Correction to: Elliptic inequalities with L^1 data in Musielak–Orlicz spaces

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Received: 21 December 2017 / Accepted: 6 July 2018 / Published online: 16 July 2018
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Correction to: Monatsh Math (2017) 183:1–33
<https://doi.org/10.1007/s00605-016-1010-1>

This erratum concerns the proof of Lemma 1 of the paper [1] which contains some mistakes, so we reformulated the content and the proof of this Lemma and we gave also some remarks at the end of this erratum.

Firstly we recall the main result of the paper [1].

1 Main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ ($N \geq 2$) and let φ and γ be two Musielak functions such that $\gamma \prec\prec \varphi$. We assume also that the function φ depends only on $N - 1$ coordinates of x . Given a measurable function $\Psi : \Omega \rightarrow \mathbb{R}$ and consider the set

$$K_\Psi = \{u \in W_0^1 L_\varphi(\Omega) : u \geq \Psi \text{ a.e in } \Omega\}.$$

This convex set is sequentially $\sigma(\Pi L_\varphi, \Pi E_{\bar{\varphi}})$ closed in $W_0^1 L_\varphi(\Omega)$ (see [2]).

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \rightarrow W^{-1} L_{\bar{\varphi}}(\Omega)$ be a mapping given by $A(u) = -\operatorname{div} a(x, u, \nabla u)$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\xi, \xi_* \in \mathbb{R}^N$ ($\xi \neq \xi_*$)

Communicated by A. Constantin.

The original article can be found online at <https://doi.org/10.1007/s00605-016-1010-1>.

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$$|a(x, s, \xi)| \leq k_1 (c(x) + \bar{\varphi}_x^{-1}(\gamma(x, k_2|s|)) + \bar{\varphi}_x^{-1}(\varphi(x, k_3|\xi|))) \quad (1)$$

$$(a(x, s, \xi) - a(x, s, \xi_*)) (\xi - \xi_*) > 0 \quad (2)$$

$$a(x, s, \xi) (\xi - \nabla v_0) \geq \alpha \varphi(x, |\xi|) \quad (3)$$

with $v_0 \in K_\Psi \cap W_0^1 E_\varphi(\Omega)$, $\alpha, k_1, k_2, k_3 > 0$ and $c(\cdot) \in E_{\bar{\varphi}}(\Omega)$.

Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$

$$g(x, s, \xi) s \geq 0 \quad (4)$$

$$|g(x, s, \xi)| \leq b(|s|) (h(x) + \varphi(x, |\xi|)) \quad (5)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and non-decreasing function and $h(\cdot)$ is a given nonnegative function in $L^1(\Omega)$.

Now, assume that

$$K_\Psi \cap W_0^1 E_\varphi(\Omega) \text{ is dense in } K_\Psi \quad (6)$$

for the modular convergence in $W_0^1 L_\varphi(\Omega)$.

Finally, we assume that

$$f \in L^1(\Omega). \quad (7)$$

We define $T_0^{1,\varphi}(\Omega)$ the set of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $T_k(u) \in W_0^1 L_\varphi(\Omega)$ for all $k \geq 0$ where $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is the truncation at height k defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

We have proved the following theorem in the paper [1]

Theorem 1 Assume that (1)–(7) hold true, then there exists $u \in T_0^{1,\varphi}(\Omega)$, such that $u \geq \Psi$, $g(x, u, \nabla u) \in L^1(\Omega)$ and

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) dx \leq \int_{\Omega} f T_k(u - v) dx$$

for all $v \in K_\Psi \cap L^\infty(\Omega)$ and for all $k \geq 0$.

To prove this theorem, we used Lemma 1 of [1] whose proof contains some mistakes, so we reformulated the content and the proof of this Lemma in this erratum

Lemma 1 Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and $\bar{\varphi}$ be two complementary Musielak–Orlicz functions which satisfy the following conditions

(i) There exists a constant $c_0 > 0$ such that $\inf_{x \in \Omega} \varphi(x, 1) \geq c_0$;

(ii) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$

we have

$$\frac{\varphi(x, t)}{\varphi(y, t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)} \quad \text{for all } t \geq 1; \quad (8)$$

$$(iii) \quad \int_E \varphi(x, \lambda) dx < \infty \quad \text{for all } \lambda \geq 0 \text{ and all measurable set } E \subset \Omega$$

$$\text{with } \text{mes}(E) < +\infty; \quad (9)$$

$$(iv) \quad \text{There exists a constant } c_1 > 0 \text{ such that } \bar{\varphi}(x, 1) \leq c_1 \text{ a.e in } \Omega. \quad (10)$$

Under these assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and $\mathfrak{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Before the proof of this lemma, we recall some results which will be used later.

