Erratum to: Threshold Boolean form for joint probabilistic constraints with random technology matrix

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This note reports a correction and clarifies the results presented in [2]. The third author of this corrigendum is not an author of the paper [2]. In the original publication, Theorem 28 (Sect. 4.2.1, p. 410) states that “*The probabilistically constrained problem PMRTM can be reformulated as the quadratic integer optimization problem QDE.*” This statement must be corrected, since problem QDE is in fact an inner approximation of the chance-constrained problem PMRTM, and the equivalence between QDE and PMRTM can only be claimed under the specific settings presented below in the modified Theorem 28.

In part (ii) of Theorem 28 in [2, p. 411], it is shown that any feasible solution for the chance-constrained problem PMRTM defined by a p -sufficient recombination ω^k is also feasible for the reformulated problem QDE. While that statement is correct, this, along with the fact any feasible solution of QDE is feasible for PMRTM (part

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Table 1 Possible realizations of ξ in Example 1

k	ω_1^k	ω_2^k	$\mathbb{P}(\xi = \omega^k)$	$F_1(\omega_1^k)$	$F_2(\omega_2^k)$	$F(\omega^k)$
1	1	2	1/3	1/3	1	1/3
2	2	1	1/3	1	1/3	1/3
3	2	2	1/3	1	1	1

(i) of Theorem 28 [2, p. 410]), is not sufficient to claim the equivalence between **QDE** and **PMRTM**. Indeed, it might be possible to find feasible solutions for the chance-constrained problem **PMRTM** that do not correspond to a p -sufficient recombination $k \in \bar{\Omega}_B^+$ and that are not feasible for **QDE**.

The following example [5] illustrates that the reformulation **QDE** proposed in [2] is not in general equivalent to **PMRTM**, but constitutes an inner approximation of the chance-constrained problem **PMRTM**.

Example 1 Let $d = 3$, and let the three possible realizations of ξ be given in Table 1, along with the joint cumulative probability function F and the marginal cumulative distribution functions F_j , $j = 1, 2$.

Set $p = 2/3$ so that in order to be feasible to the probabilistic constraint a solution x must satisfy at least two of the following three inequalities:

$$x_1 + 2x_2 \leq 3, \quad 2x_1 + x_2 \leq 3, \quad 2x_1 + 2x_2 \leq 3.$$

Because the last inequality dominates the other two, the feasible region reduces to:

$$G = \{x \in \mathbb{R}_+^2 : x_1 + 2x_2 \leq 3, 2x_1 + x_2 \leq 3\}.$$

We next present the **QDE** formulation for this example. The sets of cut points are: $C_1 = \{\omega_j^k : F_j(\omega_j^k) \geq p, k \in \Omega\} = \{2\}$ and $C_2 = \{2\}$. Thus, the only recombination is the vector $(2, 2)$, which satisfies $F((2, 2)) = 1 \geq p$, and the set $\bar{\Omega}_B^-$ is empty. Using the equation numbering in [2], we see that: (1) there is no constraint (33); (2) the constraints (34) are $\lambda_{11} = 1$ and $\lambda_{21} = 1$; and (3) the constraint (62) is:

$$x_1(2\lambda_{11}) + x_2(2\lambda_{21}) \leq 3.$$

Substituting in the only feasible λ solution we obtain:

$$X = \{x \in \mathbb{R}_+^2 : 2x_1 + 2x_2 \leq 3\}.$$

Thus, $(1, 1) \in G$ but $(1, 1) \notin X$.

Revised statement of Theorem 28 in [2]

We provide now the revised Theorem 28 which defines several conditions under which the formulation **QDE** is an exact reformulation for **PMRTM**.

Using $t_{ij} = s_{ij}\xi_j$ in [2], the joint chance constraint (3) in [2, p. 392] can be rewritten as

$$\mathbb{P}\left(\sum_{j \in J} t_{ij} x_j \leq d_i, i = 1, \dots, r\right) \geq p. \quad (1)$$

The modified Theorem 28 is given below.

