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Effective anti-plane properties of piezoelectric fibrous composites

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Abstract Anti-plane shear of piezoelectric fibrous composites is theoretically investigated. The geometry of composites is described by the 2-dimensional geometry in a section perpendicular to the unidirectional fibers. The previous constructive results obtained for scalar conductivity problems are extended to piezoelectric anti-plane problems. First, the piezoelectric problem is written in the form of the vector-matrix $\mathbb{R}$ -linear problem in a class of double periodic functions. In particular, application of the zeroth-order solution to the $\mathbb{R}$ -linear problem yields a vector-matrix extension of the famous Clausius–Mossotti approximation. The vector-matrix problem is decomposed into two scalar $\mathbb{R}$ -linear problems. This reduction allows us to directly apply all the known exact and approximate analytical results for scalar problems to establish high-order formulae for the effective piezoelectric constants. Special attention is paid to non-overlapping disks embedded in a two-dimensional background.

1 Introduction

We study piezoelectric anti-plane shear of piezoelectric fibrous composites in a stationary electromagnetic field when the electric charge induces the elastic stresses and deformations and vice versa. The geometry of the composites can be described by a two-dimensional geometry in a section perpendicular to the unidirectional fibers parallel to the x_3 -direction.

Different methods have been applied to estimate the effective constants of piezoelectric fibrous composites. The theory of homogenization of piezoelectric composites and exact formulae for the layered piezocomposites were described by Galka, Gambin, Telega and Wojnar (see the series of works [1–7]). Benveniste [8] derived a relation between the effective constants and a phase interchange connection for 2-phase piezoelectric fibrous composites in anti-plane statement. Milgrom and Shtrikman [9] studied the effective properties of a polycrystal made of uniaxial crystals. Schulgasser [10] established the exact relations for overall moduli of piezoelectric composites consisting of two and many transversely isotropic phases. Avellaneda and Swart [11] applied the effective medium theory and extended Hashin–Shtrikman bounds to isotropic polycrystals. They obtained expressions for the effective moduli in the effective medium approximations wherein each grain behaves like a sphere surrounded by a homogenized medium. Dunn and Taya [12] calculated the effective tensor of the piezoelectric 2-phase composites by application of various self-consistent theories. The effective medium approximations were developed out by Adler and Mityushev [13] in matrix form and applied to electrokinetic phenomena in porous media. The obtained results are valid for piezoelectric composites because of a common mathematical background. The application of self-consistent methods was extended by Levin et al. [14] to

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2-dimensional piezoelectric composites. One of the methods [14], based on the effective field, gave formulae for the effective constants in explicit form; the other, the effective medium, gave implicit formulae.

Duality transformations were systematically applied in [13] to vector-matrix problems. The authors imposed conditions on the statistical properties of the local quantities in order to deduce exact explicit and implicit formulae when media are described by continuous local laws. These conditions and formulae are analogous to Matheron [15] and Dykhne [16] ones deduced for scalar problems. It is worth noting that formulae obtained in the framework of the 2-dimensional theory of duality transformations [13, 15, 16] are exact, and formulae deduced by self-consistent methods [11–14] are approximate and hold only for dilute and weakly inhomogeneous composites. The theory of the self-consistent methods was systematically discussed by Kan-aun and Levin [17]; its limitations were analyzed in [18]. A method of integral equations for doubly periodic problems including anisotropic piezoelectric composites was developed in [19–21]. A rigorous mathematical study of the piezoelectric boundary value problems based on the theory of singular integral equations was presented in [22, 23].

Many results are based on the method proposed by Rayleigh [24] and developed by McPhedran et al. [25–28] for regular lattices. Infinite systems of linear algebraic equations for the multipole coefficients of complex potentials were derived and truncated to get various low-order formulae of the effective conductivity. The method of Rayleigh was extended to elastic problems by Filshtinskii [20, 29] and further by McPhedran et al. [30–33]. Mityushev et al. [34–39] extended the latter results for conductivity to arbitrary locations of disks per periodicity cell by a method of functional equations. High-order approximate analytical formulae for the effective conductivity were derived and exact formulae for regular arrays were deduced (see the survey [37]).

Guinovart et al. [40] (see also papers cited therein) extended the method of Rayleigh to piezoelectric fibrous materials. It was declared in [40] that closed-form analytical formulae are given for the effective properties. By *closed-form analytical formulae*, these authors mean the effective constants presented in a form that explicitly contains some parameters. These parameters can be numerically computed for fixed regular arrays by the truncation of infinite systems of linear algebraic equations following the method of Rayleigh. In the present paper, the terms *exact formulae* and *approximate analytical formulae* are used in a commonly accepted meaning. In particular, an effective constant σ_e is given in such a form that only σ_e is in the left part of the formula and the right part includes all the given geometrical and physical parameters explicitly in symbolic form, for instance, centers and radii of inclusions. Hence, contrary to [40], we have no any parameter in the right parts of our formulae that should be separately computed by a hidden numerical procedure.

This work is devoted to the extension of the constructive results [34–39] to piezoelectric anti-plane problems. First, the problem is written in the form of the vector-matrix $\mathbb{R}$ -linear problem [41, 42] in a class of doubly periodic functions. Application of the zeroth-order approximation yields a vector-matrix extension of the famous Clausius–Mossotti approximation (67). This result is in agree with [13] and [14] obtained in another form. The vector-matrix problem is decomposed into two scalar $\mathbb{R}$ -linear problems. This reduction allows us to directly apply all the exact and approximate analytical formulae obtained before for scalar problems [30–39]. In particular, coupled anti-plane problems presented in [40] are reduced to scalar problems. This implies that truncated systems of order N^2 for numerical solution used in [40] are reduced to systems of order N by a simple linear transformation. It has to be noted that the direct application is performed only when the physical parameter Δ defined by (74) is positive. In the case $\Delta \leq 0$, an extension of the methods [30–39] has to be made.

2 Local equations and complex potentials

The interaction of the elastic and electric fields is local in the framework of the theory of continuum and can be modeled by local partial differential equations. Let $\mathbf{E} = (E_1, E_2, E_3)$ be the electric field strength, $\mathbf{D} = (D_1, D_2, D_3)$ the electric displacement vector, σ_{ij} the stresses and $\mathbf{u}$ the elastic displacement. These values are considered as vector-functions of the spatial variables x_j ($j = 1, 2, 3$). Maxwell's equations in stationary electromagnetic problems become equations of electrostatics,

$$\nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = 0, \quad (1)$$

where the free charge density vanishes. Anti-plane problems are stated for fibers parallel to the x_3 -axis when the elastic and electric forces do not depend on x_3 ; the stress tensor has only the nonzero components σ_{13} and σ_{23} , and the displacement has the form $\mathbf{u} = (0, 0, u_3)$.

