Delta_{\mathcal{Q}}|$ (see Figures 1 and 2).

We now associate to each C_σ (respectively, $C_{\sigma,\eta}$) a regular cell complex, $\mathcal{K}(\sigma)$, (respectively, $\mathcal{K}(\sigma, \eta)$) homotopy equivalent to $\varphi^{-1}_{1,\mathcal{Q}}(h_{\mathcal{Q}}(C_\sigma))$ (respectively, $\varphi^{-1}_{1,\mathcal{Q}}(h_{\mathcal{Q}}(C_{\sigma,\eta}))$.

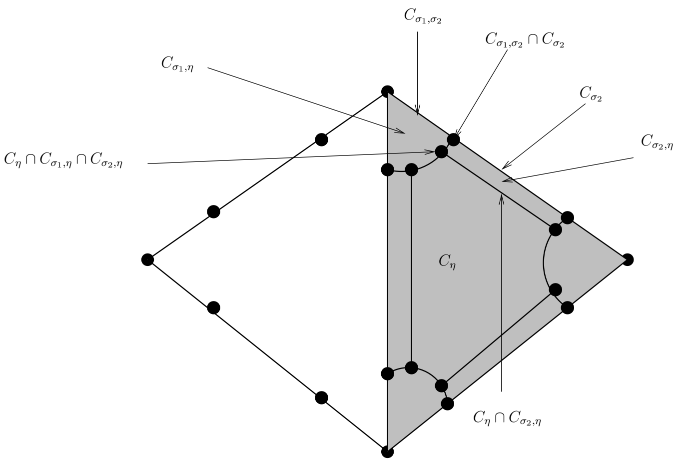

Fig. 2. The corresponding complex $\mathcal{C}(\Delta_{\mathcal{Q}})$.

For each $\sigma \in \Delta_Q$, and $\omega \in h_Q(\sigma)$, let

$$\lambda_0^\sigma(\omega) = \cdots = \lambda_{i_0}^\sigma(\omega) < \lambda_{i_0+1}^\sigma(\omega) = \cdots = \lambda_{i_1}^\sigma(\omega) < \cdots < \lambda_{i_{p-1}+1}^\sigma(\omega) = \cdots = \lambda_{i_p}^\sigma(\omega) < 0 = \cdots = \lambda_{i_{p+1}}^\sigma(\omega) < \cdots = \cdots = \lambda_k^\sigma(\omega)$$

denote the eigenvalues of ωP. Here, $\mathrm{index}(\omega P) = i_p+1$. Also, since the multiplicities of the eigenvalues do not change as ω varies over $h_Q(\sigma)$, the block structure, $[0,\ldots,i_0], [i_0+1,\ldots,i_1],\ldots,[\cdot,\ldots,k]$ also does not change as ω varies over $h_Q(\sigma)$. For $0 \leq j \leq p$, let $M^j(\sigma,\omega)$ denote the subspace of R^{k+1} orthogonal to the subspace spanned by the eigenvectors corresponding to the eigenvalues $\lambda_0^\sigma(\omega) = \cdots = \lambda_{i_0}^\sigma(\omega) < \lambda_{i_0+1}^\sigma(\omega) = \cdots = \lambda_{i_1}^\sigma(\omega) < \cdots < \lambda_{i_{p-1}+1}^\sigma(\omega) = \cdots = \lambda_{i_j}^\sigma(\omega)$, and let $M(\sigma,\omega) = M^p(\sigma,\omega)$. Since the eigenvalues vary continuously and their multiplicities do not change as ω varies over $h_Q(\sigma)$, the flag of subspaces $M^0(\sigma,\omega) \supset \cdots \supset M^p(\sigma,\omega)$ also varies continuously over $h_Q(\sigma)$.

For each $\sigma \in \Delta_Q$, and $\omega \in h_Q(\sigma)$, let $\{e_0(\sigma,\omega),\ldots,e_k(\sigma,\omega)\}$, be a continuously varying orthonormal basis of R^{k+1}, computed using a parametrized version of Gram-Schmidt orthogonalization algorithm, such that $e_i(\sigma,\omega),\ldots,e_k(\sigma,\omega)$, form an orthonormal basis of $M^p(\sigma,\omega) = M(\sigma,\omega)$, where $i = \mathrm{index}(\omega P)$. Since the number and the degrees of the polynomials defining the triangulation Δ_Q are bounded by $k^{2^{O(\ell)}}$, and the complexity of Gram-Schmidt orthogonalization is polynomial in the size of the input matrix, it is clear that the univariate representations defining the parametrized orthonormal basis $\{e_0(\sigma,\omega),\ldots,e_k(\sigma,\omega)\}$, have complexity bounded by $k^{2^{O(\ell)}}$. After having computed the orthonormal basis, $\{e_0(\sigma,\omega),\ldots,e_k(\sigma,\omega)\}$, we extend it continuously to each $C_{\sigma,\eta}$ for η with $\sigma \prec \eta$, satisfying the condition that

$$M(\sigma,\omega) \subset \mathrm{span}(e_i(\sigma,\omega),\ldots,e_k(\sigma,\omega)).$$

