Remarks on localized sharp functions on certain sets in $\mathbb{R}^n$

Jacek Dziubański¹ · Agnieszka Hejna¹

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Abstract The aim of this note is to define localized sharp functions on certain domains in $\mathbb{R}^n$ and prove L^p estimates analogue to that of Fefferman–Stein. The proofs go by modifications of the good lambda inequality.

Keywords Local maximal function · Sharp function · Good lambda inequality · Whitney decomposition

Mathematics Subject Classification 42B25 · 42B35

1 Introduction

On $\mathbb{R}^n$ let $f_{\Delta}^{\#}(x)$ and $M^{\Delta}f(x)$ denote the classical dyadic sharp function and dyadic maximal function respectively, that is,

$$f_{\Delta}^{\#}(x) = \sup_{x \in Q \in \Delta} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$
$$M^{\Delta}f(x) = \sup_{x \in Q \in \Delta} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

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✉ Jacek Dziubański
jdziuban@math.uni.wroc.pl

Agnieszka Hejna
agnieszka.hejna@math.uni.wroc.pl

¹ Instytut Matematyczny, Uniwersytet Wrocławski, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

where here and subsequently, Δ denotes the collection of all dyadic cubes in $\mathbb{R}^n$ and

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

Suppose that $f \in L^{p_0}(\mathbb{R}^n)$ for some p_0 . The well-known Fefferman–Stein inequality asserts that if $1 < p < \infty$, $1 \leq p_0 \leq p$, and $f_\Delta^\# \in L^p(\mathbb{R}^n)$, then $M^\Delta f \in L^p(\mathbb{R}^n)$ and

$$\|M^\Delta f\|_{L^p(\mathbb{R}^n)} \leq C_n(p) \|f_\Delta^\#\|_{L^p(\mathbb{R}^n)} \quad (1.1)$$

(see [2, Sect. 3], [4, Chapter 4]). The inequality (1.1) implies that for every $1 < p < \infty$ one has

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_n(p) \|f_\Delta^\#\|_{L^p(\mathbb{R}^n)}.$$

The estimate (1.1) is a consequence of the following *good lambda distributional inequality*

$$\left| \{x \in \mathbb{R}^n : M^\Delta f(x) > \lambda, f_\Delta^\#(x) \leq c\lambda\} \right| \leq a \left| \{x \in \mathbb{R}^n : M^\Delta f(x) > b\lambda\} \right|, \quad (1.2)$$

where $\lambda > 0$, $c > 0$, $0 < b < 1$, $a = 2^n c / (1 - b)$, and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (see [4]).

Let Ω be a domain in $\mathbb{R}^n$. Our goal is to define for $f \in L^1_{\text{loc}}(\Omega)$ a localized version $f^\#_{\text{loc}}$ of the sharp function which will satisfy

$$\|f\|_{L^p(\Omega)} \leq C_p \|f^\#_{\text{loc}}\|_{L^p(\Omega)} \quad \text{for } f \in L^p(\Omega). \quad (1.3)$$

By localized we mean that the cubes which are taken in the definition of $f^\#_{\text{loc}}(x)$ are contained in a bounded set $\mathcal{B}_x \subset \Omega$. So one possible definition can be taken as follows. Let $\tau : \Omega \rightarrow (0, \infty)$. For $f \in L^1_{\text{loc}}(\Omega)$ we set

$$f^\#_{\text{loc}, \tau}(x) = \sup_{x \in Q \subset \Omega, \ell(Q) < \tau(x)} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where Q is any cube (not necessarily dyadic) and $\ell(Q)$ denotes its side-length. Note that τ cannot be taken arbitrarily. For example, if $\Omega = (0, \infty)$ and τ is such that $\lim_{x \rightarrow \infty} \tau(x)/x = 0$, then taking $f(x) = \chi_{(0, R)}(x)$ we have $\|f\|_{L^p(\Omega)} = R^{1/p}$ while $\lim_{R \rightarrow \infty} R^{-1/p} \|f^\#_{\text{loc}, \tau}\|_{L^p(\Omega)} = 0$. On the other hand, we shall show that for certain sets Ω in $\mathbb{R}^n$ if $\tau(x)$ behaves like $\frac{1}{2} \text{dist}(x, \partial\Omega)$, then $f^\#_{\text{loc}, \tau}$ satisfies (1.3). Moreover, the inequality (1.3) holds for $p = 1$, provided f is supported by a bounded set and $|f| \log(2 + |f|)$ is integrable. These will be obtained by proving modifications of the good lambda inequality (see Propositions 2.4, 3.7, and 4.5).

2 Localized sharp function on $\mathbb{R}^n \setminus \{0\}$

Let $\Omega = \mathbb{R}^n \setminus \{0\}$. We define the localized sharp function on Ω as

$$f_{\text{loc}, \Omega}^{\#}(x) = \sup_{x \in K \subset \left\{ y \in \Omega : \frac{|x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|x| \right\}} \frac{1}{|K|} \int_K |f(y) - f_K| dy,$$

where the supremum is taken over all cubes K (not necessarily dyadic) contained in the set

$$\left\{ y \in \Omega : \frac{|x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|x| \right\}.$$

We now turn to define the local dyadic maximal function associated with a Whitney decomposition of Ω . For this purpose, put

$$\rho(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|).$$

Set

$$\mathcal{L} = \{Q \in \Delta : \ell(Q) = \rho(0, Q)\}.$$

The set $\mathcal{L}$ forms a Whitney covering of Ω . For every integer k we define the k -th layer L_k of $\mathcal{L}$ as

$$L_k = \left\{ Q \in \mathcal{L} : \ell(Q) = 2^{-k} \right\}.$$

Clearly, $Q \in L_k$ if and only if $2^{-m}Q \in L_{k+m}$. Figure 1 shows three k -layers for $n = 2$. Here and subsequently, $\alpha Q = \{\alpha x : x \in Q\}$, $\alpha > 0$.