Let a smooth oriented curve L locally divides the domains G^+ and G^- on the plane (x_1, x_2) occupied by different materials. The electric field strength in G^+ and G^- is expressed through the electrostatic potential $\mathbf{E}(x_1, x_2) = -\nabla\phi(x_1, x_2)$. The electric displacement vector and the stresses satisfy

$$\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \quad (2)$$

The combined effect of the anti-plane deformation and the electric fields in each section of the fiber composite is locally described by the coupled equations [19]

$$\begin{aligned} \sigma_{i3} &= c_{44} \frac{\partial u_3}{\partial x_i} - d_{12} E_i, \\ D_i &= d_{12} \frac{\partial u_3}{\partial x_i} + \epsilon_{11} E_i \quad (i = 1, 2), \end{aligned} \quad (3)$$

where d_{12} stands for the piezoelectric modulus, ϵ_{11} the permittivity and c_{44} the shear modulus.

These equations imply that the functions u_3 and ϕ are harmonic in the domains G^+ and G^- . Therefore, they can be expressed in terms of the complex potentials

$$u_3(z) = \operatorname{Re} \varphi_1(z), \quad \phi(z) = \operatorname{Re} \varphi_2(z), \quad (4)$$

where $z = x_1 + ix_2$ (i denotes the imaginary unit). The functions $\varphi_1(z)$ and $\varphi_2(z)$ are analytic in $G^\pm$ and continuously differentiable in the closures of the considered domains. The following representations take place [19]

$$\sigma_{13} - i\sigma_{23} = \sum_{j=1,2} d_{1j} \varphi_j'(z), \quad E_1 - iE_2 = -\varphi_2'(z), \quad D_1 - iD_2 = \sum_{j=1,2} d_{2j} \varphi_j'(z), \quad (5)$$

where $d_{11} = c_{44}$, $d_{22} = -\epsilon_{11}$.

The main force vector $P_1^{(AB)}$ and the electric flux $P_2^{(AB)}$ along an arc AB have the form [19]

$$\begin{aligned} P_1^{(AB)} &= \int_{AB} \sigma_{n3} ds = -\operatorname{Im} \sum_{j=1,2} d_{1j} \varphi_j(z)|_A^B, \\ P_2^{(AB)} &= \int_{AB} D_n ds = -\operatorname{Im} \sum_{j=1,2} d_{2j} \varphi_j(z)|_A^B, \end{aligned} \quad (6)$$

where $\mathbf{n} = (n_1, n_2)$ is the unit vector to AB , $\sigma_{n3} = \sigma_{13}n_1 + \sigma_{23}n_2$, and $D_n = D_1n_1 + D_2n_2$.

3 Contact conditions and $\mathbb{R}$ -linear problem

Let the mechanical and electric contact between media which occupy the domains G^+ and G^- be perfect. The perfect mechanical contact means that the normal limit stresses and the displacements from the both sides of L coincide:

$$\sigma_{13}^+ n_1 + \sigma_{23}^+ n_2 = \sigma_{13}^- n_1 + \sigma_{23}^- n_2, \quad u_3^+ = u_3^- \text{ on } L. \quad (7)$$

Here, $\mathbf{n} = (n_1, n_2)$ is the outward normal vector to L . The ideal electric contact implies that the jump of the normal component of the electric displacement and the tangent component of the electric field strength vanish:

$$D_n^+ = D_n^-, \quad E_s^+ = E_s^-, \quad (8)$$

where the tangent vector $\mathbf{s} = (-n_2, n_1)$. The superscript “+” is assigned to all the values in the domain G^+ and “−” to G^- .

Using the complex potentials, we write (7) and (8) in the form

$$\operatorname{Im} \sum_{j=1,2} d_{1j}^+ \varphi_j^+(t) = \operatorname{Im} \sum_{j=1,2} d_{1j}^- \varphi_j^-(t), \quad (9)$$

$$\operatorname{Re} \varphi_1^+(t) = \operatorname{Re} \varphi_1^-(t), \quad t \in L \quad (10)$$

and

$$\operatorname{Im} \sum_{j=1,2} d_{2j}^+ \varphi_j^+(t) = \operatorname{Im} \sum_{j=1,2} d_{2j}^- \varphi_j^-(t), \quad (11)$$

$$\operatorname{Re} \varphi_2^+(t) = \operatorname{Re} \varphi_2^-(t), \quad t \in L, \quad (12)$$

where

$$\varphi_j^+(t) = \lim_{t \rightarrow z \in G^+} \varphi_j^+(z), \quad \varphi_j^-(t) = \lim_{t \rightarrow z \in G^-} \varphi_j^-(z). \quad (13)$$

Equations (9), (11) and (12) are obtained from the first Eqs. (7) and (8) by integration along L [41].

Introduce the vector-functions

$$\Phi^\pm(z) = \begin{pmatrix} \varphi_1^\pm(z) \\ \varphi_2^\pm(z) \end{pmatrix}, \quad (14)$$

the two real relations (10) and (12) become

$$\operatorname{Re} \Phi^+(t) = \operatorname{Re} \Phi^-(t), \quad t \in L. \quad (15)$$

The conditions (9) and (11) can be written in the form

$$\operatorname{Im} \mathbf{D}_+ \Phi^+(t) = \operatorname{Im} \mathbf{D}_- \Phi^-(t), \quad t \in L, \quad (16)$$

where

$$\mathbf{D}_\pm = \begin{pmatrix} d_{11}^\pm & d_{12}^\pm \\ d_{12}^\pm & d_{22}^\pm \end{pmatrix} = \begin{pmatrix} c_{44}^\pm & d_{12}^\pm \\ d_{12}^\pm & -\epsilon_{11}^\pm \end{pmatrix}. \quad (17)$$

The two real vector-matrix conditions (15) and (16) can be cast in the complex vector form as

$$\Phi^+(t) = \frac{1}{2} \mathbf{D}_+^{-1} (\mathbf{D}_- + \mathbf{D}_+) \Phi^-(t) - \frac{1}{2} \mathbf{D}_+^{-1} (\mathbf{D}_- - \mathbf{D}_+) \overline{\Phi^-(t)}, \quad t \in L. \quad (18)$$

The matrix $\mathbf{D}_+$ is invertible since its determinant $-(c_{44}\epsilon_{11} + d_{12}^2)$ is negative.