This extension can be done in a consistent manner because the first i eigenvalues, $\lambda_0(\omega P),\ldots,\lambda_{i-1}(\omega P)$ of ωP stay negative, and $\lambda_{i-1}(\omega P) < \lambda_i(\omega P)$ for ω in any infinitesimal neighborhood of $h_Q(C_\sigma)$. Thus, the linear subspace of R^k orthogonal to the eigenspaces corresponding to the eigenvalues, $\lambda_0(\omega P),\ldots,\lambda_{i-1}(\omega P)$ is well-defined and varies continuously with ω in any infinitesimal neighborhood of $h_Q(C_\sigma)$.

The orthonormal basis

$$\{e_0(\sigma,\omega),\ldots,e_k(\sigma,\omega)\},$$

determines a complete flag of subspaces, $\mathcal{F}(\sigma,\omega)$, consisting of

$$\begin{gathered} L^0(\sigma,\omega) = 0,\\ L^1(\sigma,\omega) = \mathrm{span}(e_k(\sigma,\omega)),\\ L^2(\sigma,\omega) = \mathrm{span}(e_k(\sigma,\omega), e_{k-1}(\sigma,\omega)),\\ \vdots\\ L^{k+1}(\sigma,\omega) = \mathrm{R}^{k+1}. \end{gathered}$$

For $0 \leq j \leq k$, let $c_j^+(\sigma, \omega)$ (respectively, $c_j^-(\sigma, \omega)$) denote the $(k-j)$-dimensional cell consisting of the intersection of the $L^{k-j+1}(\sigma, \omega)$ with the unit hemisphere in R^{k+1} defined by $\{x \in \mathbf{S}^k \mid \langle x, e_j(\sigma, \omega)\rangle \geq 0\}$ (respectively, $\{x \in \mathbf{S}^k \mid \langle x, e_j(\sigma, \omega)\rangle \leq 0\}$).

The regular cell complex $\mathcal{K}(\sigma)$ (as well as $\mathcal{K}(\sigma, \eta)$) is defined as follows.

The cells of $\mathcal{K}(\sigma)$ are $\{(x, \omega) \mid x \in c_j^{\pm}(\sigma, \omega), \omega \in h_{\mathcal{Q}}(c)\}$, where $\mathrm{index}(\omega P) \leq j \leq k$, and $c \in \mathcal{C}(\Delta_{\mathcal{Q}})$ is either C_σ itself, or a cell contained in the boundary of C_σ.

Similarly, the cells of $\mathcal{K}(\sigma, \eta)$ are $\{(x, \omega) \mid x \in c_j^{\pm}(\sigma, \omega), \omega \in h_{\mathcal{Q}}(c)\}$, where $\mathrm{index}(\omega P) \leq j \leq k$, $c \in \mathcal{C}(\Delta_{\mathcal{Q}})$ is either $C_{\sigma,\eta}$ itself, or a cell contained in the boundary of $C_{\sigma,\eta}$.

Our next step is to obtain cellular subdivisions of each non-empty intersection amongst the spaces associated to the complexes constructed above, and thus obtain a regular cell complex, $\mathcal{K}(B_{\mathcal{Q}})$, homotopy equivalent to $B_{\mathcal{Q}}$.

First notice that $|\mathcal{K}(\sigma', \eta')|$ (respectively, $|\mathcal{K}(\sigma)|$) has a non-empty intersection with $|\mathcal{K}(\sigma, \eta)|$ only if $C_{\sigma',\eta'}$ (respectively, $C_{\sigma'}$) intersects $C_{\sigma,\eta}$.

Let C be some non-empty intersection amongst the C_σ's and $C_{\sigma,\eta}$'s, that is C is a cell of $\mathcal{C}(\Delta_{\mathcal{Q}})$. Then, $C \subset \eta$ for a unique simplex $\eta \in \Delta_{\mathcal{Q}}$, and

$$C = C_{\sigma_1,\eta} \cap \cdots \cap C_{\sigma_p,\eta},$$

with $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_p \prec \eta$ and $p \leq \#\mathcal{Q} + 1$.

Consider $\omega \in h_{\mathcal{Q}}(C)$. We have p different flags,

$$\mathcal{F}(\sigma_1, \omega), \ldots, \mathcal{F}(\sigma_p, \omega),$$

giving rise to p independent regular cell decompositions of $B(\omega, \eta) = M(\omega, \eta) \cap \mathbf{S}^k$.