Fig. 1 Cubes from three k -layers of $\mathcal{L}$ for $n = 2$ which are contained in $(0, \infty)^n$

For every positive integer m the partition $\mathcal{L}_m$ of Ω is obtained by dividing each cube Q from $\mathcal{L}$ into 2^{nm} dyadic cubes each of side-length $2^{-m}\ell(Q)$. Let

$$\mathcal{D} = \mathcal{L} \cup \bigcup_{m=1}^{\infty} \mathcal{L}_m.$$

The local dyadic maximal function associated with the Whitney covering $\mathcal{L}$ of Ω is defined by

$$M_{\mathcal{D}}f(x) = \sup_{x \in K \in \mathcal{D}} \frac{1}{|K|} \int_K |f(y)| dy.$$

Our goal of this section is to prove the following theorem.

Theorem 2.1 *For every $1 \leq p < \infty$ there is a constant $C > 0$ such that for every locally integrable function f on Ω for which there exists $0 < p_0 \leq p$ such that $M_{\mathcal{D}}f \in L^{p_0}(\Omega)$ one has*

$$\|M_{\mathcal{D}}f\|_{L^p(\Omega)} \leq C \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)}.$$

Corollary 2.2 *For every $1 < p < \infty$ there is a constant $C_p > 0$ such that*

$$\|f\|_{L^p(\Omega)} \leq C_p \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)} \quad \text{for } f \in L^p(\Omega).$$

Corollary 2.3 *There is a constant $C > 0$ such that if f is supported by a bounded set and $|f| \log(2 + |f|)$ is integrable, then*

$$\|f\|_{L^1(\Omega)} \leq C \|f_{\text{loc}, \Omega}^{\#}\|_{L^1(\Omega)}.$$

There is no loss of generality if we assume that all the functions under consideration take values in $\mathbb{R}$. Clearly, for almost every $x \in \Omega$ there is a unique cube $Q \in \mathcal{L}$ such that $x \in Q$. For such an x let

$$Sf(x) = |f|_Q, \quad S'f(x) = f_Q. \quad (2.1)$$

The proof of Theorem 2.1 is a consequence of the following modified version of the good lambda inequality, which is stated in the proposition below.

Proposition 2.4 (modified good lambda inequality) *For every constant $0 < b < 1$ there is a constant $C > 0$ such that for all $c, \alpha > 0$, and every locally integrable function f which satisfies $\lim_{|x| \rightarrow \infty} Sf(x) = 0$ we have*

$$\begin{aligned} & \left| \{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} \right| \\ & \leq Cc |\{x \in \Omega : M_{\mathcal{D}}f(x) > b\alpha\}| + |\{x \in \Omega : Sf(x) > b\alpha\}|, \end{aligned}$$

where here and subsequently,

$$\mathbb{S}f(x) = 2^{1-n} \cdot 3^{2n} \sum_{j=1}^{\infty} f_{\text{loc}, \Omega}^{\#}(2^j x).$$

Proof of Theorem 2.1 If we assume Proposition 2.4, the proof of the theorem is a slight modification of that in the classical case (see [2–4]). For the convenience of the reader, we provide details. We may assume that $\|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)}$ is finite. Then, by the Minkowski inequality,

$$\|\mathbb{S}f\|_{L^p(\Omega)} \leq 2^{1-n} \cdot 3^{2n} \sum_{j=1}^{\infty} 2^{-jn/p} \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)} = C_p \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)}. \quad (2.2)$$

Since $M_{\mathcal{D}}f \in L^{p_0}(\Omega)$, $\lim_{|x| \rightarrow \infty} Sf(x) = 0$. Let

$$\begin{aligned} I_R &= p \int_0^R \alpha^{p-1} |\{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha\}| d\alpha \\ &= p \int_0^R \alpha^{p-1} \left| \{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} \right| d\alpha \\ &\quad + p \int_0^R \alpha^{p-1} \left| \{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) \geq c\alpha\} \right| d\alpha. \end{aligned}$$

Applying Proposition 2.4, we obtain

$$\begin{aligned} I_R &\leq Ccp \int_0^R \alpha^{p-1} |\{x \in \Omega : M_{\mathcal{D}}f(x) > b\alpha\}| d\alpha \\ &\quad + p \int_0^R \alpha^{p-1} |\{x \in \Omega : \mathbb{S}f(x) > b\alpha\}| d\alpha \\ &\quad + p \int_0^R \alpha^{p-1} |\{x \in \Omega : f_{\text{loc}, \Omega}^{\#}(x) \geq c\alpha\}| d\alpha \\ &\leq Ccb^{-p} I_{bR} + p \int_0^R \alpha^{p-1} |\{x \in \Omega : \mathbb{S}f(x) > b\alpha\}| d\alpha \\ &\quad + p \int_0^R \alpha^{p-1} |\{x \in \Omega : f_{\text{loc}, \Omega}^{\#}(x) \geq c\alpha\}| d\alpha. \end{aligned}$$