It is convenient to introduce the vector-function

$$\Omega^-(z) = \frac{1}{2} \mathbf{D}_+^{-1} (\mathbf{D}_- + \mathbf{D}_+) \Phi^-(z), \quad z \in G^- \quad (19)$$

and the normalized matrix

$$\mathbf{R} = \mathbf{D}_+^{-1} (\mathbf{D}_- - \mathbf{D}_+) (\mathbf{D}_- + \mathbf{D}_+)^{-1} \mathbf{D}_+ = -(\mathbf{I} - \mathbf{D}_+^{-1} \mathbf{D}_-) (\mathbf{I} + \mathbf{D}_+^{-1} \mathbf{D}_-)^{-1}. \quad (20)$$

Then, (18) takes the form of the following $\mathbb{R}$ -linear conjugation condition [41]:

$$\Phi^+(t) = \Omega^-(t) - \mathbf{R} \overline{\Omega^-(t)}, \quad t \in L. \quad (21)$$

Introduce the vector-functions

$$\Psi^+(z) = \frac{d}{dz} \Phi^+(z), \quad z \in G^+, \quad \Psi^-(z) = \frac{d}{dz} \Omega^-(z), \quad z \in G^-. \quad (22)$$

Differentiate the conditions (21) along the curve L on the tangent vector $\mathbf{s}$. Then, (21) becomes (see analogous scalar manipulations in [41])

$$\Psi^+(t) = \Psi^-(t) + \overline{n(t)}^2 \mathbf{R} \overline{\Psi^-(t)}, \quad t \in L. \quad (23)$$

Here, the unit outward normal vector $\mathbf{n} = (n_1, n_2)$ is presented as the complex value $n(t) = n_1 + in_2$ when t belongs to L .

Equation (6) can be written in the vector-matrix form as

$$\mathbf{P}^{(AB)} = \begin{pmatrix} P_1^{(AB)} \\ P_2^{(AB)} \end{pmatrix} = -\operatorname{Im} \mathbf{D}_+ \Phi^+(z)|_A^B, \quad (24)$$

where for definiteness it is assumed that the arc AB lies in G^+ .

4 Double periodic statement

We consider a 2-dimensional lattice $\mathcal{Q}$ defined by two fundamental translation vectors ω_1 and ω_2 expressed in terms of complex numbers. Let $\omega_1 > 0$ and $\text{Im } \omega_2 > 0$ as it is usually assumed in the theory of elliptic functions [43]. Let the zeroth cell $Q_{(0,0)}$ be the parallelogram determined by the vertices $\pm \frac{1}{2}\omega_j$ ($j = 1, 2$). The lattice $\mathcal{Q}$ consists of the cells $Q_{(m_1, m_2)} = Q_{(0,0)} + m_1\omega_1 + m_2\omega_2$, where m_1 and m_2 run over the set $\mathbb{Z}$ of integer numbers. Let mutually disjoint domains G_k ($k = 1, 2, \dots, N$) be located in $Q_{(0,0)}$. The union of all the inclusions forms the non-connected domain $G^- = \bigcup_{k=1}^N G_k$. The multiply connected domain G^+ is defined as the complement of $G^- \cup \partial G^-$ to the open parallelogram $Q_{(0,0)}$, i.e., $G^+ = Q_{(0,0)} \setminus \bigcup_{k=1}^N (G^- \cup \partial G^-)$, where ∂G^- denotes the boundary of G^- .

We study the electrical and elastic fields in the doubly periodic composites when the domains G^+ and G^- are occupied by piezoelectric materials whose properties are described by the matrices (17). It is convenient to introduce the complex potentials $\Omega_k(z) = \Omega(z)$ and $\Psi_k(z) = \Psi(z)$ in G_k ($k = 1, 2, \dots, N$). The complex potential $\Psi_0(z) \equiv \Psi^+(z)$ is analytic in G^+ and doubly periodic, i.e., $\Psi_0(z) = \Psi_0(z + m_1\omega_1 + m_2\omega_2)$ for all $m_1, m_2 \in \mathbb{Z}$. The complex potential $\Phi_0(z) \equiv \Phi^+(z)$ is quasiperiodic, i.e.,

$$\Phi_0(z + \omega_j) - \Phi_0(z) = C_j \quad (j = 1, 2), \quad (25)$$

where $C_j = (C_{1j}, C_{2j})^T$ are constant vectors that model the external piezoelectric field applied to the composite in the following way. Let A and B from (6) have the complex coordinates z and $z + \omega_j$, respectively. Then, (25) yields

$$P_{\ell j} := P_{\ell}^{(z, z+\omega_j)} = -\text{Im} \sum_{m=1,2} d_{\ell m}^+ C_{mj} \quad (\ell, j = 1, 2). \quad (26)$$

It can be also written in the form

$$\mathbf{P}_j^{(AB)} = -\text{Im } \mathbf{D}_+ C_j \quad (j = 1, 2), \quad (27)$$

where (24) is used. The constants $P_{\ell j}$ can be expressed in terms of the average stresses and induction. First, introduce the average value of a doubly periodic function $f(x_1, x_2)$ over the periodicity cell $Q_{(0,0)}$ as the double integral

$$\langle f \rangle = \frac{1}{|Q_{(0,0)}|} \int_{Q_{(0,0)}} f(x_1, x_2) dx_1 dx_2, \quad (28)$$

where $|Q_{(0,0)}| = \omega_1 \text{Im} \omega_2$ denotes the area of the cell $Q_{(0,0)}$. Periodicity of $f(x_1, x_2)$ implies that the integral (28) can be replaced by the integral over a rectangle Q defined by the vertices $\pm \frac{1}{2}\omega_1 \pm \frac{i}{2}\text{Im} \omega_2$,

$$\langle f \rangle = \frac{1}{|Q|} \int_Q f(x_1, x_2) dx_1 dx_2 = \frac{1}{|Q|} \int_{-\frac{\omega_1}{2}}^{\frac{\omega_1}{2}} dx_1 \int_{-\frac{\text{Im} \omega_2}{2}}^{\frac{\text{Im} \omega_2}{2}} f(x_1, x_2) dx_2. \quad (29)$$

Using (6), consider, for instance, the constant

$$P_{11} = - \int_{-\frac{\omega_1}{2}}^{\frac{\omega_1}{2}} \sigma_{23}(x_1, x_2) dx_1. \quad (30)$$

Integrate (30) by x_2 from $-\frac{\text{Im} \omega_2}{2}$ to $\frac{\text{Im} \omega_2}{2}$ and divide the result by $\text{Im} \omega_2$. In accordance with (28), we obtain

$$P_{11} = -\omega_1 \langle \sigma_{23} \rangle. \quad (31)$$

Similar arguments yield the following:

$$P_{12} = -\text{Im}[\omega_2(\langle \sigma_{13} \rangle - i \langle \sigma_{23} \rangle)], \quad P_{21} = -\omega_1 \langle D_2 \rangle, \quad P_{22} = -\text{Im}[\omega_2(\langle D_1 \rangle - i \langle D_2 \rangle)]. \quad (32)$$

Let the average values $\langle \sigma_{\ell 3} \rangle$ and $\langle D_{\ell} \rangle$ ($\ell = 1, 2$) be known. Then, the constants $P_{\ell m}$ ($\ell, m = 1, 2$) are calculated by (31)–(32). Thus, we arrive at the following.