There is a unique smallest regular cell complex, $\mathcal{K}'(C, \omega)$, that refines all these cell decompositions. The cells of this cell decomposition consists of the following. Let $L \subset M(\omega, \eta)$ be any linear subspace of dimension m, $0 \leq m \leq k+1$, which is an intersection, of linear subspaces $L_1, \ldots, L_p$, where $L_i \in \mathcal{F}(\sigma_i, \omega)$, $1 \leq i \leq p$. The elements of the flags, $\mathcal{F}(\sigma_1, \omega), \ldots, \mathcal{F}(\sigma_p, \omega)$ of dimensions $m+1$, partition L into polyhedral cones of various dimensions. The union of the sets of intersections of these cones with $\mathbf{S}^k$, over all such subspaces $L \subset M(\omega, \eta)$, are the cells of $\mathcal{K}'(C, \omega)$. Figure 3 illustrates the refinement described above in case of two flags in R^3.

We now triangulate $h_{\mathcal{Q}}(C)$, using the algorithm implicit in Theorem 3.2 (Triangulation), such that the combinatorial type of the arrangement of flags,

$$\mathcal{F}(\sigma_1, \omega), \ldots, \mathcal{F}(\sigma_p, \omega)$$

and hence the cell decomposition $\mathcal{K}'(C, \omega)$, stays invariant over the image, $h_C(\theta)$, of each simplex, θ, of this triangulation. More precisely, we first compute a family of polynomials, $\mathcal{A}_C \subset \mathrm{R}[Z_1, \ldots, Z_\ell]$ whose signs at ω determine the combinatorial type of the corresponding arrangement of flags. It is easy to verify that the number and degrees of the polynomials in the family $\mathcal{A}_C$ is bounded by $k^{2^{O(\ell)}}$. We

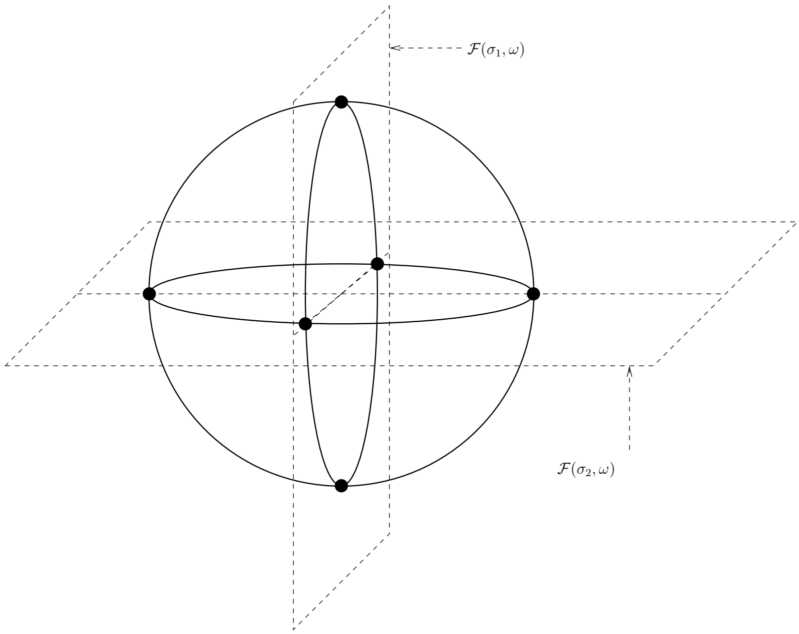

Fig. 3. The cell complex $\mathcal{K}'(C, \omega)$.

then use the algorithm implicit in Theorem 3.2 (Triangulation), with $\mathcal{A}_C$ as input, to obtain the required triangulation.

The closures of the sets

$$\{(\omega, x) \mid x \in c \in \mathcal{K}'(C, \omega), \omega \in h_{\mathcal{Q}}(h_C(\theta))\}$$

constitute a regular cell complex, $\mathcal{K}(C)$, which is compatible with the regular cell complexes $\mathcal{K}(\sigma_1), \ldots, \mathcal{K}(\sigma_p)$.

The following proposition gives an upper bound on the size of the complex $\mathcal{K}(C)$. We use the notation introduced in the previous paragraph.

Proposition 5. *For each $\omega \in h_{\mathcal{Q}}(C)$, the number of cells in $\mathcal{K}'(C, \omega)$ is bounded by $k^{O(\ell)}$. Moreover, the number of cells in the complex $\mathcal{K}(C)$ is bounded by $k^{2^{O(\ell)}}$.*

Proof. The first part of the proposition follows from the fact that there are at most $k^{\#\mathcal{Q}+1} = k^{\ell+1}$ choices for the linear space L and the number of $(m-1)$ dimensional cells contained in L is bounded by 2^{ℓ} (which is an upper bound on the number of full dimensional cells in an arrangement of at most ℓ hyperplanes). The second part is a consequence of the complexity estimate in Theorem 3.2 (Triangulation) and the bounds on number and degrees of polynomials in the family $\mathcal{A}_C$ stated above. □

We denote by $\mathcal{K}(B_Q)$, the union of all the complexes $\mathcal{K}(C)$ constructed above, noting that by construction, $\mathcal{K}(B_Q)$ is a regular cell complex.