Clearly, $I_R < \infty$, since, by assumption, $M_{\mathcal{D}}f \in L^{p_0}(\Omega)$ and $0 < p_0 \leq p$. Moreover, $I_{bR} \leq I_b$ because $0 < b < 1$. Taking c small enough such that $Ccb^{-p} < 1$ we obtain

$$\begin{aligned} I_R &\leq C'p \int_0^R \alpha^{p-1} |\{x \in \Omega : \mathbb{S}f(x) > b\alpha\}| d\alpha \\ &\quad + C'p \int_0^R \alpha^{p-1} |\{x \in \Omega : f_{\text{loc}, \Omega}^{\#}(x) \geq c\alpha\}| d\alpha. \end{aligned}$$

Fig. 2 Complementary neighborhoods for $n = 2$

Letting $R \rightarrow \infty$, we conclude

$$\|M_{\mathcal{D}} f\|_{L^p(\Omega)} \leq C_1 \|\mathbb{S} f\|_{L^p(\Omega)} + C_2 \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)} \leq C_3 \|f_{\text{loc}, \Omega}^{\#}\|_{L^p(\Omega)},$$

where in the last inequality we have used (2.2). $\square$

The remaining part of the section is devoted to proving Proposition 2.4. The following two lemmas will play a crucial role in the proof.

Lemma 2.5 *For every locally integrable function f and almost every $x \in \Omega$ one has*

$$S' f(x) \leq S' f(2x) + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \Omega}^{\#}(2x).$$

Proof It suffices to prove the lemma for $x = (x_1, x_2, \dots, x_n)$ such that $x_j > 0$ for every $j = 1, 2, \dots, n$. Let Q_1 be the unique dyadic cube from $\mathcal{L}$ which contains x . Let k be such that $2^{-(k+1)} = \ell(Q_1)$. Set $Q_2 = 2Q_1$. Let

$$Q_0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_j \leq 1 \text{ for all } j = 1, 2, \dots, n\}. \quad (2.3)$$

Then there is the unique vector $p = (p_1, p_2, \dots, p_n) \neq \mathbf{0}$, $p_j \in \{0, 1\}$, such that $Q_1 = 2^{-k-1} Q_0 + 2^{-k-1} p$. Set $K = 3 \cdot 2^{-k-1} Q_0 + 2^{-k-1} p$. Then

$$Q_1 \cup Q_2 \subset K \subset \left\{ y \in \Omega : \frac{|2x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|2x| \right\}.$$

We shall call the set $M = K \setminus (Q_1 \cup Q_2)$ the *complementary neighborhood* of the pair of cubes (Q_1, Q_2) .

Let us remark that for $n = 1$ the complementary neighborhood of two intervals is the empty set. For $n = 2$ the complementary neighborhoods are presented in Fig. 2.

We have

$$\begin{aligned} \frac{|Q_1|}{|K|} &= \frac{1}{3^n}, \quad \frac{|Q_2|}{|K|} = \frac{2^n}{3^n}, \quad \frac{|M|}{|K|} = \frac{3^n - 2^n - 1}{3^n}, \\ f_K &= \frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M. \end{aligned} \quad (2.4)$$

We consider two cases.

Case 1 $S'f(x) \geq f_M$. Then

$$\begin{aligned}
 f_{\text{loc}, \Omega}^{\#}(2x) &\geq \frac{1}{|K|} \int_K |f(y) - f_K| dy \\
 &= \frac{1}{|K|} \int_K \left| f(y) - \left(\frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) \right| dy \\
 &\geq \frac{1}{|K|} \int_{Q_1} \left| f(y) - \left(\frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) \right| dy \\
 &\geq \frac{1}{|K|} \left| \int_{Q_1} f(y) - \left(\frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) dy \right| \\
 &= \left| f_{Q_1} \frac{1}{3^n} - f_{Q_1} \frac{1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}} \right| \\
 &\geq f_{Q_1} \frac{1}{3^n} - f_{Q_1} \frac{1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}} \\
 &= f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}}.
 \end{aligned}$$

By the assumption $-f_M \geq -S'f(x) = -f_{Q_1}$, hence

$$\begin{aligned}
 f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}} &\geq f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_{Q_1} \frac{3^n - 1 - 2^n}{3^{2n}} \\
 &= f_{Q_1} \frac{2^n}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} = S'f(x) \frac{2^n}{3^{2n}} - S'f(2x) \frac{2^n}{3^{2n}},
 \end{aligned}$$

which gives the lemma.

Case 2 $S'f(x) \leq f_M$. The proof in this case is similar to that in Case 1. The only difference is that we diminish the area of integration to Q_2 instead of Q_1 . We omit the details. If $n = 1$, then $M = \emptyset$. In this case we set $f_M = 0$ and proceed as in Case 1. $\square$

Remark 2.6 If we apply the lemma to the function $-f$, we obtain the inequality

$$S'f(2x) \leq S'f(x) + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \Omega}^{\#}(2x).$$

Lemma 2.7 For every locally integrable function f and almost every $x \in \Omega$ one has

$$Sf(x) \leq Sf(2x) + 2^{1-n} \cdot 3^{2n} f_{\text{loc}, \Omega}^{\#}(2x). \quad (2.5)$$

Proof Using Lemma 2.5 to $|f|$ we get $Sf(x) \leq Sf(2x) + 2^{-n} \cdot 3^{2n} |f|_{\text{loc}, \Omega}^{\#}(2x)$. The inequality (2.5) holds because $|f|_{\text{loc}, \Omega}^{\#}(2x) \leq 2 f_{\text{loc}, \Omega}^{\#}(2x)$. $\square$

Iterating the inequality (2.5) we obtain the following corollary.

