Maximal q -Subharmonicity in $\mathbb{C}^n$

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Received: 3 October 2011 / Accepted: 22 April 2012 / Published online: 19 March 2013
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Abstract In this paper, we study maximal q -subharmonic functions in $\mathbb{C}^n$. We prove that maximality of q -subharmonic functions is a local notion and give a condition to check the maximality of C^2 q -subharmonic functions.

Keywords q -Subharmonic functions · Maximal q -subharmonic functions

Mathematics Subject Classification (2010) 32U05

1 Introduction

Let Ω be an open set in $\mathbb{C}^n$. An upper semi-continuous function $u : \Omega \rightarrow [-\infty, +\infty)$ is called plurisubharmonic on Ω if for every complex line ℓ of $\mathbb{C}^n$, $u|_{\ell \cap \Omega}$ is a subharmonic function on $\ell \cap \Omega$.

The set of plurisubharmonic functions on Ω is denoted by $PSH(\Omega)$.

Now as in [2] and [7], there is a class of plurisubharmonic functions playing an important role in pluripotential theory. This is a class of maximal plurisubharmonic functions. We recall the following definition given in [2].

Definition 1 A plurisubharmonic function u on Ω is called maximal plurisubharmonic (briefly, $u \in MPSH(\Omega)$) if for every $v \in PSH(\Omega)$, $v \leq u$ outside a compact subset of Ω implies $v \leq u$ on Ω .

As in [2] and [7], a locally bounded plurisubharmonic function u is in $MPSH(\Omega)$ if and only if it satisfies the homogeneous complex Monge–Ampère equation $(dd^c u)^n = 0$.

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Moreover, a recent result in [3] claimed that maximality is a local notion for locally bounded plurisubharmonic functions (see Corollary 1 in [3]). Also in [3], Blocki extended the above result for the class $\mathcal{E}(\Omega)$ introduced and investigated by Cegrell in [4] recently.

The aim of this paper is to extend the class of maximal plurisubharmonic functions to the class of maximal q -subharmonic functions, where q is an integer with $1 \leq q \leq n$ and to give some results about maximal q -subharmonic functions. Notice that for studying maximality of plurisubharmonic functions one often approaches by using the complex Monge–Ampère operator. But defining the complex Monge–Ampère operator for q -subharmonic functions is impossible. Hence one needs to find another approach for studying maximal q -subharmonic functions. In this paper, by using a new method, we prove that the maximality of q -subharmonic functions is equivalent to a local notion. We also provide a condition to check when a C^2 q -subharmonic function u is maximal q -subharmonic.

The note is organized as follows. Besides the introduction, the note has two sections. Section 2 is devoted to study q -subharmonic functions and to establish some results concerning this class. Section 3 deals with maximal q -subharmonic functions and proves the local property of this class.

2 q -Subharmonic Functions in $\mathbb{C}^n$

First, we recall the following definition of q -subharmonic functions which has been introduced by H. Ahn and N.Q. Dieu in [1] (also see [5]).

Definition 2 Let Ω be an open set in $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$, $u \not\equiv -\infty$ is called q -subharmonic if for every q -dimensional complex plane L in $\mathbb{C}^n$, $u|_L$ is a subharmonic function on $L \cap \Omega$. This means that for every compact subset $K \Subset L \cap \Omega$ and every continuous harmonic function h on K such that $u \leq h$ on ∂K it follows that $u \leq h$ on K .

The set of q -subharmonic functions on Ω is denoted by $SH_q(\Omega)$.

Compared with subharmonic and plurisubharmonic functions in potential theory and pluripotential theory, it is easy to see that 1-subharmonic functions are plurisubharmonic and n -subharmonic functions are subharmonic.

The following basic properties of q -subharmonic functions can be proved in the same way as for subharmonic functions.