Proposition 6. *$|\mathcal{K}(B_Q)|$ is homotopy equivalent to B_Q.*

Proof. We first show that B_Q is homotopy equivalent to a subset $B'_Q \subset B_Q$ as follows. For each simplex $\sigma \in \Delta_Q$ of the largest dimension ℓ, we use the retraction used in the proof of Proposition 4, to retract $\varphi_1^{-1}(h_Q(\mathrm{relint}(C_\sigma)))$ to the set $\{(\omega, x) \mid \omega \in \mathrm{relint}(C_\sigma), x \in B(\omega, \sigma)\}$. In this way we obtain a semi-algebraic set, X'_ℓ, which is a deformation retract of $\mathrm{Ext}(B_Q, \mathrm{R}\langle\varepsilon_0, \ldots, \varepsilon_{\ell-1}\rangle)$. Let $X_\ell = \lim_{\varepsilon_{\ell-1}} X'_\ell$. Notice that in the definition of X'_ℓ, if we replace $\varepsilon_{\ell-1}$ by a variable t and denote the corresponding set by $X'_{\ell,t}$, then for all $0 < t < t'$, $X'_{\ell,t} \subset X'_{\ell,t'}$ and each $X_{\ell,t}$ is closed and bounded. It then follows (see Lemma 16.17 in [**9**]) that $\mathrm{Ext}(X_\ell, \mathrm{R}\langle\varepsilon_0, \ldots, \varepsilon_{\ell-1}\rangle)$ has the same homotopy type as X'_ℓ, and hence X_ℓ has the same homotopy type as $\mathrm{Ext}(B_Q, \mathrm{R}\langle\varepsilon_0, \ldots, \varepsilon_{\ell-2}\rangle)$.

Now repeat the process using the $(\ell - 1)$-dimensional simplices and so on, to finally obtain $X_0 = B'_Q$, which by construction has the same homotopy type as B_Q. Finally, (again using Lemma 16.17 in [**9**]) we also have that $X_0 = \lim_{\varepsilon_0} |\mathcal{K}(B_Q)|$ and $\mathrm{Ext}(X_0, \mathrm{R}\langle\varepsilon_0, \ldots, \varepsilon_{\ell-1}\rangle)$ has the same homotopy type as $|\mathcal{K}(B_Q)|$. □

We also have,

Proposition 7. *The number of cells in the cell complex $\mathcal{K}(B_Q)$ is bounded by $k^{2^{O(\ell)}}$.*

Proof. The proposition is a consequence of Proposition 5 and the fact that the number of cells in the complex $\mathcal{C}(\Delta_Q)$ is bounded by $k^{2^{O(\ell)}}$. □

We now define,

$$\mathcal{M}_Q^\bullet = C^\bullet(\mathcal{K}(B_Q),$$

where $C^\bullet(\mathcal{K}(B_Q)$ is the cellular co-chain complex of the regular cell complex $\mathcal{K}(B_Q)$.

Let $\mathcal{H}(T_Q)$ (resp. $\mathcal{H}(B_Q)$) be a suitably fine cellular subdivision of T_Q (resp. B_Q) and let

$$\varphi'_{2,Q} : C_\bullet(\mathcal{H}(B_Q)) \to C_\bullet(\mathcal{H}(T_Q)),$$

be the homomorphism induced by a cellular map, which is a cellular approximation of $\varphi_{2,Q}$.

Let $\varphi_Q : |\mathcal{K}(B_Q)| \to B_Q$ denote the homotopy equivalence shown to exist by Proposition 6 above and let

$$\varphi'_Q : C_\bullet(\mathcal{K}'(B_Q)) \to C_\bullet(\mathcal{H}(B_Q)),$$

be the homomorphism induced by a cellular approximation to φ_Q, where $\mathcal{K}'(B_Q)$ is a cellular subdivision of the complex $\mathcal{K}(B_Q)$.

Since, each cell of $\mathcal{K}(B_Q)$ is a union of cells of $\mathcal{K}'(B_Q)$, there is a natural homomorphism

$$\theta_Q : C_\bullet(\mathcal{K}(B_Q)) \to C_\bullet(\mathcal{K}'(B_Q))$$

obtained by sending each p-dimensional cell of $\mathcal{K}(B_Q)$ to the sum of p-dimensional cells of $\mathcal{K}'(B_Q)$ contained in it, for every $p \geq 0$. It is a standard fact that θ_Q and its dual, $\check{\theta}_Q$, are quasi-isomorphisms.

Let

$$\psi_Q = \check{\theta}_Q \circ \check{\varphi}'_Q \circ \check{\varphi}'_{2,Q} : C^\bullet(\mathcal{H}(T_Q)) \to C^\bullet(\mathcal{K}(B_Q)),$$

where $\check{\varphi}'_Q$ (resp. $\check{\varphi}'_{2,Q}$) is the dual homomorphism of φ'_Q (resp. $\varphi'_{2,Q}$).