Boundary value problem: Given constants $P_{\ell j}$ ($\ell, j = 1, 2$). To find the vector-function $\Phi^+(z)$ analytic in G^+ and continuously differentiable in the closure of G^+ , $\Phi^+(z)$ fulfills the quasiperiodicity conditions (25), where the constants $C_{\ell j}$ satisfy (26). To find the vector-functions $\Omega_k(z)$ analytic in G_k ($k = 1, 2, \dots, N$) and continuously differentiable in $G_k \cup \partial G_k$, the boundary values of these vector-functions satisfy the $\mathbb{R}$ -linear condition

$$\Phi_0(t) = \Omega_k(t) - \mathbf{R} \overline{\Omega_k(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, N). \quad (33)$$

This boundary value problem can be considered as a vector-matrix generalization of the scalar problem discussed in [44]. This problem in the theory of composites provides the following standard method to compute the effective piezoelectric constants [19]. First, the averaged values $\langle \sigma_{\ell 3} \rangle$ and $\langle D_{\ell} \rangle$ ($\ell = 1, 2$) are fixed. After, the constants $P_{\ell j}$ are calculated with (31)–(32) and the problem (33) is solved in a class of quasiperiodic functions. Here, the jumps C_j from (25) satisfy the conditions (26). Then, the averaged deformations $\langle \frac{\partial u_3}{\partial x_{\ell}} \rangle$ and electric fields $\langle E_{\ell} \rangle$ ($\ell = 1, 2$) are computed as linear combinations of $\langle \sigma_{\ell 3} \rangle$ and $\langle D_{\ell} \rangle$. This linear dependence determines the effective piezoelectric tensor $\mathbf{P}$. More precisely, few independent problems should be solved to determine all the components of $\mathbf{P}$. For instance, two problems should be solved for macroscopically isotropic composites.

We propose another method to determine $\mathbf{P}$, which is convenient in symbolic computations. Let the host medium and inclusions be made from the same material. This implies that $\mathbf{D}_+ = \mathbf{D}_-$ and $\mathbf{R} = \mathbf{0}$ by (20). Then, the $\mathbb{R}$ -linear condition (33) becomes the condition of analytic continuation of $\Phi_0(z)$ into G_k ($k = 1, 2, \dots, N$). Therefore, this degenerate case corresponds to the problem when a quasi-periodic vector-function $\Phi_0(z)$ analytic in $Q_{(0,0)}$ has to be found. The components of $\Phi_0(z)$ have to be linear functions $az + b$. It is convenient to take two linearly independent solutions of the degenerate problem,

$$\Phi_0^{(1)}(z) = (z, 0)^T \quad \text{and} \quad \Phi_0^{(2)}(z) = (0, z)^T. \quad (34)$$

Then, (25) yields the following:

$$C_j^{(1)} = \omega_j(1, 0)^T \quad \text{and} \quad C_j^{(2)} = \omega_j(0, 1)^T. \quad (35)$$

Now, we come back to the general condition (33) and solve the problem (33), (25) with the prescribed jumps (35). After, we calculate the averaged fields and symbolically compute $\mathbf{P}$ by the averaged piezoelectric law.

5 Effective elastic constants for piezoelectric problems

This section is devoted to the calculation of the effective piezoelectric tensor $\mathbf{P}$ by the method outlined at the end of the previous section. For simplicity, it is assumed that the inclusions G_k are distributed in such a way that the considered composite is isotropic in macroscale in the plane perpendicular to the fibers. The average equations (3) for macroscopically isotropic composites take the form

$$\begin{aligned} \langle \sigma_{i3} \rangle &= c_{44}^* \langle \frac{\partial u_3}{\partial x_i} \rangle - d_{12}^* \langle E_i \rangle, \\ \langle D_i \rangle &= d_{12}^* \langle \frac{\partial u_3}{\partial x_i} \rangle + \epsilon_{11}^* \langle E_i \rangle \quad (i = 1, 2). \end{aligned} \quad (36)$$

Equation (36) determines the effective piezoelectric tensor $\mathbf{P}$. For macroscopically isotropic composites, $\mathbf{P}$ can be presented by the matrix

$$\mathbf{P} = \begin{pmatrix} c_{44}^* & d_{12}^* \\ d_{12}^* & -\epsilon_{11}^* \end{pmatrix}. \quad (37)$$

The signs in the second column correspond to Eq. (36). They are chosen for convenience of the further calculations. Moreover, it makes the tensor $\mathbf{P}$ symmetric. Equation (36) can be written in vector-matrix form as

$$\begin{pmatrix} \langle \sigma_{i3} \rangle \\ \langle D_i \rangle \end{pmatrix} = \mathbf{P} \begin{pmatrix} \langle \frac{\partial u_3}{\partial x_i} \rangle \\ -\langle E_i \rangle \end{pmatrix} \quad (i = 1, 2). \quad (38)$$

In order to find the components of $\mathbf{P}$, we calculate two sets of the average values from (36) corresponding to two different external fields for $i = 1$. First, using the representations (4) and the definition (14) of the vector-functions $\Phi_k(z)$, we calculate the integral as follows:

$$\begin{pmatrix} \langle \sigma_{13} \rangle \\ \langle D_1 \rangle \end{pmatrix} = \mathbf{D}_+ \int_{G^+} \operatorname{Re} \Phi'_0(z) dx_1 dx_2 + \mathbf{D}_- \sum_{k=1}^n \int_{G_k} \operatorname{Re} \Phi'_k(z) dx_1 dx_2. \quad (39)$$

The vector-functions $\Phi_k(z)$ and $\Omega_k(z)$ are related by the equation (see (19))

$$\Omega_k(z) = \frac{1}{2} \mathbf{D}_+^{-1} (\mathbf{D}_- + \mathbf{D}_+) \Phi_k(z), \quad z \in G_k. \quad (40)$$

In order to transform (39), we use Green's formula as follows:

$$\int_Q \operatorname{Re} w'(z) dx_1 dx_2 = \int_{\partial Q} \operatorname{Re} w(z) dx_2, \quad (41)$$

where $z = x_1 + ix_2$. The double integrals from (39) can be reduced to the linear integrals as

$$\int_{G^+} \operatorname{Re} \Phi'_0(z) dx_1 dx_2 = \int_{\partial Q_{(0,0)}} \operatorname{Re} \Phi_0(t) dx_2 - \sum_{k=1}^n \int_{\partial G_k} \operatorname{Re} \Phi_0(t) dx_2 \quad (42)$$

and

$$\int_{G_k} \operatorname{Re} \Phi'_k(z) dx_1 dx_2 = \int_{\partial G_k} \operatorname{Re} \Phi_k(t) dx_2, \quad (43)$$

where $t = x_1 + ix_2 \in \partial G_k$. The integral on $\partial Q_{(0,0)}$ from (42) can be calculated through the increments (25)

$$\int_{\partial Q_{(0,0)}} \operatorname{Re} \Phi_0(t) dx_2 = \operatorname{Re} C_1 \operatorname{Im} \omega_2. \quad (44)$$

Formula (39) becomes

$$\begin{pmatrix} \langle \sigma_{23} \rangle \\ \langle D_2 \rangle \end{pmatrix} = \mathbf{D}_+ \operatorname{Re} C_1 \operatorname{Im} \omega_2 + \sum_{k=1}^n \int_{\partial G_k} (\mathbf{D}_- \operatorname{Re} \Phi_k(t) - \mathbf{D}_+ \operatorname{Re} \Phi_0(t)) dx_2. \quad (45)$$