Corollary 2.8 Assume that a locally integrable function f on Ω satisfies

$$\lim_{|x| \rightarrow \infty} Sf(x) = 0.$$

Then

$$Sf(x) \leq \mathbb{S}f(x) \text{ for every } x \in \Omega.$$

Proof of Proposition 2.4 The proof of the proposition is a modification of that of the classical good lambda inequality (cf. [1, 4]). Let $\{Q_j\}$ be the partition of the set $\{x \in \Omega : M_{\mathcal{D}}f(x) > b\alpha\}$ which consists of maximal dyadic cubes Q_j from $\mathcal{D}$ which satisfy $|f|_{Q_j} > b\alpha$. Obviously, the cubes Q_j have disjoint interiors. Further,

$$\begin{aligned} \{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} &\subset \{x \in \Omega : M_{\mathcal{D}}f(x) > b\alpha\} = \bigcup_j Q_j, \\ \{x \in \Omega : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} &= \bigcup_j \{x \in Q_j : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\}. \end{aligned}$$

Thus, it suffices to show that either

$$\left| \{x \in Q_j : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} \right| \leq cC|Q_j| \quad (2.6)$$

or

$$\{x \in Q_j : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\} \subset \{x \in \Omega : \mathbb{S}f(x) > b\alpha\}. \quad (2.7)$$

Assume that the set $A_j = \{x \in Q_j : M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\}$ is not empty, otherwise there is nothing to prove. Fix $x_0 \in A_j$. We consider two cases.

Case 1 $Q_j \in \mathcal{L}$. Then $b\alpha < |f|_{Q_j} = Sf(y) \leq \mathbb{S}f(y)$ for every $y \in Q_j$, where in the last inequality we have used Corollary 2.8. Thus $Q_j \subset \{x \in \Omega : \mathbb{S}f(x) > b\alpha\}$ and (2.7) holds in this case.

Case 2 $Q_j \in \mathcal{L}_m$ for $m \geq 1$. In this case the proof follows the pattern from [4]. Indeed, first observe that for every $Q \in \mathcal{D}$ such that $Q_j \subsetneq Q$ one has $|f|_Q \leq b\alpha$. Thus, for every $x \in Q_j$ such that $M_{\mathcal{D}}f(x) > \alpha$ one has $M_{\mathcal{D}}\{(f - f_Q)\chi_{Q_j}\}(x) > (1 - b)\alpha$. Let $\tilde{Q}_j$ be the parent of Q_j . Clearly, $\tilde{Q}_j \in \mathcal{D}$ and

$$A_j = \{x \in Q_j : M_{\mathcal{D}}(f - f_{\tilde{Q}_j})\chi_{Q_j}(x) > (1 - b)\alpha, M_{\mathcal{D}}f(x) > \alpha, f_{\text{loc}, \Omega}^{\#}(x) < c\alpha\}.$$

Since $M_{\mathcal{D}}$ satisfies the weak type (1,1) inequality with the constant $C' = 1$, we have

$$\begin{aligned} |A_j| &\leq \frac{1}{(1 - b)\alpha} \int_{Q_j} |f - f_{\tilde{Q}_j}| dx \leq \frac{1}{(1 - b)\alpha} \int_{\tilde{Q}_j} |f - f_{\tilde{Q}_j}| dx \leq \frac{|\tilde{Q}_j|}{(1 - b)\alpha} f_{\text{loc}, \Omega}^{\#}(x_0) \\ &\leq \frac{|\tilde{Q}_j|}{(1 - b)\alpha} c\alpha = \frac{2^nc}{1 - b} |Q_j|, \end{aligned}$$

so (2.6) holds in this case with $C = 2^n(1 - b)^{-1}$. $\square$

Fig. 3 A part of the partition $\tilde{\mathcal{L}}$ for $n = 2$ with the five layers $\tilde{L}_{-1}, \tilde{L}_0, \dots, \tilde{L}_3$

3 Localized sharp function for $(0, \infty)^n$

In this section $\tilde{\Omega} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_j > 0, j = 1, 2, \dots, n\}$ denotes the generalized first quoter in $\mathbb{R}^n$. The distance of $x \in \tilde{\Omega}$ from the boundary is given by $\tilde{\rho}(x, \partial\tilde{\Omega}) = \min\{x_j : j = 1, 2, \dots, n\}$. We define the partition $\tilde{\mathcal{L}}$ of $\tilde{\Omega}$:

$$\tilde{\mathcal{L}} = \{Q \in \Delta : Q \subset \tilde{\Omega} \text{ and } \ell(Q) = \tilde{\rho}(Q, \partial\tilde{\Omega})\}.$$

Clearly,

$$\tilde{\mathcal{L}} = \bigcup_{k \in \mathbb{Z}} \tilde{L}_k, \text{ where } \tilde{L}_k = \{Q \in \tilde{\mathcal{L}} : \ell(Q) = 2^{-k}\}.$$