Proposition 8. *For* $0 \leq i \leq k-1$, *the induced homomorphisms*,

$$\psi_Q^* : H^i(C^\bullet(\mathcal{H}(T_Q))) \to H^i(\mathcal{M}_Q^\bullet)$$

are isomorphisms.

Proof. The proof is clear since ψ_Q is a composition of quasi-isomorphisms. □

Now let, $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}$ with $\#\mathcal{A} = \#\mathcal{B} + 1 < k$.

The simplicial complex $\Delta_\mathcal{B}$ is a subcomplex of $\Delta_\mathcal{A}$ and hence, $\mathcal{K}(B_\mathcal{B})$ is a subcomplex of $\mathcal{K}(B_\mathcal{A})$ and thus there exists a natural homomorphism (induced by restriction),

$$\varphi_{\mathcal{A},\mathcal{B}} : \mathcal{M}_\mathcal{A}^\bullet \to \mathcal{M}_\mathcal{B}^\bullet.$$

The complexes $\mathcal{M}_\mathcal{A}^\bullet$, $\mathcal{M}_\mathcal{B}^\bullet$, and the homomorphisms, $\varphi_{\mathcal{A},\mathcal{B}}$, $\psi_\mathcal{A}$, $\psi_\mathcal{B}$ satisfy

Proposition 9. *The diagram*

$$\begin{array}{ccc} \mathcal{M}_\mathcal{A}^\bullet & \xrightarrow{\varphi_{\mathcal{A},\mathcal{B}}} & \mathcal{M}_\mathcal{B}^\bullet \\ \uparrow \psi_\mathcal{A} & & \uparrow \psi_\mathcal{B} \\ C^\bullet(\mathcal{H}(T_\mathcal{A})) & \xrightarrow{r} & C^\bullet(\mathcal{H}(T_\mathcal{B})) \end{array}$$

is commutative, *where* r *is the restriction homomorphism.*

Proof. Clear from the construction. □

It follows from Proposition 9 that the diagram (6.1) is also commutative.

We denote by

$$\check{\varphi}_{\mathcal{B},\mathcal{A}} : \check{\mathcal{M}}^\bullet_{\mathcal{B}} \to \check{\mathcal{M}}^\bullet_{\mathcal{A}}$$

the homomorphism dual to $\varphi_{\mathcal{A},\mathcal{B}}$. We denote by $\mathcal{D}^{\bullet,\bullet}_{\mathcal{P}}$ the double complex defined by:

$$\mathcal{D}^{p,q}_{\mathcal{P}} = \bigoplus_{\mathcal{Q}\subset\mathcal{P},\#\mathcal{Q}=p+1} \check{\mathcal{M}}^q_{\mathcal{Q}}.$$

The vertical differentials,

$$d : \mathcal{D}^{p,q}_{\mathcal{P}} \to \mathcal{D}^{p,q-1}_{\mathcal{P}},$$

are induced componentwise from the differentials of the individual complexes $\check{\mathcal{M}}^\bullet_{\mathcal{Q}}$. The horizontal differentials,

$$\delta : \mathcal{D}^{p,q}_{\mathcal{P}} \to \mathcal{D}^{p+1,q}_{\mathcal{P}},$$

are defined as follows: for $a \in \mathcal{D}^{p,q}_{\mathcal{P}} = \oplus_{\#\mathcal{Q}=p+1}\check{\mathcal{M}}^q_{\mathcal{Q}}$, for each subset $\mathcal{Q} = \{P_{i_0}, \ldots, P_{i_{p+1}}\} \subset \mathcal{P}$ with $i_0 < \cdots < i_{p+1}$, the $\mathcal{Q}$-th component of $\delta a \in \mathcal{D}^{p+1,q}_{\mathcal{P}}$ is given by,

$$(\delta a)_{\mathcal{Q}} = \sum_{0\le j\le p+1} \check{\varphi}_{\mathcal{Q}_j,\mathcal{Q}}(a_{\mathcal{Q}_j}),$$

where $\mathcal{Q}_j = \mathcal{Q} \setminus \{P_{i_j}\}$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \oplus_{\#\mathcal{Q}=1}\check{\mathcal{M}}^3_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=2}\check{\mathcal{M}}^3_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=3}\check{\mathcal{M}}^3_{\mathcal{Q}} & \longrightarrow & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \oplus_{\#\mathcal{Q}=1}\check{\mathcal{M}}^2_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=2}\check{\mathcal{M}}^2_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=3}\check{\mathcal{M}}^2_{\mathcal{Q}} & \longrightarrow & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \oplus_{\#\mathcal{Q}=1}\check{\mathcal{M}}^1_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=2}\check{\mathcal{M}}^1_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=3}\check{\mathcal{M}}^1_{\mathcal{Q}} & \longrightarrow & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \oplus_{\#\mathcal{Q}=1}\check{\mathcal{M}}^0_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=2}\check{\mathcal{M}}^0_{\mathcal{Q}} & \xrightarrow{\delta} & \oplus_{\#\mathcal{Q}=3}\check{\mathcal{M}}^0_{\mathcal{Q}} & \longrightarrow & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have the following theorem.