Proposition 1 Let Ω be an open set in $\mathbb{C}^n$ and $1 \leq q \leq n$. Then the following hold:

1. $SH_q(\Omega)$ is a convex cone.
2. If $\{u_\alpha\}$, $\alpha \in A$ is a family of q -subharmonic functions and $u = \sup_{\alpha \in A} u_\alpha < +\infty$, u is upper semi-continuous then u is a q -subharmonic function.
3. If $\{u_j\}_{j=1}^\infty$ is a decreasing sequence of q -subharmonic functions then so is $u = \lim_{j \rightarrow +\infty} u_j$.
4. If u is a q -subharmonic function in Ω then $u_\varepsilon := u * \varrho_\varepsilon$ is smooth q -subharmonic in Ω_ε , where $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$ and $\varrho_\varepsilon = \varrho(z/\varepsilon)/|\varepsilon|^{2n}$, ϱ is a nonnegative smooth function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \varrho dV_n = 1$. Moreover, $u * \varrho_\varepsilon$ decreasingly tends to u when $\varepsilon \downarrow 0$.
5. If χ is a convex increasing function in $\mathbb{R}$ and u is q -subharmonic in Ω then so is $\chi \circ u$.
6. If u is a q -subharmonic function then for any unitary change of coordinates $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$, the function $u \circ \varphi \in SH_q(\Omega)$.

7. Let u be a q -subharmonic function in Ω and G relatively compact open subset of Ω and $v \in SH_q(G)$ satisfying $\lim_{G \ni \xi \rightarrow z \in \partial G} v(\xi) \leq u(z)$ for all $z \in \partial G$. Then the function

$$w(z) = \begin{cases} \max(u(z), v(z)), & z \in G, \\ u(z), & z \in \Omega \setminus G \end{cases}$$

is q -subharmonic on Ω .

Now we give the following.

Proposition 2 Let u be an upper-semicontinuous function on $\Omega \subset \mathbb{C}^n$ and $u \in L^1(\Omega, \text{loc})$, where $L^1(\Omega, \text{loc})$ denotes the set of locally integrable functions on Ω . Then the following statements are equivalent:

- (a) u is a q -subharmonic function in Ω .
- (b) $i \partial \bar{\partial} u \wedge \omega^{q-1} \geq 0$ in the sense of currents, where $\omega := i \partial \bar{\partial} |z|^2$.

In particular, if $u \in C^2(\Omega)$ then $u \in SH_q(\Omega)$ if and only if its complex Hessian has the sum of q smallest eigenvalues nonnegative at each point.

Proof First, we assume that $u \in C^2(\Omega)$. Let $u \in SH_q(\Omega)$ and $z_0 \in \Omega$. We can choose a system of coordinates $(z_1, \dots, z_n)$ of $\mathbb{C}^n$ such that the Hessian $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0))$ is diagonal. Assume that $L \subset \mathbb{C}^n$ is a q -dimension complex plane of $\mathbb{C}^n$ with $z_0 \in L$. Then by the hypothesis $u|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$ and so $\sum_{k \in K} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k}(z_0) \geq 0$ for all $K = \{k_1, k_2, \dots, k_q\} \subset \{1, 2, \dots, n\}$. It follows that $i \partial \bar{\partial} u(z_0) \wedge \omega^{q-1} \geq 0$. Hence, $i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0$ for all $z \in \Omega$.

Conversely, assume that $i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0$ for all $z \in \Omega$. Let L be a q -dimensional subspace of $\mathbb{C}^n$. Since $i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0$, $z \in \Omega$ it follows that $u \in SH(\Omega \cap L)$. Hence, $u \in SH_q(\Omega)$. Thus the conclusion is true in the case $u \in C^2(\Omega)$.

Assume that u is as in the statement of the proposition. By putting $u_\varepsilon = u * \varrho_\varepsilon$ and applying the above results to u_ε , we obtain the desired conclusion. $\square$

Example 1 We give an example of a q -subharmonic function which is not plurisubharmonic. Let $d > 1$ and $1 < q \leq d$. Consider the function

$$\varphi(z) = |z|^2 - q|z_1|^2 = \sum_{j=1}^d |z_j|^2 - q|z_1|^2, \quad z \in \mathbb{C}^d.$$

It is easy to see that $\sum_{j=1}^q \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j}(z) = 0$ and by (b) of Proposition 2 it follows that φ is q -subharmonic. However, φ is not plurisubharmonic. Indeed, let $\ell = \{(z_1, 0, \dots, 0)\} \subset \mathbb{C}^d$ be a complex line. Then $\varphi|_\ell = (1 - q)|z_1|^2$ is not subharmonic, and the desired conclusion follows.