Theorem 28 *The quadratic integer optimization problem **QDE** is in general an inner approximation of the probabilistically constrained problem **PMRTM**, and is equivalent to the probabilistically constrained problem **PMRTM** if any of the following conditions are satisfied:*

(i) *Each inequality in (1) has exactly one monomial:*

$$\mathbb{P}(x_j t_{ij} \leq d_i, i = 1, \dots, r) \geq p, \quad (2)$$

(ii) *Each inequality in (1) contains exactly one random variable:*

$$\mathbb{P}\left(t_i \sum_{j=1}^m x_j \leq d_i, i = 1, \dots, r\right) \geq p, \quad (3)$$

with $t_i = t_{ij}$, $j \in J$ for each i .

(iii) *Each inequality in (1) contains exactly one decision variable:*

$$\mathbb{P}\left(x_{j'} \sum_{j=1}^m t_{ij} \leq d_i, i = 1, \dots, r\right) \geq p, \quad (4)$$

with $j' \in J$.

(iv) *Each inequality in (1) is the product of a sum of random variables by a sum of decision variables:*

$$\mathbb{P}\left(\sum_{k=1}^m t_{ik} \sum_{j=1}^m x_j \leq d_i, i = 1, \dots, r\right) \geq p, \quad (5)$$

(v) *The system of inequalities in (1) includes any combination of stochastic inequalities of the form described in (i) to (iv).*

Proof It was shown in the original manuscript [2] that: (1) any solution feasible for **QDE** is feasible for **PMRTM**, and (2) any solution feasible for **PMRTM** defined by $k \in \bar{\Omega}^+$ is feasible for **QDE**. What remains to be demonstrated is that any solution feasible for **PMRTM** is always feasible for **QDE** when the chance constraint takes a form presented in (i) to (v).

- (i): $x_j t_{ij} \leq d_i$ can be rewritten as: (a) $\frac{x_j}{d_i} \leq \frac{1}{\xi_{ij}} \Leftrightarrow -\frac{x_j}{d_i} \geq -\frac{1}{t_{ij}}$ if d_i and t_{ij} have the same sign, and (b) $\frac{x_j}{d_i} \geq \frac{1}{t_{ij}}$ if d_i and t_{ij} have opposite signs. This corresponds to the case of a joint chance constraint with random right-hand side vector. When the uncertainty is the random right-hand side only, it is well known (see, e.g., [1,3]) that there is no $k' \notin \bar{\Omega}_B^+$ such that $\mathbb{P}(\xi \leq \omega^{k'}) \geq p$ with $\omega^{k'} \leq \omega^k$ and $\omega^{k'} \neq \omega^k$ for any $k \in \bar{\Omega}_B^+$. Hence, under condition (i), the feasible area defined by **PMRTM** is included in the feasible area defined by **QDE**, which is the result that we set out to prove.
- (ii): As in (i), we have a single random variable in each stochastic inequality.
- (iii): Let $\vartheta_i = \sum_{j=1}^m t_{ij}$, $i = 1, \dots, r$. Constraint (4) becomes

$$\mathbb{P}(\vartheta_i x_i \leq d_i, i = 1, \dots, r) \geq p. \quad (6)$$

As ξ_{ij} is discretely distributed with finite support, so is ϑ_i . As (6) has the same form as (2), the proof of (i) applies here too.

- (iv): The probabilistic constraint (5) can be rewritten as

$$\mathbb{P}\left(\vartheta_i \sum_{j=1}^m x_j \leq d_i, i = 1, \dots, r\right) \geq p, \quad (7)$$

which is similar to (3).

- (v): The proofs of (i)–(iv) essentially show that each stochastic inequality is equivalent to a collection of inequalities of type (i). $\square$

The proof is obtained with the same approach used in [4] in the case of joint chance constraints including quadratic stochastic inequalities of the form $\sum_{j_1, j_2 \in J} s_{ij_1 j_2} x_{j_1} x_{j_2} \xi_{ij_1 j_2} \leq d_i, i = 1, \dots, r$ instead of the linear ones considered in [2].

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