Theorem 6.1. *For* $0 \le i \le k$,

$$H^i(S) \cong H^i(\mathrm{Tot}^\bullet(\mathcal{D}^{\bullet,\bullet}_{\mathcal{P}})).$$

Proof. For each i, $1 \leq i \leq s$, let $S_i \subset \mathbf{S}^k$ denote, the set defined on $\mathbf{S}^k$ by $P_i \leq 0$. Then, $S = \cap_{i=1}^s S_i$. Choosing a suitably fine triangulation of $\cup_{i=1}^s S_i$ consider the Mayer-Vietoris double complex, $\mathcal{N}^{\bullet,\bullet}$, as described in Section 3.5. The homomorphisms,

$$\bigoplus_{\mathcal{Q}\subset\mathcal{P},\#\mathcal{Q}=p+1} \check{\psi}_{\mathcal{Q}} : \bigoplus_{\mathcal{Q}\subset\mathcal{P},\#\mathcal{Q}=p+1} \check{\mathcal{M}}^q_{\mathcal{Q}} \longrightarrow \bigoplus_{\mathcal{Q}\subset\mathcal{P},\#\mathcal{Q}=p+1} C_q(\mathcal{H}(T_{\mathcal{Q}}))$$

give a homomorphism of the double complexes,

$$\psi : \mathcal{D}^{\bullet,\bullet}_{\mathcal{P}} \longrightarrow \mathcal{N}^{\bullet,\bullet}. \tag{6.2}$$

By Proposition 8, ψ induces isomporphisms between the E_1 terms of the two spectral sequences, obtained by taking homology with respect to the vertical differentials. Theorem 3.1 then implies that ψ induces isomorphisms in the associated spectral sequences. But, the second spectral sequence converges to the homology of S by (3.1). The theorem is an immediate consequence. □

7. Algorithms for quadratic forms

In this section we describe the algorithm for computing the top Betti numbers of a basic semi-algebraic set defined by quadratic forms. We first describe an algorithm for computing the complexes $\check{\mathcal{M}}^{\bullet}_{\mathcal{Q}}$ described in the previous section.

ALGORITHM 1 (Build Complex for Unions).

INPUT:

(A) An integer ℓ, $0 \leq \ell \leq k$.
(B) A quadratic map $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \to \mathrm{R}^s$ given by s homogeneous quadratic polynomials, $P_1, \ldots, P_s \in \mathrm{R}[X_0, \ldots, X_k]$.

OUTPUT:

(A) For each subset $\mathcal{Q} \subset \mathcal{P} = \{P_1, \ldots, P_s\}$, $\#\mathcal{Q} \leq \ell + 2$ a description of the complex $\check{\mathcal{M}}_{\mathcal{Q}}$, consisting of a basis for each term of the complex and matrices (in this basis) for the differentials.
(B) For each $\mathcal{Q}' \subset \mathcal{Q}$, with $\#\mathcal{Q} = \#\mathcal{Q}'+1$, matrices for the homomorphisms,

$$\check{\varphi}_{\mathcal{Q}',\mathcal{Q}} : \check{\mathcal{M}}^{\bullet}_{\mathcal{Q}'} \to \check{\mathcal{M}}^{\bullet}_{\mathcal{Q}}.$$

PROCEDURE

Step 1 For each subset $\mathcal{Q} = \{P_{i_1}, \ldots, P_{i_{\ell+2}}\} \subset \mathcal{P}$, with $\#\mathcal{Q} = \ell + 2$, let $P_{\mathcal{Q}}$ be the quadratic map corresponding to the subset $\mathcal{Q}$.
Let $Z_{\mathcal{Q}} = (Z_{i_1}, \ldots, Z_{i_{\ell+2}})$ and let $M_{\mathcal{Q}}$ be the symmetric matrix corresponding to the quadratic form $Z_{\mathcal{Q}} \cdot P_{\mathcal{Q}} = Z_{i_1} P_{i_1} + \cdots + Z_{i_{\ell+2}} P_{i_{\ell+2}}$. The entries of $M_{\mathcal{Q}}$ depend linearly on $Z_{i_1}, \ldots, Z_{i_{\ell+2}}$. Let,

$$F(Z_{\mathcal{Q}}, T) = \det(M_{\mathcal{Q}} + T \cdot I_{k+1}) = T^{k+1} + C_k T^k + \cdots + C_0,$$

where each $C_i \in \mathrm{R}[Z_{i_1}, \ldots, Z_{i_{\ell+2}}]$ is a polynomial of degree at most $k+1$. Let $\mathcal{A}_{\mathcal{Q}} = \{C_0, \ldots, C_k\}$.