3 Maximal q -Subharmonic Functions

The following definition is similar as in the situation of maximal plurisubharmonic functions presented in [2] and [7].

Definition 3 A function $u \in SH_q(\Omega)$ is said to be *maximal q -subharmonic* if for every $v \in SH_q(\Omega)$, $v \leq u$ outside a compact subset of Ω implies that $v \leq u$ in Ω .

The set of all maximal q -subharmonic functions in Ω is denoted by $MSH_q(\Omega)$.

We give the following.

Proposition 3 Let Ω be an open subset in $\mathbb{C}^n$ and let $u \in SH_q(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$. Then the following conditions are equivalent:

- (a) $u \in MSH_q(\Omega)$.
- (b) $u + g \in MSH_q(\Omega)$ for all pluriharmonic functions g in Ω .
- (c) For every open subset $G \Subset \Omega$ and every $v \in SH_q(G)$ such that $\limsup_{G \ni w \rightarrow z \in \partial G} (v - u)(w) \leq 0$ it follows that $v \leq u$ in G .
- (d) For every open subset $G \Subset \Omega$ and every $v \in SH_q(\Omega)$ we have

$$\sup_G (v - u) \leq \sup_{\Omega \setminus G} (v - u). \quad (1)$$

Proof From Definition 3, we infer that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) is obvious. We prove that (a) $\Leftrightarrow$ (d). Assume that $u \in MSH_q(\Omega)$. Let $G \Subset \Omega$ and $v \in SH_q(\Omega)$. Put $M = \sup_{\Omega \setminus G} (v - u)$. If $M = +\infty$ then (1) holds. Now assume that $M < +\infty$. Then $v - M \in SH_q(\Omega)$ and $v - M \leq u$ on $\Omega \setminus G$. Hence, by (a), we get $v - M \leq u$ on Ω and (1) follows.

Conversely, let $v \in SH_q(\Omega)$ and $G \Subset \Omega$ such that $v \leq u$ on $\Omega \setminus G$. Take an open subset G_1 such that $G \Subset G_1 \Subset \Omega$. We have $v \in SH_q(G_1)$. By (d), we have

$$\sup_G (v - u) \leq \sup_{G_1 \setminus G} (v - u) \leq 0,$$

and the desired conclusion follows. $\square$

Now we are in a position to prove the local property of maximal q -subharmonic functions. Namely, we have the following.

Theorem 1 Let $\Omega \subset \mathbb{C}^n$ be an open set, q be an integer with $1 \leq q \leq n$ and $u \in SH_q(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$. Then u is maximal q -subharmonic if and only if u is local maximal q -subharmonic in Ω (i.e., for every $z \in \Omega$ there is an open neighborhood $V_z \subset \Omega$ of z such that $u|_{V_z}$ is maximal q -subharmonic on V_z).

Proof The proof of the necessity is obvious. Now we give the proof of the sufficiency. Assume that $G \Subset \Omega$ and v is a q -subharmonic function on Ω such that $v \leq u$ on $\Omega \setminus G$. We have to prove that $v \leq u$ on G . Choose $z^j \in \Omega$, $j = 1, \dots, m$ and open subsets $K_j \Subset V_{z^j} \Subset \Omega$, $j = 1, \dots, m$ such that $z^j \in V_{z^j}$, $G \subset \bigcup_{j=1}^m K_{z^j}$ and u is maximal q -subharmonic on V_{z^j} for all $j = 1, \dots, m$.

We split the proof into two steps.