For two complementary Musielak–Orlicz functions φ and $\bar{\varphi}$, we have [4]:

(i) The Young inequality:

$$ts \leq \varphi(x, t) + \bar{\varphi}(x, s) \quad \text{for all } t, s \geq 0, x \in \Omega. \quad (11)$$

(ii) The Hölder inequality:

$$\left| \int_\Omega u(x) v(x) dx \right| \leq \|u\|_{\varphi, \Omega} \|v\|_{\bar{\varphi}, \Omega}, \quad \text{for all } u \in L_\varphi(\Omega), v \in L_{\bar{\varphi}}(\Omega). \quad (12)$$

Remark 1 The Young inequality implies the inequality

$$\|u\|_{\varphi, \Omega} \leq \rho_{\varphi, \Omega}(u) + 1 \quad \text{for all } u \in L_\varphi(\Omega). \quad (13)$$

Proof of Lemma 1. Since every bounded Lipschitz domain is the union of a finite number of domains, each of which is a starlike with respect to an open ball (see [3]), then we can assume that Ω is a bounded starlike domain relative to some ball $B_r = \{x \in \mathbb{R}^N / |x| < r\}$.

So, for each $\varepsilon > 0$ we have

$$(1 - \rho\varepsilon)\Omega + \varepsilon B \subset \Omega, \quad (14)$$

where $\rho = r^{-1}$ and B is the unit ball of $\mathbb{R}^N$.

Let $J(\cdot)$ be a measurable function with support in the ball $B_r = B(0, r)$ and define

$$J_\varepsilon(x) = \frac{1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right).$$

Let $(\omega^k)_k$ a finite partition of Ω such that $\text{diam } \omega^k \leq \varepsilon (k = 1, 2, 3, \dots, n = n(\varepsilon))$ and put $\Omega^k = \alpha_\varepsilon(\omega^k - B_{\varepsilon r})$ with $\alpha_\varepsilon = (1 - \rho\varepsilon)^{-1}$. For $\varepsilon \leq \frac{r}{2}$ and $m = 2(1 + 2r)$ we have $\alpha_\varepsilon \in]1, 2]$ and $\text{diam } \Omega^k \leq m\varepsilon$. Consider $f \in L_\varphi(\Omega)$ and define the family of operators

$$K_\varepsilon f(x) = \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy,$$

where

$$\Theta_\varepsilon = \left\{ z \in \Omega / |f(z)| \leq \frac{1}{m\varepsilon} \right\}.$$

We prove that the operators K_ε are uniformly bounded from $L_\varphi(\Omega)$ into $L_\varphi(\Omega)$.

For this end, we extend f by zero to $\mathbb{R}^N$ and assume that $\|f\|_{\varphi, \Omega} \leq 1$, we shall prove that

$$\rho_{\varphi, \Omega}(K_\varepsilon f) = \int_{\Omega} \varphi(x, |K_\varepsilon f(x)|) dx \leq c \quad (15)$$

for all ε such that $0 < \varepsilon \leq \varepsilon^0 \leq 1$ and $c > 0$ independent of f .

For each $k > 0$, we choose $y_k \in \Omega^k$ and put $\varphi_k(t) = \varphi(y_k, t)$, $\forall t \geq 0$. We have

$$\begin{aligned} \rho_{\varphi, \Omega}(K_\varepsilon f) &= \sum_{k=1}^{n(\varepsilon)} \int_{\omega^k} \varphi \left(x, \left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right| \right) dx \\ &= \sum_{k=1}^{n(\varepsilon)} \int_{\omega^k} A_k(x, \varepsilon) \varphi_k \left(\left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right| \right) dx, \end{aligned} \quad (16)$$

where

$$A_k(x, \varepsilon) = \frac{\varphi(x, \left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right|)}{\varphi_k(\left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right|)}$$

We shall prove the uniform estimate

$$A_k(x, \varepsilon) \leq c, \quad x \in \omega^k, \quad (17)$$

where $c > 0$ does not depend on $x \in \omega^k$, k and $\varepsilon \in (0, \varepsilon^0)$ with some $\varepsilon^0 > 0$.

