Erratum to: Risk-averse dynamic programming for Markov decision processes

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The Original publication of the article suffers from a technical inaccuracy associated with verification of measurability of the optimal decision rules. This erratum corrects the gap, by introducing additional conditions of *joint* continuity and measurability of the functions and multifunctions involved.

We use the terminology and notation of [4].

A controlled kernel Q is *setwise (strongly) continuous*, if for all bounded measurable functions v on $\mathcal{X}$, and all convergent sequences $\{(x_k, u_k)\}$, $k = 1, 2, \dots$, one has

$$\lim_{k \rightarrow \infty} \int_{\mathcal{X}} v(y) Q(dy|x_k, u_k) = \int_{\mathcal{X}} v(y) Q(dy|x, u), \quad (1)$$

where $x = \lim_{k \rightarrow \infty} x_k$ and $u = \lim_{k \rightarrow \infty} u_k$. In the product space $\mathcal{X} \times \mathcal{M}$, we always consider the product topology of strong convergence in $\mathcal{X}$ and the above setwise convergence of measures in $\mathcal{M}$. Observe that in the case when $Q(\cdot|x, u) \in \mathcal{M} \subset \mathcal{Y} = \mathcal{L}_q(\mathcal{X}, \mathcal{B}, P_0)$ for all (x, u) , the setwise continuity and the weak* continuity [with $v \in \mathcal{V} = \mathcal{L}_p(\mathcal{X}, \mathcal{B}, P_0)$ in (1)] are equivalent.

We make the following general assumptions (for $t = 1, 2, \dots$):

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- G1.** The transition kernels $Q_t(\cdot, \cdot)$ are strongly continuous;
G2. The conditional risk measures ρ_t are Markov and the multifunctions $\mathcal{A}_t(\cdot, \cdot) \equiv \partial_\varphi \sigma_t(0, \cdot, \cdot)$ are lower semicontinuous;
G3. The functions $c_t(\cdot, \cdot)$ are bounded and lower semicontinuous;
G4. The multifunctions $U_t(\cdot)$ are measurable and compact-valued.

First, we establish semicontinuity of a risk transition mapping with respect to the state–control pair.

Proposition 1 *Suppose (G1)–(G3) and let $v \in \mathcal{V}$ be bounded. Then the mapping $(x, u) \mapsto \sigma_t(v, x, Q_t(x, u))$ is lower semicontinuous on $\text{graph}(U_t)$.*

Proof Consider the dual representation of the risk transition mapping

$$\sigma_t(v, x, Q_t(x, u)) = \max_{\mu \in \mathcal{A}_t(x, Q_t(x, u))} \int_{\mathcal{X}} v(y) \mu(dy). \quad (2)$$

By (G1) and (G2), the multifunction $(x, u) \mapsto \mathcal{A}_t(x, Q_t(x, u))$ is lower semicontinuous. Owing to condition (G3), the function $(x, u, \mu) \mapsto \int_{\mathcal{X}} v(y) \mu(dy)$ is lower semicontinuous on $\text{graph}(U_t) \times \mathcal{M}$. The assertion follows now from [1, Theorem 1.4.16], whose proof remains valid in our setting as well. $\square$

If the risk transition mapping $\sigma(\cdot, \cdot, \cdot)$ is continuous, then its subdifferential $\partial_\varphi \sigma_t(0, \cdot, \cdot)$ is upper semicontinuous. However, in Proposition 1, we assume *lower* semicontinuity of the mapping $(x, m) \mapsto \partial_\varphi \sigma(0, x, m)$, which is not trivial and should be verified for each case.

Example 1 Let us verify the lower semicontinuity assumption for the multifunction $\mathcal{A}$ given in [4, Example 4] with $r = 1$:

$$\mathcal{A}(x, m) = \left\{ \mu \in \mathcal{M} : \exists (h \in \mathcal{L}_\infty(\mathcal{X}, \mathcal{B}, P_0)) \frac{d\mu}{dm} = 1 + h - \langle h, m \rangle, \right. \\ \left. \|h\|_\infty \leq \kappa, h \geq 0 \right\}. \quad (3)$$

Consider an arbitrary $\mu \in \mathcal{A}(x, m)$ and suppose $x_k \rightarrow x$, $m_k \rightarrow m$, as $k \rightarrow \infty$. We need to find $\mu_k \in \mathcal{A}(x_k, m_k)$ such that $\mu_k \rightarrow \mu$. Let h be the function for which, according to (3), $\frac{d\mu}{dm} = 1 + h - \int h(z) m(dz)$. We define μ_k by specifying their Radon–Nikodym derivatives: $\frac{d\mu_k}{dm_k} = 1 + h - \int h(z) m_k(dz)$. By construction, $\mu_k \in \mathcal{A}(x_k, m_k)$. Then, for any bounded measurable function v , we obtain

$$\begin{aligned} \int_{\mathcal{X}} v(y) \mu_k(dy) &= \int_{\mathcal{X}} v(y) \left(1 + h(y) - \int_{\mathcal{X}} h(z) m_k(dz) \right) m_k(dy) \\ &= \int_{\mathcal{X}} v(y) (1 + h(y)) m_k(dy) - \int_{\mathcal{X}} h(z) m_k(dz) \int_{\mathcal{X}} v(y) m_k(dy). \end{aligned}$$

As $m_k \rightarrow m$, we conclude that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathcal{X}} v(y) \mu_k(dy) &= \int_{\mathcal{X}} v(y)(1 + h(y)) m(dy) - \int_{\mathcal{X}} h(z) m(dz) \int_{\mathcal{X}} v(y) m(dy) \\ &= \int_{\mathcal{X}} v(y) \mu(dy), \end{aligned}$$

which is the strong convergence of μ_k to μ .