Step 1. We prove that if K_j , $j = 1, \dots, m$ are open subsets in Ω and $\tilde{K}_j$, $j = 1, \dots, m$ are relatively compact open subsets such that $K_j \Subset \tilde{K}_j \Subset V_j \Subset \Omega$, $j = 1, \dots, m$ then the following inequality holds

$$\sup_{\bigcup_{j=1}^m K_j} (v - u) \leq \sup_{\bigcup_{j=1}^m \tilde{K}_j} (v - u) \leq \sup_{\bigcup_{j=1}^m V_j \setminus \bigcup_{j=1}^m K_j} (v - u). \quad (2)$$

It suffices to prove that (2) holds for $m = 2$. Let $V = V_1 \cup V_2$. To get a contradiction, without loss of generality we may assume that

$$\sup_{V \setminus K_1 \cup K_2} (v - u) < 0 < \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u) < +\infty.$$

For $\varepsilon > 0$, put $u_{(\varepsilon)}(z) = u(z) - \varepsilon|z|^2$, $z \in V$ and choose ε sufficiently small such that

$$\sup_{V \setminus K_1 \cup K_2} (v - u_{(\varepsilon)}) < 0 < \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) < +\infty.$$

Then there exists a sequence $\{p_j\} \subset \tilde{K}_1 \cup \tilde{K}_2$, $p_j \rightarrow p \in \tilde{K}_1 \cup \tilde{K}_2$ such that

$$\sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) = \lim_j (v - u_{(\varepsilon)})(p_j).$$

Now we consider the following two cases.

Case 1. $p \notin \overline{K}_1 \cup \overline{K}_2$. Then $p \in (V \setminus \overline{K}_1) \cap (V \setminus \overline{K}_2)$. Hence there exists j_0 such that $p_j \in (V \setminus \overline{K}_1) \cap (V \setminus \overline{K}_2)$ for every $j \geq j_0$. It follows that

$$\begin{aligned} 0 &< \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) = \lim_j (v - u_{(\varepsilon)})(p_j) \leq \sup_{(V \setminus \overline{K}_1) \cap (V \setminus \overline{K}_2)} (v - u_{(\varepsilon)}) \\ &\leq \sup_{(V \setminus K_1) \cap (V \setminus K_2)} (v - u_{(\varepsilon)}) = \sup_{V \setminus (K_1 \cup K_2)} (v - u_{(\varepsilon)}) < 0, \end{aligned}$$

and we get a contradiction.

Case 2. $p \in \overline{K}_1 \cup \overline{K}_2$. We may assume that $p \in \overline{K}_1 \subseteq \tilde{K}_1$. We choose balls $\tilde{\mathbb{B}} = \mathbb{B}(p, \frac{r}{2}) \subset \mathbb{B}_0 = \mathbb{B}(p, r) \subset \mathbb{B}_1 = \mathbb{B}(p, r_1) \Subset \mathbb{B}_2 = \mathbb{B}(p, r_2) \Subset \tilde{K}_1$. Moreover, we may assume that $p_j \in \tilde{\mathbb{B}}$ for all $j \geq 1$. For all j we can write

$$u_{(\varepsilon)}(z) = u(z) - \varepsilon|z|^2 = u(z) + \varepsilon|p_j|^2 - 2\varepsilon \operatorname{Re}\langle z, p_j \rangle - \varepsilon|z - p_j|^2.$$

Note that since $\operatorname{Re}\langle z, p_j \rangle$ is pluriharmonic and u is a maximal q -subharmonic function in V_1 , hence, by (b) of Proposition 3, $u_{(\varepsilon)}(z) + \varepsilon|z - p_j|^2 = u(z) + \varepsilon|p_j|^2 - 2\varepsilon \operatorname{Re}\langle z, p_j \rangle$ is maximal q -subharmonic in V_1 . From (d) of Proposition 3 we get

$$(v - u_{(\varepsilon)})(p_j) \leq \sup_{z \in \mathbb{B}_2 \setminus \overline{\mathbb{B}}_1} (v(z) - u_{(\varepsilon)}(z) - \varepsilon|z - p_j|^2).$$