Similarly to the previous section, for every positive integer m the partition $\tilde{\mathcal{L}}_m$ consists of dyadic cubes which are obtained by dividing each cube Q from $\tilde{\mathcal{L}}$ into 2^{mn} dyadic cubes each of the side-length $2^{-m}\ell(Q)$ (Fig. 3). Set

$$\tilde{\mathcal{D}} = \tilde{\mathcal{L}} \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{L}}_m.$$

Define the local maximal dyadic function $M_{\tilde{\mathcal{D}}}$ and localized sharp function $f_{\text{loc}, \tilde{\Omega}}^{\#}$ associated with the Whitney covering $\tilde{\mathcal{L}}$ of $\tilde{\Omega}$ as

$$M_{\tilde{D}}f(x) = \sup_{x \in Q \in \tilde{D}} |f|_Q,$$

$$f_{\text{loc}, \tilde{\Omega}}^{\#}(x) = \sup_{x \in K \subset \{y \in \tilde{\Omega} : \frac{1}{4}\tilde{\rho}(x, \partial\tilde{\Omega}) \leq \tilde{\rho}(y, \partial\tilde{\Omega}) \leq 4\tilde{\rho}(x, \partial\tilde{\Omega})\}} \frac{1}{|K|} \int_K |f(y) - f_K| dy,$$

where the supremum is taken over all cubes (not necessarily dyadic).

It turns out that the following theorem analogue to Theorem 2.1 holds.

Theorem 3.1 *For every $1 \leq p < \infty$ there is a constant $C > 0$ such that for every locally integrable function f on $\tilde{\Omega}$ for which there exists $0 < p_0 \leq p$ such that $M_{\tilde{D}}f \in L^{p_0}(\tilde{\Omega})$ one has*

$$\|M_{\tilde{D}}f\|_{L^p(\tilde{\Omega})} \leq C \|f_{\text{loc}, \tilde{\Omega}}^{\#}\|_{L^p(\tilde{\Omega})}.$$

Corollary 3.2 *For every $1 < p < \infty$ there is a constant $C > 0$ such that*

$$\|f\|_{L^p(\tilde{\Omega})} \leq C \|f_{\text{loc}, \tilde{\Omega}}^{\#}\|_{L^p(\tilde{\Omega})} \quad \text{for } f \in L^p(\tilde{\Omega}).$$

The remaining part of this section is devoted to proving Theorem 3.1.

Similarly to the previous section [see (2.1)] we set

$$\tilde{S}f(x) = |f|_Q \quad \text{and} \quad \tilde{S}'f(x) = f_Q, \quad (3.1)$$

where Q is the unique cube from $\tilde{\mathcal{L}}$ which contains x (such a Q is well-defined for almost every x). Let k be such that $Q \in \tilde{L}_k$. Our goal is to define the successors x' and $Q' \in \tilde{L}_{k-1}$ of x and Q respectively in such a way that $x' \in Q'$ and

$$\tilde{S}f(x) \leq \tilde{S}f(x') + 2^{1-n} 3^{2n} f_{\text{loc}, \tilde{\Omega}}^{\#}(x').$$

To this end, observe that there is a unique vector $q = (q_1, q_2, \dots, q_n)$, where q_j are non-negative integers such that at least one q_j equals 0, and $Q = 2^{-k}Q_0 + 2^{-k}(q + \mathbf{1})$, where here and subsequently, $\mathbf{1} = (1, 1, \dots, 1)$. Consider the coordinates x_j of x for which $q_j = 0$. There is no loss of generality if we assume these are the first m coordinates, $m \in \{1, 2, \dots, n\}$. So $Q = 2^{-k}Q_0 + 2^{-k}(1, 1, \dots, 1, 1 + q_{m+1}, \dots, 1 + q_n)$, $q_{m+1}, \dots, q_n \geq 1$. Define

$$x' = F(x) = (2x_1, 2x_2, \dots, 2x_m, x_{m+1}, \dots, x_n) \quad \text{for } x \in Q. \quad (3.2)$$

Then, for almost every x , the point x' belongs to the unique $Q' \in L_{k-1}$ and

$$Q' = 2^{-k+1}Q_0 + 2^{-k+1}(q' + \mathbf{1}),$$

$$q' = (0, \dots, 0, q'_{m+1}, \dots, q'_n), \quad q'_j = \lfloor (q_j - 1)/2 \rfloor, \quad j = m+1, \dots, n.$$

Fig. 4 Cubes which contain points $x, F(x), F(F(x)), \dots$ for $n = 2$

Lemma 3.3 For every $f \in L^1_{\text{loc}}(\tilde{\Omega})$ and almost every $x \in \tilde{\Omega}$ one has

$$\tilde{S}'f(x) \leq \tilde{S}'f(x') + 2^{-n} \cdot 3^{2n} f^{\#}_{\text{loc}, \tilde{\Omega}}(x'), \quad (3.3)$$

$$\tilde{S}f(x) \leq \tilde{S}f(x') + 2^{1-n} \cdot 3^{2n} f^{\#}_{\text{loc}, \tilde{\Omega}}(x'), \quad (3.4)$$

$$\tilde{S}'f(x') \leq \tilde{S}'f(x) + 2^{-n} \cdot 3^{2n} f^{\#}_{\text{loc}, \tilde{\Omega}}(x). \quad (3.5)$$