Recall that a *normal integrand* is a function $f : \mathcal{X} \times U \rightarrow \mathbb{R} \cup \{+\infty\}$ such that that its epigraphical mapping $x \mapsto \{(u, \alpha) \in \mathcal{U} \times \mathbb{R} : f(x, u) \leq \alpha\}$ is a closed-valued and measurable multifunction [3, sec. 14.D].

Proposition 2 Suppose (G1)–(G4) and let $v : \mathcal{X} \rightarrow \mathbb{R}$ be measurable and bounded. Then for all $t = 1, 2 \dots$ the functions

$$\psi_t(x) = \inf_{u \in U_t(x)} \{c_t(x, u) + \sigma_t(v, x, Q_t(x, u))\}, \quad x \in \mathcal{X},$$

are measurable and bounded, and measurable selectors π_t of U_t exist, such that

$$\psi_t(x) = c_t(x, \pi_t(x)) + \sigma_t(v, x, Q_t(x, \pi_t(x))), \quad \forall x \in \mathcal{X}.$$

Proof Let us fix t and consider the function $f_t : \mathcal{X} \times U \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as follows:

$$f_t(x, u) = \begin{cases} c_t(x, u) + \sigma_t(v, x, Q_t(x, u)) & \text{if } u \in U_t(x), \\ +\infty & \text{otherwise.} \end{cases}$$

Owing to Proposition 1, $f_t(\cdot, \cdot)$ is lower semicontinuous and is thus a normal integrand [3, Ex. 14.31]. It follows from [3, Thm. 14.37] that the function $\psi_t(x) = \inf_u f_t(x, u)$ is measurable and that the optimal solution mapping $\Psi_t(x) = \{u \in U_t(x) : \psi_t(x) = f_t(x, u)\}$ is measurable. By (G4), the set $U_t(x)$ is compact, and thus $\Psi_t(x) \neq \emptyset$ for all $x \in \mathcal{X}$. Ψ_t is also compact-valued. By virtue of [2], a measurable selector π_t of Ψ_t exists. Let us recall the dual representation (2) again:

$$\psi_t(x) = c_t(x, \pi_t(x)) + \max_{\mu \in \mathcal{A}_t(x, Q_t(x, \pi_t(x)))} \int_{\mathcal{X}} v(y) \mu(dy).$$

As the set $\mathcal{A}_t(x, Q_t(x, \pi_t(x)))$ contains only probability measures, and the functions $c_t(\cdot, \cdot)$ and $v(\cdot)$ are bounded, the function $\psi_t(\cdot)$ is bounded as well. $\square$

In [4, Thm. 2], the assumptions (i)–(v) have to be replaced by the following conditions:

- (i) The general conditions (G1)–(G4) are satisfied;
- (ii) The function $c_{T+1}(\cdot)$ is measurable and bounded.

The assertion is the same as before, with the additional claim that the value functions $v_t(\cdot)$ are measurable and bounded.

The proof of [4, Thm. 2] follows the original line of argument. The only difference is that at the bottom of page 247, where [3, Thm. 14.37] is invoked, we need to invoke Proposition 2 from this note. The conditions (i)–(ii) above have to be assumed also in [4, Cor. 2], which uses Theorem 2.

In [4, Thm. 4], the assumptions (i)–(iv) have to be replaced by the following conditions:

- (i') The general conditions (G1)–(G4) are satisfied;
- (ii') The conditional risk measures ρ_t , $t = 1, 2, \dots$, are stationary.

The assertion is the same as before, with the additional claim that the value function $\hat{v}(\cdot)$ is measurable and bounded.

The proof of [4, Thm. 4] remains unchanged; the conditions (G1)–(G4) are needed because Theorem 2 is invoked. Originally, [4, Thm. 4] assumed also nonnegativity of the cost function $c(\cdot, \cdot)$, but this condition not needed, if (G3) is assumed, as explained on page 251 of [4]. The conditions (i')–(ii') above have to be assumed also in [4, Thms. 5–7 and Cor. 3], all of which use Theorem 4.

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References

1. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
2. Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors. Bulletin de l'Académie Polonaise de Sciences, Série Mathématiques, Astronomie et Physique **13**, 397–411 (1965)
3. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (1998)
4. Ruszczyński, A.: Risk-averse dynamic programming for Markov decision processes. Math. Progr. Ser. B **125**, 235–261 (2010)