On the other hand, because $z \in \mathbb{B}_2 \setminus \overline{\mathbb{B}}_1$, it follows that $|z - p_j|^2 \geq \frac{r^2}{4}$ for all $j \geq 1$. Hence $-u_{(\varepsilon)}(z) - \varepsilon|z - p_j|^2 \leq -u_{(\varepsilon)}(z) - \varepsilon \frac{r^2}{4}$ for all $z \in \mathbb{B}_2 \setminus \overline{\mathbb{B}}_1$ and for all $j \geq 1$. It follows that

$$\begin{aligned} (v - u_{(\varepsilon)})(p_j) &\leq \sup_{z \in \mathbb{B}_2 \setminus \overline{\mathbb{B}}_1} (v(z) - u_{(\varepsilon)}(z) - \varepsilon|z - p_j|^2) \leq \sup_{z \in \mathbb{B}_2 \setminus \overline{\mathbb{B}}_1} (v - u_{(\varepsilon)})(z) - \frac{\varepsilon r^2}{4} \\ &\leq \sup_{\tilde{K}_1} (v - u_{(\varepsilon)}) - \frac{\varepsilon r^2}{4} \leq \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) - \frac{\varepsilon r^2}{4}. \end{aligned}$$

Letting $j \rightarrow \infty$, we infer that

$$\sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) \leq \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) - \frac{\varepsilon r^2}{4},$$

and we get a contradiction. Hence, (2) is proved.

Step 2. Let $x_0 \in G$. We need to prove

$$v(x_0) \leq u(x_0). \quad (3)$$

By Step 1, we get

$$\begin{aligned} (v - u)(x_0) &\leq \sup_G (v - u) \leq \sup_{\bigcup_{j=1}^m K_{z_j}} (v - u) \\ &\leq \sup_{(\bigcup_{j=1}^m V_{z_j}) \setminus (\bigcup_{j=1}^m K_{z_j})} (v - u) \leq \sup_{\Omega \setminus G} (v - u) \leq 0. \end{aligned}$$

This shows that (3) is true and the proof is complete. $\square$

From the above theorem we get the following useful corollary.

Corollary 1 Assume that $u \in \mathcal{C}(\Omega)$. Then $u \in MSH_q(\Omega)$ if and only if for every open subset $G \Subset \Omega$ and every $v \in SH_q(G) \cap C^2(G)$ the following holds

$$\sup_K (v - u) \leq \sup_{G \setminus K} (v - u), \quad (4)$$

where $K \Subset G$ is an arbitrary relatively compact open subset of G .

Proof From Theorem 1 and (d) of Proposition 3, it follows that the necessity is clear. Thus it suffices to prove that if (4) holds then $u \in MSH_q(\Omega)$. Let $v \in SH_q(\Omega)$, $G \Subset \Omega$. By (d) of Proposition 3, we have to prove

$$\sup_G (v - u) \leq \sup_{\Omega \setminus G} (v - u).$$

To get a contradiction, we assume that

$$\sup_{\Omega \setminus G} (v - u) < 0 < \delta < \sup_G (v - u).$$

Choose open subsets G_1, G_2 of Ω such that $G \Subset G_1 \Subset G_2 \Subset \Omega$. Since $u \in \mathcal{C}(\Omega)$ so we have $\sup_{G_2} |u_\varepsilon - u| < \delta$ for all ε sufficiently small, where $u_\varepsilon = u * \varrho_\varepsilon$. Moreover, because $v - u < 0$ on $\Omega \setminus G$, so we have

$$v_\varepsilon < u_\varepsilon < u + \delta \quad \text{on } G_2 \setminus G_1$$

for all ε sufficiently small. Hence by (4) we have

$$\begin{aligned} \sup_G (v - u) &\leq \sup_G (v_\varepsilon - u) \leq \sup_{G_1} (v_\varepsilon - u) \\ &\leq \sup_{G_2 \setminus G_1} (v_\varepsilon - u) \leq \delta < \sup_G (v - u), \end{aligned}$$

and we get a contradiction. Thus,

$$\sup_G (v - u) \leq \sup_{\Omega \setminus G} (v - u),$$

and Proposition 3 implies that $u \in MSH_q(\Omega)$. The proof is complete. $\square$

Compared to Proposition 1.4.9 in [2] we have the following.

Corollary 2 Assume that $\{u_j\}_{j=1}^\infty$ is a decreasing sequence of maximal q -subharmonic functions in Ω . Then $u = \lim_{j \rightarrow \infty} u_j$ either is a maximal q -subharmonic function or $\equiv -\infty$ on Ω .