Step 2 Using the algorithm implicit in Theorem 3.2 (Triangulation), compute a semi-algebraic triangulation,

$$h_{\mathcal{Q}} : \Delta_{\mathcal{Q}} \to \Omega_{\mathcal{Q}},$$

respecting the family $\mathcal{A}_{\mathcal{Q}} \cup \mathrm{Elim}_T(\{F(Z_{\mathcal{Q}}, T)\})$ (see [**9**] for the definition of Elim), such that for any subset $\mathcal{Q}' \subset \mathcal{Q}$, $\Delta_{\mathcal{Q}'}$ is a sub-complex of $\Delta_{\mathcal{Q}}$.

Step 3 Construct the cell complex $\mathcal{C}(\Delta_{\mathcal{Q}})$.

Step 4 For each cell $C \in \mathcal{C}(\Delta_{\mathcal{Q}})$, compute using a parametrized version of Gram-Schmidt orthogonalizations, as well as the the algorithm implicit in Theorem 3.2 (Triangulation), the cell complex $\mathcal{K}(C)$ and thus obtain a description of $\mathcal{K}(B_{\mathcal{Q}})$.

Step 5 Compute the matrices corresponding to the differentials in the complex $\mathcal{M}^\bullet_{\mathcal{Q}} = C^\bullet(\mathcal{K}(B_{\mathcal{Q}}))$.

Step 6 For $\mathcal{Q}' \subset \mathcal{Q} \subset \mathcal{P}$ with $\#\mathcal{Q} = \#\mathcal{Q}' + 1 < k$, compute the matrices for the homomorphisms of complexes,

$$\check{\varphi}_{\mathcal{Q}',\mathcal{Q}} : \check{\mathcal{M}}^\bullet_{\mathcal{Q}'} \to \check{\mathcal{M}}^\bullet_{\mathcal{Q}}.$$

in the following way.

The simplicial complex $\mathcal{K}(B_{\mathcal{Q}'})$ is a subcomplex of $\mathcal{K}(B_{\mathcal{Q}})$ by construction. Compute the matrix for the restriction homomorphism,

$$\varphi_{\mathcal{Q},\mathcal{Q}'} : C^\bullet(\mathcal{K}(B_{\mathcal{Q}})) \to C^\bullet(\mathcal{K}(B_{\mathcal{Q}'})).$$

and output the matrix for the dual homomorphism.

Complexity Analysis: The complexity of Step 1 is $\sum_{i=0}^{\ell+2} \binom{s}{i} k^{O(i)}$. The complexity of Step 2 is $\sum_{i=0}^{\ell+2} \binom{s}{i} k^{2^{O(\min(\ell,s))}}$, using the complexity of the algorithm for triangulating semi-algebraic sets. It follows from Proposition 7 that the complexities of all the remaining steps are also bounded by $\sum_{i=0}^{\ell+2} \binom{s}{i} k^{2^{O(\min(\ell,s))}}$.

Proof of Correctness.. It follows from Descarte's rule of signs (see Remark 2.42, page 41 in [**9**]) that for any $z \in \Omega_{\mathcal{Q}}$, $\mathrm{index}(zP_{\mathcal{Q}})$ is equal to the number of sign variations in the sequence $C_0(z), \ldots, C_k(z), +1$. Thus, the signs of the polynomials $\mathcal{A}_{\mathcal{Q}} = \{C_0, \ldots, C_k\}$ determine the index of $zP_{\mathcal{Q}}$. Hence for any simplex σ of $\Delta_{\mathcal{Q}}$, $\mathrm{index}(\omega P_{\mathcal{Q}})$ stays invariant as ω varies over $h_{\mathcal{Q}}(\sigma)$.

The correctness of the algorithm is now a consequence of Proposition 6 and Proposition 9. □

Let $P_1, \ldots, P_s \in \mathrm{R}[X_0, \ldots, X_k]$ be homogeneous quadratic polynomials, and consider the set $S \subset \mathbf{S}^k$ defined by, $S = \{x \in \mathbf{S}^k \mid P_1(x) \leq 0, \ldots, P_s \leq 0\}$.

We will also denote for $1 \leq i \leq s$, by S_i the set defined by $\{x \in \mathbf{S}^k \mid P_i(x) \leq 0.\}$ Clearly, $S = \cap_{1 \leq i \leq s} S_i$.

ALGORITHM 2 (Computing the highest ℓ Betti Numbers: the homogeneous case).