Application of (15), (40) and (20) to the latter integrand yields

$$\mathbf{D}_- \operatorname{Re} \Phi_k(t) - \mathbf{D}_+ \operatorname{Re} \Phi_0(t) = 2\mathbf{R} \operatorname{Re} \Omega_k(t). \quad (46)$$

Then, (45) becomes

$$\begin{pmatrix} \langle \sigma_{13} \rangle \\ \langle D_1 \rangle \end{pmatrix} = \mathbf{D}_+ \left[\operatorname{Re} C_1 \operatorname{Im} \omega_2 + 2\mathbf{R} \sum_{k=1}^n J_k \right], \quad (47)$$

where

$$J_k = \int_{\partial G_k} \operatorname{Re} \Omega_k(t) dx_2. \quad (48)$$

Application of similar arguments yields the following:

$$\begin{pmatrix} \langle \frac{\partial u_3}{\partial x_1} \rangle \\ -\langle E_1 \rangle \end{pmatrix} = \operatorname{Re} C_1 \operatorname{Im} \omega_2. \quad (49)$$

Formulae (38), (47) and (49) produce formulae for the effective tensor $\mathbf{P}$ in terms of the integrals (48) in the following way. Substitute two vectors (35) into (47) and (49) instead of C_1 and C_2 . Then, we obtain two vectors

$$\operatorname{Re} C_1^{(1)} \operatorname{Im} \omega_2 = |Q_{(0,0)}|(1, 0)^T \quad (50)$$

and

$$\operatorname{Re} C_1^{(2)} \operatorname{Im} \omega_2 = |Q_{(0,0)}|(0, 1)^T. \quad (51)$$

The corresponding local fields can be averaged and Eq. (38) applied to these fields implies that

$$\mathbf{V} = \mathbf{P}\mathbf{S}, \quad (52)$$

where

$$\mathbf{V} = \begin{pmatrix} \langle \sigma_{13}^{(1)} \rangle & \langle \sigma_{13}^{(2)} \rangle \\ \langle D_1^{(1)} \rangle & \langle D_1^{(2)} \rangle \end{pmatrix}, \quad \mathbf{S} = |Q_{(0,0)}| \mathbf{I}, \quad (53)$$

where $\mathbf{I}$ is the identity matrix. The elements of $\mathbf{V}$ are given by (47) with the constant vectors C_1 and C_2 given by (35). It follows from (52) that

$$\mathbf{P} = |Q_{(0,0)}|^{-1} \mathbf{V}. \quad (54)$$

Consider two $\mathbb{R}$ -linear problems (33) stated at the end of the Sect. 4

$$\Phi_0^{(\ell)}(t) = \Omega_k^{(\ell)}(t) - \overline{\mathbf{R} \Omega_k^{(\ell)}(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, n; \ell = 1, 2) \quad (55)$$

with the quasi-periodicity conditions (25) when the jumps of $\Phi_0^{(\ell)}(z)$ are given by (35)

$$\Phi_0^{(1)}(z + \omega_j) - \Phi_0^{(1)}(z) = \omega_j(1, 0)^T, \quad \Phi_0^{(2)}(z + \omega_j) - \Phi_0^{(2)}(z) = \omega_j(0, 1)^T \quad (j = 1, 2). \quad (56)$$

Here, ℓ is the number of the $\mathbb{R}$ -linear problem ($\ell = 1, 2$); k is the number of the inclusion; and j is the number of the jump condition in each ℓ th problem.

Introduce the vector-functions

$$\Psi_0^{(\ell)}(z) = \frac{d}{dz} \Phi_0^{(\ell)}(z), \quad \Psi_k^{(\ell)}(z) = \frac{d}{dz} \Omega_k^{(\ell)}(z) \quad (57)$$

and the matrices

$$\mathbf{\Psi}_k(z) = \left(\Psi_k^{(1)}(z), \Psi_k^{(2)}(z) \right), \quad k = 1, 2, \dots, n. \quad (58)$$

Differentiation of (55) on a tangent parameter of ∂G_k yields [41]

$$\Psi_0^{(\ell)}(t) = \Psi_k^{(\ell)}(t) + \overline{\mathbf{R} n^2(t) \Psi_k^{(\ell)}(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, n), \quad (59)$$

where $n(t)$ is the normal outward vector to ∂G_k expressed in terms of complex values. Differentiation of (56) implies double periodicity of $\Psi_0^{(\ell)}(z)$. Using (48) and (41), we introduce the matrix

$$\mathbf{J}_k = -\frac{1}{2i} \int_{\partial G_k} (\Omega_k^{(1)}(t), \Omega_k^{(2)}(t)) d\bar{t} = \int_{G_k} \mathbf{\Psi}_k(z) dx_1 dx_2. \quad (60)$$

Then, (54) can be written in the extended form as

$$\mathbf{P} = \mathbf{D}_+ \left(\mathbf{I} + \frac{2}{|Q_{(0,0)}|} \mathbf{R} \sum_{k=1}^n \mathbf{J}_k \right) \quad (61)$$

This formula (61) can be considered as an extension of Mityushev's formula [34, 37, 44] to piezoelectric fiber composites.

Example 1 Let mutually disjoint disks $G_k = \{z \in \mathbb{C} : |z - a_k| < r\}$ ($k = 1, 2, \dots, n$) be located in $Q_{(0,0)}$. The normal vector to the circle $|t - a_k| = r$ has the form $n(t) = \frac{1}{r}(t - a_k)$. Then, (59) becomes

$$\Psi_0^{(\ell)}(t) = \Psi_k^{(\ell)}(t) + \left(\frac{r}{t - a_k}\right)^2 \mathbf{R} \overline{\Psi_k^{(\ell)}(t)}, \quad |t - a_k| = r \quad (k = 1, 2, \dots, n). \quad (62)$$

The integral (60) can be calculated by the mean value theorem for harmonic functions

$$\mathbf{J}_k = \pi r^2 \Psi_k(a_k). \quad (63)$$

Let ν denote the concentration of disks in the periodicity cell $Q_{(0,0)}$

$$\nu = \frac{n\pi r^2}{|Q_{(0,0)}|}. \quad (64)$$

Substitution of (63) into (61) yields the following:

$$\mathbf{P} = \mathbf{D}_+ \left(\mathbf{I} + 2\nu \mathbf{R} \frac{1}{n} \sum_{k=1}^n \Psi_k(a_k) \right). \quad (65)$$

For dilute and weakly inhomogeneous composites [18], the vector-functions $\Psi_k^{(\ell)}(z)$ satisfying (62) are approximated by constant vectors obtained by differentiation of (34). Hence, the matrices $\Psi_k(z)$ are approximated by the unit matrix:

$$\Psi_k(z) \approx \mathbf{I}. \quad (66)$$

Substitution of these approximations into (65) and application of the simple rational approximation in ν yield the following:

$$\mathbf{P} \approx \mathbf{D}_+ (\mathbf{I} + \nu \mathbf{R}) (\mathbf{I} - \nu \mathbf{R})^{-1}. \quad (67)$$

The latter formula can be considered as an extension of the famous Clausius–Mossotti approximation to piezoelectric fiber composites. Direct computations show that the matrix in the right part of (67) is symmetric.