Proof Assume that $u \not\equiv -\infty$. By (d) of Proposition 3, it is enough to prove that for every open subset $G \Subset \Omega$ and every $v \in SH_q(\Omega)$ we have

$$\sup_G(v - u) \leq \sup_{\Omega \setminus G}(v - u). \quad (5)$$

Let $\{b_k\}_{k=1}^\infty \subset G$ be such that $\sup_G(v - u) = \lim_{k \rightarrow \infty} (v - u)(b_k)$. Fix a k . Since $u_j \in MSH_q(\Omega)$ so by (d) of Proposition 3 we have

$$(v - u_j)(b_k) \leq \sup_G(v - u_j) \leq \sup_{\Omega \setminus G}(v - u_j) \leq \sup_{\Omega \setminus G}(v - u)$$

for every $j = 1, 2, \dots$. Letting $j \rightarrow \infty$, we get

$$(v - u)(b_k) \leq \sup_{\Omega \setminus G}(v - u).$$

Hence we have

$$\sup_G(v - u) = \lim_{k \rightarrow \infty} (v - u)(b_k) \leq \sup_{\Omega \setminus G}(v - u).$$

Thus (5) is proved and the desired conclusion follows. $\square$

The following fact is well known.

Lemma 1 Assume that $A = (a_{jk})_{j,k=1}^n$ is a complex $n \times n$ -matrix such that $A = \overline{A}^t$. Put $B = (a_{jk})_{j,k=1}^{n-1}$. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of A and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-1}$ the eigenvalues of B . Then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n.$$

In particular, if the matrix B has one nonnegative eigenvalue then so does the matrix A .

Proof Without loss of generality, we may assume that $B = \text{diag}(\mu_1, \dots, \mu_{n-1})$ and

$$A = \begin{bmatrix} \mu_1 & 0 & \dots & 0 & \overline{x}_1 \\ 0 & \mu_2 & \dots & 0 & \overline{x}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mu_{n-1} & \overline{x}_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & b \end{bmatrix}.$$

First, we prove that

$$\det A = \mu_1 \mu_2 \dots \mu_{n-1} b - \sum_{j=1}^{n-1} \mu_1 \dots \widehat{\mu_j} \dots \mu_{n-1} |x_j|^2. \quad (6)$$

Indeed, we have

$$\begin{aligned} \det A &= \mu_1 \begin{vmatrix} \mu_2 & \dots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_{n-1} & \bar{x}_{n-1} \\ x_2 & \dots & x_{n-1} & b \end{vmatrix} + (-1)^n x_1 \begin{vmatrix} 0 & \dots & 0 & \bar{x}_1 \\ \mu_2 & \dots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_{n-1} & \bar{x}_{n-1} \end{vmatrix} \\ &= \mu_1 \begin{vmatrix} \mu_2 & \dots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_{n-1} & \bar{x}_{n-1} \\ x_2 & \dots & x_{n-1} & b \end{vmatrix} - |x_1|^2 \mu_2 \cdots \mu_{n-1}. \end{aligned}$$

Hence by induction we get

$$\det A = \mu_1 \mu_2 \cdots \mu_{n-1} b - \sum_{j=1}^{n-1} \mu_1 \cdots \widehat{\mu_j} \cdots \mu_{n-1} |x_j|^2.$$

Thus, (6) follows.

Now consider the equation $f(\lambda) = \det(A - \lambda I) = 0$. By (6), we get

$$f(\lambda) = \prod_{j=1}^{n-1} (\mu_j - \lambda)(b - \lambda) - \sum_{j=1}^{n-1} (\mu_1 - \lambda) \cdots \widehat{\mu_j} \cdots (\mu_{n-1} - \lambda) |x_j|^2.$$

Since $f(\mu_j) \cdot f(\mu_{j+1}) \leq 0$ for all $j = 1, 2, \dots, n-2$, and moreover $f(-\infty) \cdot f(\mu_1) \leq 0$, $f(\mu_{n-1}) \cdot f(+\infty) \leq 0$, hence there exist n solutions λ_j , $j = 1, 2, \dots, n$ of the equation $f(\lambda) = 0$ such that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

The proof is complete. $\square$

We need the following fact.