By using the Hölder inequality, we have

$$\begin{aligned} F(x, \varepsilon) &= \left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right| \leq \frac{M}{\varepsilon^N} \int_{\Omega} |\chi_{B_{\varepsilon r}}(y) f(\alpha_\varepsilon y)| dy \\ &\leq \frac{M}{\varepsilon^N} \|f\|_{\varphi} \|\chi_{B_{\varepsilon r}}\|_{\bar{\varphi}}, \end{aligned}$$

where $M = \sup_{B_r} |J(y)|$. From (10) and (13) and for $0 < \varepsilon \leq |B_r|^{-\frac{1}{N}} = \varepsilon_1^0$, we get

$$\|\chi_{B_{\varepsilon r}}\|_{\bar{\varphi}} \leq c_1 |B_{\varepsilon r}| + 1 \leq c_1 + 1$$

Hence

$$F(x, \varepsilon) \leq \frac{c_2}{\varepsilon^N}.$$

Under the assumption that $0 < \varepsilon \leq \frac{1}{2m} = \varepsilon_2^0$, we have

$$\begin{aligned} A_k(x, \varepsilon) &= \frac{\varphi(x, F(x, \varepsilon))}{\varphi(y_k, F(x, \varepsilon))} \leq (F(x, \varepsilon))^{\frac{A}{\log(\frac{1}{m\varepsilon})}} \\ &\leq (c_2 \varepsilon^{-N})^{\frac{A}{\log(\frac{1}{m\varepsilon})}} \leq c_2^{\frac{A}{\log(\frac{1}{m})}} (\varepsilon^{-N})^{\frac{A}{\log(\frac{1}{m\varepsilon})}}. \end{aligned}$$

Then

$$A_k(x, \varepsilon) \leq c = c_3 e^{2NA}, \quad c_3 = c_2^{\frac{A}{\log(\frac{1}{m})}}$$

for $x \in \omega^k$ and $0 < \varepsilon \leq \frac{1}{m^2} = \varepsilon_3^0$.

Consequently, for $0 < \varepsilon \leq \varepsilon^0$ where $\varepsilon^0 = \min_{1 \leq k \leq 3} \varepsilon_k^0$, we obtain the estimate

$$A_k(x, \varepsilon) \leq c \quad \text{with} \quad c = c_3 e^{2NA}. \quad (18)$$

Turning to (16) and using (18) and Jensen's inequality, we get

$$\begin{aligned} \rho_{\varphi, \Omega}(K_\varepsilon f) &\leq c \sum_{k=1}^{n(\varepsilon)} \int_{\omega^k} \varphi_k \left(\left| \int_{\Omega} J_\varepsilon(x-y) f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon y) dy \right| \right) dx \\ &\leq c \sum_{k=1}^{n(\varepsilon)} \int_{\{|y| < \varepsilon r\}} |J_\varepsilon(y)| dy \int_{\omega^k} \varphi_k(|f \chi_{\Theta_\varepsilon}(\alpha_\varepsilon(x-y))|) dx \\ &\leq c \sum_{k=1}^{n(\varepsilon)} \int_{\{|y| < r\}} |J(y)| dy \int_{\alpha_\varepsilon(\omega^k - \varepsilon y)} \varphi_k(|f \chi_{\Theta_\varepsilon}(z)|) dz \\ &\leq c \sum_{k=1}^{n(\varepsilon)} \int_{\Omega^k \cap \Theta_\varepsilon} \varphi_k(|f(z)|) dz \\ &\leq c \sum_{k=1}^{n(\varepsilon)} \int_{\Omega^k \cap \Theta_\varepsilon} \frac{\varphi(y_k, |f(z)|)}{\varphi(z, |f(z)|)} \varphi(z, |f(z)|) dz \\ &\leq c \sum_{k=1}^{n(\varepsilon)} \left(\frac{1}{m\varepsilon} \right)^{\frac{A}{\log \frac{1}{m\varepsilon}}} \int_{\Omega^k} \varphi(z, |f(z)|) dz \leq c', \end{aligned}$$

where we used the fact that the covering by ω^k has a finite multiplicity (that is, each point $x \in \Omega$ belongs simultaneously not more than to a finite number n_0 of the sets ω^k , $n_0 < 1 + (1 + 2r)^N$ in this case).

Now, we shall prove that there exists $\lambda > 0$, such that

$$\lim_{\varepsilon \rightarrow 0} \rho_{\varphi, \Omega} \left(\frac{K_\varepsilon f - f}{\lambda} \right) = 0, \quad f \in L_\varphi(\Omega). \quad (19)$$

By using the uniform boundedness of the operators K_ε from $L_\varphi(\Omega)$ into $L_\varphi(\Omega)$ and the Banach–Steinhaus theorem, it is sufficient to prove (19) for the characteristic function χ_E for all bounded measurable set $E \subset \Omega$.