INPUT: *A quadratic map* $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \to \mathrm{R}^s$ *given by a set*, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_0, \ldots, X_k]$, *of s homogeneous quadratic polynomials.*
OUTPUT: $b_k(S), \ldots, b_{k-\ell}(S)$, *where S is the set defined by*

$$S = \bigcap_{P \in \mathcal{P}} \{x \in \mathbf{S}^k \mid P(x) \le 0\}.$$

PROCEDURE
Step 1 *Using Algorithm* 1 *compute the truncated complex* $\mathcal{D}_\ell^{\bullet,\bullet}$, *i.e.*

$$\begin{aligned} \mathcal{D}_\ell^{p,q} &= \mathcal{D}^{p,q}, \quad 0 \le p \le \ell+1, \quad k-\ell-1 \le q \le k, \\ &= 0, \qquad \qquad \textit{otherwise}, \end{aligned}$$

Step 2 *Compute using linear algebra, the ranks of*

$$H^i(\mathrm{Tot}^\bullet(\mathcal{D}_\ell^{\bullet,\bullet})), \quad k-\ell+1 \le i \le k.$$

Step 3 *For each* i, $k-\ell \le i \le k$, *output*, $b_i(S) = \mathrm{rank}(H^i(\mathrm{Tot}^\bullet(\mathcal{D}_\ell^{\bullet,\bullet})))$.

COMPLEXITY ANALYSIS: The number of algebraic operations is clearly bounded by $\sum_{i=0}^{\ell+2} \binom{s}{i} k^{2^{O(\min(\ell,s))}}$ using the complexity analysis of Algorithm 1.

Proof of Correctness. The correctness of the algorithm is a consequence of the correctness of Algorithm 1 and Theorem 6.1. □

Remark 7.1. Suppose that (using Notation from Algorithm 2) $\mathcal{P}' \subset \mathcal{P}$ and

$$S' = \bigcap_{P \in \mathcal{P}'} \{x \in \mathbf{S}^k \mid P(x) \le 0\},$$

and letting $\mathcal{D}_\ell'^{\bullet,\bullet}$ denote the corresponding complex for S', it is clear from the definition that there is a homomorphism, $\Phi_{\mathcal{P},\mathcal{P}'} : \mathcal{D}_\ell^{\bullet,\bullet} \to \mathcal{D}_\ell'^{\bullet,\bullet}$ defined as follows.
For

$$\varphi = \bigoplus_{\mathcal{Q} \subset \mathcal{P}, \#\mathcal{Q}=p+1} \varphi_{\mathcal{Q}} \in \mathcal{D}_\ell^{p,q} = \bigoplus_{\mathcal{Q} \subset \mathcal{P}, \#\mathcal{Q}=p+1} \check{\mathcal{M}}_{\mathcal{Q}}^q,$$

$$\Phi_{\mathcal{P},\mathcal{P}'}(\varphi) = \bigoplus_{\mathcal{Q} \subset \mathcal{P}', \#\mathcal{Q}=p+1} \varphi_{\mathcal{Q}}.$$

Recall from 6.2 that there exists,

$$\psi : \mathcal{D}_\ell^{\bullet,\bullet} \longrightarrow \mathcal{N}_\ell^{\bullet,\bullet}$$

which induces an isomorphism, $\psi^* : H^*(\mathrm{Tot}^\bullet(\mathcal{D}_\ell^{\bullet,\bullet})) \longrightarrow H^*(\mathrm{Tot}^\bullet(\mathcal{N}_\ell^{\bullet,\bullet}))$.

Denoting by $\mathcal{N}'^{\bullet,\bullet}_{\ell}$ the (truncated) Mayer-Vietoris complex for S' and by $i_{\mathcal{P},\mathcal{P}'}$: $\mathcal{N}^{\bullet,\bullet}_{\ell} \to \mathcal{N}'^{\bullet,\bullet}_{\ell}$ the inclusion homomorphism, we have the following commutative diagram.

$$\begin{array}{ccc} H^*(\mathrm{Tot}^\bullet(\mathcal{D}^{\bullet,\bullet}_{\ell})) & \xrightarrow{\Phi^*_{\mathcal{P},\mathcal{P}'}} & H^*(\mathrm{Tot}^\bullet(\mathcal{D}'^{\bullet,\bullet}_{\ell})) \\ \Big\downarrow \psi^* & & \Big\downarrow \psi'^* \\ H^*(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}_{\ell}))) & \xrightarrow{i_*} & H^*(\mathrm{Tot}^\bullet(\mathcal{N}'^{\bullet,\bullet}_{\ell}))) \end{array}$$

Note that $H^*(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet})) \cong H_*(S)$ and $H^*(\mathrm{Tot}^\bullet(\mathcal{N}'^{\bullet,\bullet})) \cong H_*(S')$.

It is clear that Algorithm 2 can be easily modified to output the complex $\mathcal{D}^{\bullet,\bullet}_{\ell}$, by outputting the matrices corresponding to the vertical and horizontal homomorphisms in the chosen bases. Furthermore, given a subset $\mathcal{P}' \subset \mathcal{P}$, Algorithm 2 can be made to output both the complexes $\mathcal{D}^{\bullet,\bullet}_{\ell}$ and $\mathcal{D}'^{\bullet,\bullet}_{\ell}$ along with the matrices defining the homomorphism $\Phi_{\mathcal{P},\mathcal{P}'}$ with the same complexity bounds.