Lemma 2 *Let $u \in C^2(\Omega)$ and assume that its complex Hessian has least one nonnegative eigenvalue at each point. Then for every open subset $G \Subset G_1 \Subset \Omega$ we have*

$$\sup_G u \leq \sup_{G_1 \setminus G} u.$$

Proof From the hypothesis and Lemma 2.6 in [6], it follows that u is an $(n-1)$ -plurisubharmonic function in Ω . The desired conclusion of the lemma follows from Lemma 2.7 in [6]. $\square$

Next we give a condition under which a C^2 q -subharmonic function is maximal q -subharmonic.

Theorem 2 *Let Ω be an open set in $\mathbb{C}^n$ and let q be an integer with $1 \leq q \leq n-1$. Assume that $u \in C^2(\Omega)$. Then $u \in MSH_q(\Omega)$ if and only if its complex Hessian has the sum of q smallest eigenvalues equal to 0 at each point.*

Proof Suppose that we arrange the eigenvalues of $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z))_{j,k=1,\dots,n}$ as follows: $\lambda_1(z) \leq \lambda_2(z) \leq \dots \leq \lambda_n(z)$. We have to prove that $u \in MSH_q(\Omega)$ if and only if $\lambda_1(z) + \lambda_2(z) + \dots + \lambda_q(z) = 0$ for all $z \in \Omega$.

Necessity. Assume that $u \in MSH_q(\Omega)$. Then by Proposition 2 it follows that $\lambda_1(z) + \lambda_2(z) + \dots + \lambda_q(z) \geq 0$ for all $z \in \Omega$. To get a contradiction, we assume that there exists $a \in \Omega$ such that $\lambda_1(a) + \lambda_2(a) + \dots + \lambda_q(a) > 0$. Choose $\varepsilon > 0$ such that

$$\lambda_1(a) + \lambda_2(a) + \dots + \lambda_q(a) > \varepsilon q > 0.$$

Hence there exists $r_\varepsilon > 0$ such that $u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \in SH_q(\mathbb{B}(a, r_\varepsilon))$. It is clear that

$$u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \leq u \quad \text{in } \Omega \setminus \mathbb{B}(a, r_\varepsilon).$$

Since $u \in MSH_q(\Omega)$ so it follows that

$$u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \leq u \quad \text{in } \mathbb{B}(a, r_\varepsilon),$$

and we get a contradiction. Hence $\lambda_1(z) + \lambda_2(z) + \dots + \lambda_q(z) = 0$ for all $z \in \Omega$.

Sufficiency. Assume that $\lambda_1(z) + \lambda_2(z) + \dots + \lambda_q(z) = 0$ for all $z \in \Omega$. By Corollary 1, it is enough to prove that for every open subset $G \Subset \Omega$ and every $v \in SH_q(G) \cap C^2(G)$ we have

$$\sup_K (v - u) \leq \sup_{G \setminus K} (v - u)$$

for all open subsets $K \Subset G$. Moreover, by Lemma 2, it suffices to check that the matrix $(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}(z) - \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z))_{j,k=1}^n$ has at least one nonnegative eigenvalue at each $z \in G$.

Let $a \in G$ and fix a . By a unitary change of coordinates, we can assume that $\lambda_j(a) = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(a)$ and $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) = 0$ for all $j \neq k$. Let $L = a + \mathbb{C}^q \times (0, \dots, 0) \subset \mathbb{C}^n$. Then L is a q -dimensional complex subspace in $\mathbb{C}^n$ containing a . By the hypothesis, we have $v|_{L \cap G} \in SH(L \cap G)$. Since $\Delta u|_{L \cap \Omega}(a) = \sum_{j=1}^q \lambda_j(a) = 0$ it follows that

$$\Delta(v - u)|_{L \cap G}(a) = \Delta v|_{L \cap G}(a) \geq 0.$$

Hence the matrix $(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}(a) - \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a))_{j,k=1}^q$ has at least one nonnegative eigenvalue. Thus, by Lemma 1, the matrix $(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}(a) - \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a))_{j,k=1}^n$ has at least one nonnegative eigenvalue, and the desired conclusion follows. $\square$

Acknowledgements The authors express many thanks to referees for useful remarks which led to the improvement of the exposition of the paper.

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