6 High-order approximate analytical formulae

The $\mathbb{R}$ -linear vector-matrix problem (33) is the key of the high-order approximate analytical formulae for the effective piezoelectric tensor $\mathbf{P}$ determined by (61). Scalar $\mathbb{R}$ -linear problems and their application to the effective conductivity tensor were discussed in [34,37,44]. In order to use these constructive results for the scalar problem, one can try to decompose the vector-matrix problem (33) into two scalar ones.

The matrix $\mathbf{R}$ is defined by (20) through two positively determined matrices $\mathbf{D}_+$ and $\mathbf{D}_-$. Let it be presented in the form

$$\mathbf{R} = \mathbf{T} \Lambda \mathbf{T}^{-1}, \quad (68)$$

where $\mathbf{T}$ is a non-singular matrix, and Λ is diagonal

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \quad (69)$$

Introduce the contrast parameters

$$\rho_1 = \frac{d_{11}^- - d_{11}^+}{d_{11}^- + d_{11}^+}, \quad \rho_2 = \frac{d_{22}^- - d_{22}^+}{d_{22}^- + d_{22}^+}, \quad \rho_3 = \frac{d_{12}^- - d_{12}^+}{d_{12}^- + d_{12}^+}, \quad (70)$$

and the constants

$$c = \frac{d_{11}^- + d_{11}^+}{d_{12}^- + d_{12}^+}, \quad e = -\frac{d_{22}^- + d_{22}^+}{d_{12}^- + d_{12}^+}, \quad h = ce. \quad (71)$$

Then, the matrix $\mathbf{R}$ can be written in the form

$$\mathbf{R} = \frac{1}{1+h} \begin{pmatrix} \rho_3 + h\rho_1 & -e(\rho_2 - \rho_3) \\ c(\rho_1 - \rho_3) & \rho_3 + h\rho_2 \end{pmatrix}. \quad (72)$$

It is assumed that the matrix $\mathbf{R}$ is not singular; hence,

$$\det \mathbf{R} = \frac{1}{1+h} (h\rho_1\rho_2 + \rho_3^2) \neq 0. \quad (73)$$

Let the constant

$$\Delta = h[h(\rho_1 - \rho_2)^2 - 4(\rho_1 - \rho_3)(\rho_2 - \rho_3)] \quad (74)$$

be positive. In this case, the matrix $\mathbf{T} = \{t_{m\ell}\}$ is real and has the form

$$\mathbf{T} = \begin{pmatrix} \frac{h(\rho_1 - \rho_2) - \sqrt{\Delta}}{2c(\rho_1 - \rho_3)} & \frac{h(\rho_1 - \rho_2) + \sqrt{\Delta}}{2c(\rho_1 - \rho_3)} \\ 1 & 1 \end{pmatrix}. \quad (75)$$

The inverse matrix $\mathbf{T}^{-1} = \{t_{m\ell}^*\}$ is given by

$$\mathbf{T}^{-1} = \frac{1}{2\sqrt{\Delta}} \begin{pmatrix} -c(\rho_1 - \rho_3) & h(\rho_1 - \rho_2) + \sqrt{\Delta} \\ c(\rho_1 - \rho_3) & -h(\rho_1 - \rho_2) + \sqrt{\Delta} \end{pmatrix}. \quad (76)$$

The real elements of the matrix (69) read as follows:

$$\begin{aligned} \lambda_1 &= \frac{1}{2(1+h)} [h(\rho_1 + \rho_2) + 2\rho_3 - \sqrt{\Delta}], \\ \lambda_2 &= \frac{1}{2(1+h)} [h(\rho_1 + \rho_2) + 2\rho_3 + \sqrt{\Delta}]. \end{aligned} \quad (77)$$

Hence, the similarity relation (68) between $\mathbf{R}$ and Λ holds.

The eigenvalues (77) always satisfy the inequality

$$|\lambda_j| \leq 1 \quad (j = 1, 2). \quad (78)$$

In order to prove this, we consider the equivalent relation

$$|h(\rho_1 + \rho_2) + 2\rho_3 \pm \sqrt{\Delta}| \leq 2(h+1). \quad (79)$$

Further, consider the case $h(\rho_1 + \rho_2) + 2\rho_3 \geq 0$. In the opposite case, the proof is similar. It is sufficient to demonstrate that $h(\rho_1 + \rho_2) + 2\rho_3 + \sqrt{\Delta} \leq 2(h+1)$. The latter inequality for positive Δ given by (74) is equivalent to

$$h^2(\rho_1 - \rho_2)^2 - 4h(\rho_1 - \rho_3)(\rho_2 - \rho_3) \leq [2(h+1 - \rho_3) - h(\rho_1 + \rho_2)]^2. \quad (80)$$

Use of simple transformations yields the following:

$$h(\rho_1 + \rho_2 - \rho_1\rho_2) \leq h + (1 - \rho_3)^2. \quad (81)$$

This inequality follows the form

$$\rho_1 + \rho_2 - \rho_1\rho_2 = 1 - (1 - \rho_1)(1 - \rho_2) \leq 1.$$

Therefore, (78) is proved.

Substitute the representation (68) into (59) and multiply the result by $\mathbf{T}^{-1}$

$$\mathbf{T}^{-1}\Psi_0^{(\ell)}(t) = \mathbf{T}^{-1}\Psi_k^{(\ell)}(t) + \Lambda\mathbf{T}^{-1} \overline{n^2(t)\Psi_k^{(\ell)}(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, n; \ell = 1, 2). \quad (82)$$

Introduce the vector-functions

$$\omega_k^{(\ell)}(z) = \mathbf{T}^{-1}\Psi_k^{(\ell)}(z) \quad (83)$$

and the matrix

$$\boldsymbol{\omega}_k(z) = \left(\omega_k^{(1)}(z), \omega_k^{(2)}(z) \right). \quad (84)$$

Then, (82) becomes

$$\omega_0^{(\ell)}(t) = \omega_k^{(\ell)}(t) + \Lambda \overline{n^2(t) \omega_k^{(\ell)}(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, n; \ell = 1, 2) \quad (85)$$

since the matrix $\mathbf{T}$ is real. The vector-matrix problem (85) can be decoupled onto the scalar problems

$$\omega_0^{(m,\ell)}(t) = \omega_k^{(m,\ell)}(t) + \lambda_m \overline{n^2(t) \omega_k^{(m,\ell)}(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, n; \ell, m = 1, 2), \quad (86)$$

where $\omega_k^{(m,\ell)}(t)$ is the m th coordinate of the vector $\omega_k^{(\ell)}(t)$ (or the element (m, ℓ) of the matrix $\boldsymbol{\omega}_k(t)$).