Note that for $\varepsilon \leq \frac{1}{m}$ we have $K_\varepsilon(\chi_E) = \int_\Omega J_\varepsilon(x - y)\chi_E(\alpha_\varepsilon y) dy$.

We have for $\lambda > 0$,

$$\rho_{\varphi, \Omega} \left(\frac{K_\varepsilon \chi_E - \chi_E}{\lambda} \right) = \int_\Omega \varphi \left(x, \frac{1}{\lambda} \left| \int_{B_r} J(y) [\chi_E(\alpha_\varepsilon(x - \varepsilon y)) - \chi_E(x)] dy \right| \right) dx.$$

By using Jensen's inequality (with respect to the variable y) and the Fubini's theorem, we obtain

$$\begin{aligned} \rho_{\varphi, \Omega} \left(\frac{K_\varepsilon \chi_E - \chi_E}{\lambda} \right) &\leq \int_\Omega \left(\int_{B_r} |J(y)| \varphi \left(x, \frac{1}{\lambda} (|\chi_E(\alpha_\varepsilon(x - \varepsilon y)) - \chi_E(x)|) \right) dy \right) dx \\ &\leq \int_{B_r} |J(y)| \left(\int_\Omega \varphi \left(x, \frac{1}{\lambda} (|\chi_E(\alpha_\varepsilon(x - \varepsilon y)) - \chi_E(x)|) \right) dx \right) dy \end{aligned}$$

since

$$|J(y)| \int_\Omega \varphi \left(x, \frac{1}{\lambda} (|\chi_E(\alpha_\varepsilon(x - \varepsilon y)) - \chi_E(x)|) \right) dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and

$$\begin{aligned} |J(y)| \int_\Omega \varphi \left(x, \frac{1}{\lambda} (|\chi_E(\alpha_\varepsilon(x - \varepsilon y)) - \chi_E(x)|) \right) dx \\ \leq |J(y)| \int_\Omega \varphi \left(x, \frac{2}{\lambda} \right) \leq c. \end{aligned}$$

Consequently, by the Lebesgue dominated convergence theorem, we get

$$\rho_{\varphi, \Omega} \left(\frac{K_\varepsilon \chi_E - \chi_E}{\lambda} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for some } \lambda > 0.$$

Finally, there exists $\lambda > 0$, such that

$$\lim_{\varepsilon \rightarrow 0} \rho_{\varphi, \Omega} \left(\frac{K_\varepsilon f - f}{\lambda} \right) = 0, \quad f \in L_\varphi(\Omega).$$

Now, let $f \in W_0^m L_\varphi(\Omega)$ and extending it by zero to $\mathbb{R}^N$. We will show that there exists $\lambda > 0$, such that

$$\bar{\rho}_{\varphi, \Omega} \left(\frac{K_\varepsilon f - f}{\lambda} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

We have

$$\bar{\rho}_{\varphi, \Omega} \left(\frac{K_{\varepsilon} f - f}{\lambda} \right) = \sum_{|j| \leq m} \rho_{\varphi, \Omega} \left(\frac{\alpha_{\varepsilon}^{|j|} K_{\varepsilon} D^j f - D^j f}{\lambda} \right).$$

Applying the same smoothing procedure as above, for $1 \leq j \leq N$, there exists $\lambda_j > 0$, such that

$$\rho_{\varphi, \Omega} \left(\frac{\alpha_{\varepsilon}^{|j|} K_{\varepsilon} D^j f - D^j f}{\lambda_j} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Then, by taking $\lambda = \max_{1 \leq j \leq N} \lambda_j$, we obtain the result. $\square$

Remark 2 All expressions of Musielak–Orlicz functions given in the example 1 of the paper object of this erratum have been given near infinity.

Remark 3 The condition $\gamma \prec\prec \varphi$ listed in the paragraph 4 must be replaced by the following one:

$\forall \varepsilon > 0$, there exists a nonnegative integrable function h , such that

$$\gamma(x, t) \leq \varphi(x, \varepsilon t) + h(x). \text{ for all } t \geq 0 \text{ and for a. e. } x \in \Omega.$$

Remark 4 We must define $E_{\varphi}(\Omega)$ as the subset of $L_{\varphi}(\Omega)$ of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $\int_{\Omega} \varphi(x, \frac{|u(x)|}{\lambda}) dx < +\infty$ for all $\lambda > 0$.

Concerning the proof of the equality $(E_{\varphi}(\Omega))' = L_{\bar{\varphi}}(\Omega)$, we are preparing it in collaboration with Sidi Mohamed Douiri in an article entitled “On some properties of the Musielak–Orlicz spaces.”

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