Proof Define (non-dyadic) cube $K'' = [2^{-k}, 2^{-k+2}]^m \times I_{m+1} \times \dots \times I_n \subset \tilde{\Omega}$, where

$$I_j = \left[2^{-k+1}q'_j + 2^{-k+2} - 3 \cdot 2^{-k}, 2^{-k+1}q'_j + 2^{-k+2} \right].$$

We have $\ell(K'') = 3 \cdot 2^{-k}$, $Q \cup F(Q) \cup Q' \subset K''$. Moreover, K'' is taken into account if we compute $f^{\#}_{\text{loc}, \tilde{\Omega}}(x')$ and $f^{\#}_{\text{loc}, \tilde{\Omega}}(x)$. Set $M = K'' \setminus (Q \cup Q')$. We have [(cf. (2.4)]

$$\frac{|Q|}{|K''|} = \frac{1}{3^n}, \quad \frac{|Q'|}{|K''|} = \frac{2^n}{3^n}, \quad \frac{|M|}{|K''|} = \frac{3^n - 2^n - 1}{3^n}, \quad (3.6)$$

hence the proof of the lemma is the same as those of Lemmata 2.5 and 2.7. $\square$

Corollary 3.4 Assume that $f \in L^1_{\text{loc}}(\tilde{\Omega})$ and $\lim_{m \rightarrow \infty} \tilde{S}f(F^m(x)) = 0$ for almost every $x \in \tilde{\Omega}$, where $F^m(x) = F(F^{m-1}(x))$ (Fig. 4). Then

$$\tilde{S}f(x) \leq \tilde{S}f(x),$$

where

$$\tilde{\mathbb{S}}f(x) = 2^{1-n} \cdot 3^{2n} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \sum_{m=0}^{\infty} f_{\text{loc}, \tilde{\Omega}}^{\#} \left(2^{k_1+m} x_1, 2^{k_2+m} x_2, \dots, 2^{k_n+m} x_n \right).$$

Proof It suffices to apply Lemma 3.3 and note that

$$2^{1-n} 3^{2n} \sum_{m=1}^{\infty} f_{\text{loc}, \tilde{\Omega}}^{\#}(F^m(x)) \leq \tilde{\mathbb{S}}f(x).$$

Remark 3.5 Let us note that $\lim_{m \rightarrow \infty} \tilde{\mathbb{S}}f(F^m(x)) = 0$ for $f \in L^p(\tilde{\Omega})$. This is a consequence of the fact that $\tilde{\rho}(F^m(x), \partial \tilde{\Omega}) \rightarrow \infty$ and $\ell(Q^m) \rightarrow \infty$, where Q^m is the unique cube from $\tilde{\mathcal{L}}$ such that $F^m(x) \in Q^m$.

Lemma 3.6 For every $1 \leq p < \infty$ there is a constant $C > 0$ such that for every $f \in L_{\text{loc}}^1(\tilde{\Omega})$ one has

$$\|\tilde{\mathbb{S}}f\|_{L^p(\tilde{\Omega})} \leq C \|f_{\text{loc}, \tilde{\Omega}}^{\#}\|_{L^p(\tilde{\Omega})}.$$

Proof This follows from the Minkowski inequality and the summability of the series

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \sum_{m=0}^{\infty} 2^{-k_1/p-m/p} 2^{-k_2/p-m/p} \cdots 2^{-k_n/p-m/p}.$$

□

Proposition 3.7 For every constant $0 < b < 1$ there is a constant $C > 0$ such that for all $c, \alpha > 0$, and every $f \in L_{\text{loc}}^1(\tilde{\Omega})$ satisfying

$$\lim_{m \rightarrow \infty} \tilde{\mathbb{S}}f(F^m(x)) = 0$$

we have

$$\begin{aligned} & \left| \{x \in \tilde{\Omega} : M_{\tilde{\mathcal{D}}}f(x) > \alpha, f_{\text{loc}, \tilde{\Omega}}^{\#}(x) < c\alpha\} \right| \\ & \leq Cc \left| \{x \in \tilde{\Omega} : M_{\tilde{\mathcal{D}}}f(x) > b\alpha\} \right| + \left| \{x \in \tilde{\Omega} : \tilde{\mathbb{S}}f(x) > b\alpha\} \right|. \end{aligned}$$

Proof The proof is identical to that of Proposition 2.4, and uses Corollary 3.4 instead of Corollary 2.8. □

Proof of Theorem 3.1 The theorem follows from Lemma 3.6 and Proposition 3.7. Its proof is identical to that of Theorem 2.1. □

Fig. 5 Cubes from the set $\tilde{\mathcal{L}}''$ for $n = 2$ (the picture presents cubes from the layers $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_5$)

4 Localized sharp function for cube

In this section we consider the cube $(0, 2)^n \subset \mathbb{R}^n$ and its Whitney decomposition $\tilde{\mathcal{L}}''$ which is defined in the following way. Let $\tilde{\mathcal{L}}'$ be the restriction of the decomposition $\tilde{\mathcal{L}}$ defined in the previous section into the unit cube $(0, 1]^n$. Let us denote by $\tilde{\mathcal{L}}''$ the set of cubes which is obtained from $\tilde{\mathcal{L}}'$ under the action of the group G of transformation generated by the reflections with respect to planes $x_j = 1$. Let $\tilde{L}_k$ be the set of cubes from $\tilde{\mathcal{L}}''$ of the side-length 2^{-k} . Clearly, $\tilde{\mathcal{L}}'' = \tilde{L}_1 \cup \tilde{L}_2 \cup \dots$ (Fig. 5).