The method [34, 37, 44] can be applied to the scalar problems (86) since (78) holds. Following Example 1, consider non-overlapping circular inclusions. Then, (86) becomes

$$\omega_0^{(m,\ell)}(t) = \omega_k^{(m,\ell)}(t) + \lambda_m \left(\frac{r}{t - a_k} \right)^2 \overline{\omega_k^{(m,\ell)}(t)}, \quad |t - a_k| = r \quad (k = 1, 2, \dots, n; \ell, m = 1, 2). \quad (87)$$

Introduce the matrix

$$\boldsymbol{\psi} := \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \frac{1}{n} \sum_{k=1}^n \boldsymbol{\Psi}_k(a_k). \quad (88)$$

The effective tensor $\mathbf{P}$ has the form (65) where by (83)–(84)

$$\boldsymbol{\psi} = \mathbf{T} \frac{1}{n} \sum_{k=1}^n \boldsymbol{\omega}_k(a_k). \quad (89)$$

In the zeroth approximation, $\boldsymbol{\Psi}_k(z) \approx \mathbf{I}$ (see (66)). Then, the zeroth approximation for $\boldsymbol{\omega}_k(z)$ is determined by (83)

$$\boldsymbol{\omega}_k(z) \approx \mathbf{T}^{-1}. \quad (90)$$

Therefore, we arrive at four scalar problems (86) ($\ell, m = 1, 2$) with the zeroth approximations

$$\omega_k^{(m,\ell)}(z) \approx t_{m\ell}^*, \quad (91)$$

where $t_{m\ell}^*$ denote the elements of the matrix $\mathbf{T}^{-1}$. The scalar problem (86) and (91) is formally coincided to the conductivity problem with the contrast parameter λ_m and the external flux expressed by the vector $t_{m\ell}^*(1, 0)^T$. The problems (87) and (91) were solved in [34, 37, 44] for arbitrary locations of non-overlapping inclusions $|z - a_k| < r$ ($k = 1, 2, \dots, n$) in the periodicity cell. For macroscopically isotropic composites, the ratio σ_e of the effective conductivity to the conductivity of the host material was written in the form [34, 37]

$$\sigma_e = 1 + 2\lambda_m \nu P(\lambda_m), \quad (92)$$

where the solution of the problem (87) and $P(\lambda_m)$ is related by the equation

$$\frac{1}{n} \sum_{k=1}^n \omega_k^{(m,\ell)}(a_k) = t_{m\ell}^* P(\lambda_m). \quad (93)$$

The function $P(\lambda_m)$ can be explicitly written in terms of the series introduced by Mityushev [36] (called by him the generalized Eisenstein–Rayleigh series).

Substitution of (93) into (89) yields

$$\boldsymbol{\psi} = \begin{pmatrix} t_{11}t_{11}^*P(\lambda_1) + t_{12}t_{21}^*P(\lambda_2) & t_{11}t_{12}^*P(\lambda_1) + t_{12}t_{22}^*P(\lambda_2) \\ t_{21}t_{11}^*P(\lambda_1) + t_{21}t_{12}^*P(\lambda_2) & t_{21}t_{12}^*P(\lambda_1) + t_{22}t_{22}^*P(\lambda_2) \end{pmatrix}.$$

Application of the explicit formulae (75)–(76) to the elements $\psi_{m\ell}$ of the matrix (88) yields

$$\begin{aligned}\psi_{11} &= \frac{1}{2\sqrt{\Delta}} \left\{ [h(\rho_2 - \rho_1) + \sqrt{\Delta}]P(\lambda_1) - [h(\rho_2 - \rho_1) - \sqrt{\Delta}]P(\lambda_2) \right\}, \\ \psi_{12} &= \frac{e}{\sqrt{\Delta}}(\rho_2 - \rho_3)[P(\lambda_1) - P(\lambda_2)], \\ \psi_{21} &= \frac{e}{\sqrt{\Delta}}(\rho_3 - \rho_1)[P(\lambda_1) - P(\lambda_2)], \\ \psi_{22} &= \frac{1}{2\sqrt{\Delta}} \left\{ [h(\rho_2 - \rho_1) + \sqrt{\Delta}]P(\lambda_2) - [h(\rho_2 - \rho_1) - \sqrt{\Delta}]P(\lambda_1) \right\}.\end{aligned}\quad (94)$$

Formula (65) can be written in the form

$$\mathbf{P} = \mathbf{D}_+ (\mathbf{I} + 2\nu \mathbf{R} \boldsymbol{\psi}), \quad (95)$$

Symbolic computations of $P(\lambda)$ for various locations of the disks by the method [34, 37, 44] and substitution of the results into (94)–(95) yield analytical formulae for the effective tensor $\mathbf{P}$. For instance, $P(\lambda)$ can be approximated by [45]

$$P(\lambda) = 1 + \lambda\nu - \lambda^3\nu^3 + \left(\frac{\lambda\nu}{\pi}\right)^2 \left[(1 + 2\lambda\nu)e_{22} - \frac{2\nu}{\pi}(1 + 2\lambda\nu)e_{33} + \frac{3\nu^2}{\pi^2}e_{44} + \left(\frac{\lambda\nu}{\pi}\right)^2 e_{2222} \right] + O(\nu^5), \quad (96)$$

where

$$\begin{aligned}e_{pp} &= \frac{(-1)^p}{n^{p+1}} \sum_{m=1}^n \sum_{\substack{k=1 \\ k \neq m}}^n |E_p(a_k - a_m)|^2 \quad (p = 2, 3, 4), \\ e_{2222} &= \frac{1}{n^5} \sum_{k=1}^n \left| \sum_{\substack{m=1 \\ m \neq k}}^n \sum_{\substack{\ell=1 \\ \ell \neq m}}^n E_2(a_k - a_m) \overline{E_2(a_m - a_\ell)} \right|^2.\end{aligned}\quad (97)$$

Here, $E_p(z)$ denotes the Eisenstein function of order p . An exact formula for the effective conductivity of regular arrays of disks, hence for $P(\lambda)$, is written in [37].

Example 2 Consider a random composite when the disks obey the uniform non-overlapping distribution. Then, $P(\lambda)$ can be approximated up to $O(\nu^6)$ by the following expression [39]:

$$\begin{aligned}P(\lambda) &\approx 1 + \nu\lambda + \nu^2\lambda^2(2.4922 - 0.0344\lambda - 0.07315\lambda^2 - 0.3998\lambda^3) \\ &\quad + \nu^3\lambda^2(-3.4145 + 3.6826\lambda + 3.15395\lambda^2 + 2.2585\lambda^3) \\ &\quad + \nu^4\lambda^2(2.107 - 6.2109\lambda - 5.22995\lambda^2 - 3.9301\lambda^3) \\ &\quad + \nu^5\lambda^2(-0.1731 + 3.5434\lambda + 3.3554\lambda^2 + 2.9949\lambda^3).\end{aligned}\quad (98)$$

It is shown in [39] that the precision $O(\nu^5)$ in (98) is sufficient to give excellent results for $|\lambda| < 1$. For $|\lambda|$ close to unity, it is better to use Padé approximations [39].

