Correction to: Explicit upper bound for the average number of divisors of irreducible quadratic polynomials

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Abstract Consider the divisor sum $\sum_{n \leq N} \tau(n^2 + 2bn + c)$ for integers b and c . We improve the explicit upper bound of this average divisor sum in certain cases, and as an application, we give an improvement in the maximal possible number of $D(-1)$ -quadruples. The new tool is a numerically explicit Pólya–Vinogradov inequality, which has not been formulated explicitly before but is essentially due to Frolenkov–Soundararajan.

Keywords Number of divisors · Quadratic polynomial · Character sums

Mathematics Subject Classification Primary 11N56; Secondary 11D09

Let $\tau(n)$ denote the number of positive divisors of the integer n . In [3], we provided an explicit upper bound for the sum $\sum_{n=1}^N \tau(n^2 + 2bn + c)$ under certain conditions on the discriminant, and we gave an application for the maximal possible number of $D(-1)$ -quadruples.

The aim of this addendum is to announce improvements in the results from [3]. We start with sharpening of Theorem 2 [3].

The original article can be found online at <https://doi.org/10.1007/s00605-017-1061-y>.

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Theorem 2A Let $f(n) = n^2 + 2bn + c$ for integers b and c , such that the discriminant $\delta := b^2 - c$ is nonzero and square-free, and $\delta \not\equiv 1 \pmod{4}$. Assume also that for $n \geq 1$ the function $f(n)$ is nonnegative. Then for any $N \geq 1$ satisfying $f(N) \geq f(1)$, and $X := \sqrt{f(N)}$, we have the inequality

$$\begin{aligned} \sum_{n=1}^N \tau(n^2 + 2bn + c) &\leq \frac{2}{\zeta(2)} L(1, \chi) N \log X \\ &\quad + \left(2.332 L(1, \chi) + \frac{4M_\delta}{\zeta(2)} \right) N + \frac{2M_\delta}{\zeta(2)} X \\ &\quad + 4\sqrt{3} \left(1 - \frac{1}{\zeta(2)} \right) M_\delta \frac{N}{\sqrt{X}} \\ &\quad + 2\sqrt{3} \left(1 - \frac{1}{\zeta(2)} \right) M_\delta \sqrt{X} \end{aligned}$$

where $\chi(n) = \left(\frac{4\delta}{n} \right)$ for the Kronecker symbol $\left(\frac{\cdot}{\cdot} \right)$ and

$$M_\delta = \begin{cases} \frac{4}{\pi^2} \delta^{1/2} \log 4\delta + 1.8934\delta^{1/2} + 1.668, & \text{if } \delta > 0; \\ \frac{1}{\pi} |\delta|^{1/2} \log 4|\delta| + 1.6408|\delta|^{1/2} + 1.0285, & \text{if } \delta < 0. \end{cases}$$

In the case of the polynomial $f(n) = n^2 + 1$, we can give an improvement to Corollary 3 from [3].

Corollary 3A For any integer $N \geq 1$, we have

$$\sum_{n \leq N} \tau(n^2 + 1) < \frac{3}{\pi} N \log N + 3.0475N + 1.3586\sqrt{N}.$$

Just as in [2, 3], we have an application of the latter inequality in estimating the maximal possible number of $D(-1)$ -quadruples, whereas it is conjectured there are none. We can reduce this number from $4.7 \cdot 10^{58}$ in [2] and $3.713 \cdot 10^{58}$ in [3] to the following bound.

Corollary 4A There are at most $3.677 \cdot 10^{58}$ $D(-1)$ -quadruples.

The improvements announced above are achieved by using more powerful explicit estimates than the ones used in [3]. More precisely, the results are obtained when instead of Lemma 2 and Lemma 3 from [3] we plug in the proof the following stronger results.

Lemma 2A For any integer $N \geq 1$ we have

$$\sum_{n \leq N} \mu^2(n) = \frac{N}{\zeta(2)} + E_1(N),$$

where $|E_1(N)| \leq \sqrt{3} \left(1 - \frac{1}{\xi(2)}\right) \sqrt{N} < 0.6793\sqrt{N}$.

Proof This is inequality (10) from Moser and MacLeod [4]. $\square$

The following numerically explicit Pólya–Vinogradov inequality is essentially proven by Frolenkov and Soundararajan [1], though it was not formulated explicitly. It supersedes the main result of Pomerance [5], which was formulated as Lemma 3 in [3].

Lemma 3A *Let*

$$M_\chi := \max_{L, P} \left| \sum_{n=L}^P \chi(n) \right|$$

for a primitive character χ to the modulus $q > 1$. Then

$$M_\chi \leq \begin{cases} \frac{2}{\pi^2} q^{1/2} \log q + 0.9467 q^{1/2} + 1.668, & \chi \text{ even}; \\ \frac{1}{2\pi} q^{1/2} \log q + 0.8204 q^{1/2} + 1.0285, & \chi \text{ odd}. \end{cases}$$

Proof Both inequalities for M_χ are shown to hold by Frolenkov and Soundararajan in the course of the proof of their Theorem 2 [1] as long as a certain parameter L satisfies $1 \leq L \leq q$ and $L = \lceil \pi^2/4\sqrt{q} + 9.15 \rceil$ for χ even, $L = \lceil \pi\sqrt{q} + 9.15 \rceil$ for χ odd. Thus both inequalities for M_χ hold when $q > 25$.

Then we have a look of the maximal possible values of M_χ when $q \leq 25$ from a data sheet, provided by Leo Goldmakher. It represents the same computations of Bober and Goldmakher used by Pomerance [5]. We see that the right-hand side of the bounds of Frolenkov–Soundararajan for any $q \leq 25$ is larger than the maximal value of M_χ for any primitive Dirichlet character χ of modulus q . This proves the lemma. $\square$

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