We define the partition $\tilde{\mathcal{L}}''_1$ by dividing each cube K from $\tilde{\mathcal{L}}''$ into 2^n dyadic cubes each of the side-length $2^{-1}\ell(K)$. Inductively, $\tilde{\mathcal{L}}''_{m+1}$ is defined by dividing each cube K from $\tilde{\mathcal{L}}''_m$ into 2^n dyadic cubes of side-length $2^{-1}\ell(K)$. Set

$$\tilde{\mathcal{D}} = \tilde{\mathcal{L}}'' \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{L}}''_m, \quad M_{\tilde{\mathcal{D}}} f(x) = \sup_{x \in K \in \tilde{\mathcal{D}}} \frac{1}{|K|} \int_K |f(x)| dx.$$

The localized sharp function is defined by

$$f_{\text{loc}, (0, 2)^n}^{\#}(x) = \sup_{x \in K \subset \mathcal{B}_x} \frac{1}{|K|} \int_K |f(y) - f_K| dy,$$

where

$$\mathcal{B}_x = \left\{ y \in (0, 2)^n : \frac{1}{4} \text{dist}(x, \partial[0, 2]^n) \leq \text{dist}(y, \partial[0, 2]^n) \leq 4 \text{dist}(x, \partial[0, 2]^n) \right\}$$

and the supremum is taken over all cubes K not necessarily dyadic.

Our aim of this section is to prove the following theorem.

Fig. 6 Cubes K and $\bigcup \text{Pre}(K)$ for $n = 2$

Theorem 4.1 For every $1 \leq p < \infty$ there is a constant $C_p > 0$ such that if $f \in L^1((0, 2)^n)$, $\int_{(0, 2)^n} f(x) dx = 0$ and $M_{\tilde{\mathcal{Q}}} f \in L^1((0, 2)^n)$, then

$$\|M_{\tilde{\mathcal{Q}}} f\|_{L^p((0, 2)^n)} \leq C_p \|f_{\text{loc}, (0, 2)^n}^{\#}\|_{L^p((0, 2)^n)}.$$

The proof requires preparations.

For each $K \in \tilde{\mathcal{L}}''$ there is a unique $\sigma \in G$ such that $\sigma(K) \subset [0, 1]^n$. Therefore in our considerations we shall deal with cubes contained in $[0, 1]^n$ and then use the group action for other cubes.

From now on, let $Q_1 = \frac{1}{2}Q_0 + \frac{1}{2}\mathbf{1}$.

For $x \in (0, 1]^n$ let $F(x)$ be defined by (3.2). Clearly, for every $R \in \tilde{\mathcal{L}}' \cap \tilde{L}_k$ with $k \geq 2$ there is a unique $K \in \tilde{\mathcal{L}}' \cap \tilde{L}_{k-1}$ such that $F(R) \subset K$. For $K \in \tilde{\mathcal{L}}'$ we set

$$\text{Pre}(K) = \{R \in \tilde{\mathcal{L}}' : F^j(R) \subset K \text{ for a certain positive integer } j\}.$$

Figure 6 shows examples of $\bigcup \text{Pre}(K)$ for $n = 2$. We have

$$\left| \bigcup \text{Pre}(K) \right| \leq (2^n - 1)|K|. \quad (4.1)$$

Lemma 4.2 Let f be an integrable function on $(0, 2)^n$. Assume that $K \in \tilde{\mathcal{L}}' \cap \tilde{L}_m$. Then

$$\begin{aligned} \left| \bigcup \text{Pre}(K) \right| f_K &\leq \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} |R| f_R + 3^{2n} \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} |R| \inf_{y \in R} f_{\text{loc}, (0, 2)^n}^{\#}(y) \\ &+ \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left| \bigcup \text{Pre}(R) \right| f_R. \end{aligned}$$

Proof Set $C_n = 3^{2n}2^{-n}$. By the same arguments we used to prove (3.5), we get $f_K \leq f_R + C_n \inf_{y \in R} f_{\text{loc}, (0,2)^n}^\#(y)$ for $R \in \tilde{L}_{m+1}$ such that $F(R) \subset K$. Hence,

$$\begin{aligned} & \left| \bigcup \text{Pre}(K) \right| f_K \\ &= \sum_{R \in \tilde{L}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left(|R| f_R + \left| \bigcup \text{Pre}(R) \right| f_R \right) \\ &\leq \sum_{R \in \tilde{L}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left(|R| f_R + C_n |R| \inf_{y \in R} f_{\text{loc}, (0,2)^n}^\#(y) \right) \\ &\quad + \sum_{R \in \tilde{L}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left(C_n \left| \bigcup \text{Pre}(R) \right| \inf_{y \in R} f_{\text{loc}, (0,2)^n}^\#(y) \right) \\ &\quad + \sum_{R \in \tilde{L}' \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left| \bigcup \text{Pre}(R) \right| f_R, \end{aligned}$$

which, by (4.1), finishes the proof. $\square$

Corollary 4.3 Assume that f is an integrable function on $(0, 2)^n$. Then

$$f_{Q_1} \leq \int_{(0,1]^n} f(y) dy + 3^{2n} \int_{(0,1]^n} f_{\text{loc}, (0,2)^n}^\#(y) dy. \quad (4.2)$$