Example 3 Consider equal ellipses located in the square lattice formed by $\omega_1 = 2$ and $\omega_2 = 2i$ in such a way that one ellipse lies exactly in one cell. Let the angle between the major semi-axis and the axis OX_1 be a random uniformly distributed variable. Then, $P(\lambda)$ can be approximated up to $O(\nu^2)$ by the following formula [38]:

$$P(\lambda) = \left[1 - \lambda^2 \left(\frac{1-q}{1+q} \right)^2 - \lambda\nu \frac{1+q^2}{2q} \right]^{-1} + O(\nu^2), \quad (99)$$

where q denotes the ratio of the semi-axes of ellipses.

Fig. 1 Dependence of the tensor $\mathbf{P}$ given by (37) on the concentration ν for ellipses from Example 3. Data are for c_{44}^* (solid line), d_{12}^* (dashed line), ϵ_{11}^* (dotted line)

Fig. 2 Dependence of $\mathbf{P}^{-1}$ on ν . Data are for the (1, 1)—component (solid line); (1, 2)—component (dashed line); (2, 2)—component (dotted line)

The tensor $\mathbf{P}^{-1}$ was computed in [21] for tetragonal array of ellipses whose axes were parallel to the coordinate axes. Let the matrix be made of PZT-4 piezoceramics with the parameters $c_{44} = 2.56 \times 10^{10} \text{ N/m}^2$, $\epsilon_{11}/\epsilon_0 = 729$ (ϵ_0 stands for the vacuum permittivity) and $e_{15} = 12.7 \text{ C/m}^2$; the fibers of PP-2 with the parameters $c_{44} = 10 \times 10^{10} \text{ N/m}^2$, $\epsilon_{11}/\epsilon_0 = 500$ and $e_{15} = 0 \text{ C/m}^2$. The numerical results of [21] were presented for $q = 0.5$.

Consider a similar numerical example with the same physical properties and the same ellipses, but randomly oriented in the square array. In this case, Eq. (77) gives the real eigenvalues $\lambda_1 = -0.186$ and $\lambda_2 = 0.592$. Computations performed with formulae (95) and (99) are displayed in Figs. 1 and 2. The components $\mathbf{P}_{11}^{-1}$ and $\mathbf{P}_{12}^{-1}$ presented in Fig. 2 and in Figs 2–4 from [21] have close numerical values. The component $\mathbf{P}_{22}^{-1}$ from [21] increases with ν as a parabolic function; $\mathbf{P}_{22}^{-1}$ in Fig. 2 depends rather linearly on ν .

Example 4 Checkerboard 2-dimensional composites can be considered as a limit case of the square inclusions. Application of Dykhne's formula [16] to (92) with $\nu = \frac{1}{2}$ yields

$$P(\lambda) = \frac{1}{\lambda} \left(\sqrt{\frac{1+\lambda}{1-\lambda}} - 1 \right). \quad (100)$$

General exact formulae for vector-matrix problems obtained in the framework of the theory of duality transformation can be found in [13].

7 Discussion

The present work is devoted to the extension of the constructive results [34–39] to piezoelectric anti-plane problems by the use of the vector-matrix $\mathbb{R}$ -linear problem [41,42] in a class of doubly periodic functions. The vector-matrix extension (67) of the Clausius–Mossotti approximation is obtained (cf. [13] and [14]). The key point to get high-order approximations formulae for the effective constants is the decomposition of the vector-matrix problem into scalar $\mathbb{R}$ -linear problems. Further, the known exact and approximate analytical

formulae obtained before for scalar problems [30–39] are directly applied. This direct application is described when the physical parameter Δ introduced by (74) in Sect. 6 is positive.

We now discuss the physical restrictions on Δ and the case $\Delta \leq 0$. The dimensionless contrast parameters (70) satisfy the inequality $-1 \leq \rho_j \leq 1$ ($j = 1, 2, 3$). The dimensionless constants c and e from (71) are always negative and h is positive. It follows from accessible data for the piezoelectric materials [21, 22] that Δ from (74) can be positive, negative and equal to zero. Consider below these cases and simple examples to demonstrate that all these cases take place.

1. The case $\Delta > 0$ is investigated in Sect. 6. Consider an example when the latter inequality is possible. Let $\rho_1 = \rho_3 > \rho_2$. Then, $\Delta = h^2(\rho_3 - \rho_2)^2 > 0$ and $\lambda_1 = \frac{h\rho_2 + \rho_1}{h+1}$, $\lambda_2 = \rho_3$.
2. The case $\Delta < 0$ takes place, for instance, for $\rho_1 = \rho_2 > \rho_3$. Then, $\Delta = -4h(\rho_1 - \rho_3)^2 < 0$,

$$\lambda_1 = \frac{h\rho_1 + \rho_3 - i\sqrt{h}(\rho_1 - \rho_3)}{h+1} \quad (101)$$

and $\lambda_2 = \overline{\lambda_1}$. In the general case $\Delta < 0$, the complex values λ_j ($j = 1, 2$) always satisfy (78) because (68) and (73) imply that $|\lambda_1|^2 = \det \mathbf{R} \leq 1$.

Following Sect. 6, one could try to derive $\mathbb{R}$ -linear scalar problems by the following decomposition:

$$\mathbf{R} = \tilde{\mathbf{T}} \Lambda \tilde{\mathbf{T}}^{-1}. \quad (102)$$

However, such a complex matrix $\tilde{\mathbf{T}}$ does not exist. The decomposition (68) with the complex matrix $\mathbf{T}$ yields the special $\mathbb{R}$ -linear vector-matrix problem

$$\omega_0(t) = \omega_k(t) - \begin{pmatrix} 0 & \lambda_1 \\ \frac{0}{\lambda_1} & 0 \end{pmatrix} \overline{\omega_k(t)}, \quad t \in \partial G_k \quad (k = 1, 2, \dots, N). \quad (103)$$

Here, the following relation is used:

$$\Lambda \mathbf{T}^{-1} \bar{\mathbf{T}} = \begin{pmatrix} 0 & \lambda_1 \\ \frac{0}{\lambda_1} & 0 \end{pmatrix}, \quad (104)$$

where λ_1 is given by (101). The problem (103) was not investigated in [34, 37, 44]. However, the result [46] suggests that the $\mathbb{R}$ -linear vector-matrix problem (103) can be solved by similar methods.

3. The case $\Delta = 0$ is equivalent to $h(\rho_1 - \rho_2)^2 = 4(\rho_1 - \rho_3)(\rho_2 - \rho_3)$ that obviously can happen. In this case, the diagonal matrix Λ has to be changed by the Jordan normal form.

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