Proof Observe that $1 = |Q_1| + \left| \bigcup \text{Pre}(Q_1) \right|$. Thus

$$f_{Q_1} = |Q_1| f_{Q_1} + \left| \bigcup \text{Pre}(Q_1) \right| f_{Q_1}.$$

By iterating Lemma 4.2 we obtain that for every positive integer $m \geq 2$ one has

$$\begin{aligned} f_{Q_1} &\leq \sum_{j=1}^m \sum_{K \in \tilde{L}' \cap \tilde{L}_j} |K| f_K + 3^{2n} \sum_{j=2}^m \sum_{K \in \tilde{L}' \cap \tilde{L}_j} |K| \inf_{y \in K} f_{\text{loc}, (0,2)^n}^\#(y) \\ &\quad + \sum_{K \in \tilde{L}' \cap \tilde{L}_m} \left| \bigcup \text{Pre}(K) \right| f_K. \end{aligned}$$

Letting $m \rightarrow \infty$, we obtain the corollary, since the last summand tends to 0. $\square$

Corollary 4.4 There is a constant $C' > 0$ such that for every integrable function f on $(0, 2)^n$ such that $\int_{(0,2)^n} f(x) dx = 0$ one has

$$\sum_{\sigma \in G} f_{\sigma(Q_1)} \leq C' \int_{(0,2)^n} f_{\text{loc}, (0,2)^n}^\#(y) dy, \quad (4.3)$$

$$\left| f_{\bigcup_{\sigma \in G} \sigma(Q_1)} \right| = 2^{-n} \left| \sum_{\sigma \in G} f_{\sigma(Q_1)} \right| \leq C' \int_{(0,2)^n} f_{\text{loc}, (0,2)^n}^\#(y) dy, \quad (4.4)$$

$$|f|_{\bigcup_{\sigma \in G} \sigma(Q_1)} \leq C' \int_{(0,2)^n} f_{\text{loc}, (0,2)^n}^{\#}(y) dy, \quad (4.5)$$

$$|f|_{\sigma(Q_1)} \leq C' \int_{(0,2)^n} f_{\text{loc}, (0,2)^n}^{\#}(y) dy. \quad (4.6)$$

Proof Clearly, $\sum_{\sigma \in G} \int_{\sigma((0,1]^n)} f dy = \int_{(0,2)^n} f dy = 0$. Hence (4.3) follows from (4.2). The inequality (4.4) is a direct consequence of (4.3). Further we write

$$\begin{aligned} \int_{\bigcup \sigma(Q_1)} |f| dy &\leq \int_{\bigcup \sigma(Q_1)} |f - f_{\bigcup \sigma(Q_1)}| dy + |f_{\bigcup \sigma(Q_1)}| \\ &\leq \inf_{x \in \bigcup \sigma(Q_1)} f_{\text{loc}, (0,2)^n}^{\#}(x) + |f_{\bigcup \sigma(Q_1)}| \end{aligned}$$

and apply (4.4) to obtain (4.5) and then (4.6). $\square$

Assume that $f \in L^1((0,2)^n)$. For $x \in (0,1]^n$ we define the function $\mathbf{S}f(x)$ as follows

$$\mathbf{S}f(x) = |f|_{Q_1} + 2^{1-n} 3^{2n} \sum_{j=0}^{k-1} f_{\text{loc}, (0,2)^n}^{\#}(F^j(x)) \text{ for } x \in K \in \tilde{L}_k \cap \tilde{L}'.$$

Proposition 4.5 *For every constant $0 < b < 1$ there is a constant $C > 0$ such that for all $c, \alpha > 0$, and every $f \in L^1((0,2)^n)$ we have*

$$\begin{aligned} &\left| \{x \in (0,1]^n : M_{\tilde{\mathcal{D}}} f(x) > \alpha, f_{\text{loc}, (0,2)^n}^{\#}(x) < c\alpha\} \right| \\ &\leq Cc \left| \{x \in (0,1]^n : M_{\tilde{\mathcal{D}}} f(x) > b\alpha\} \right| + \left| \{x \in (0,1]^n : \mathbf{S}f(x) \geq b\alpha\} \right|. \end{aligned}$$

Proof For $x \in (0,1]^n$ let $\tilde{\mathbf{S}}f(x)$ be defined by (3.1). The same arguments we used to prove Lemma 3.3 give

$$\tilde{\mathbf{S}}f(x) \leq \tilde{\mathbf{S}}(F(x)) + 2^{1-n} 3^{2n} f_{\text{loc}, (0,2)^n}^{\#}(F(x)) \text{ for } x \in (0,1]^n \setminus Q_1. \quad (4.7)$$

Iteration of (4.7) leads to $\tilde{\mathbf{S}}f(x) \leq \mathbf{S}f(x)$ for $x \in (0,1]^n$. Now the proof is the same as that of Proposition 2.4. $\square$

Proof of Theorem 4.1 For $f \in L_{\text{loc}, (0,2)^n}^1((0,2)^n)$ and $\sigma \in G$ let $f_{\sigma}(x) = f(\sigma(x))$. Since $M_{\tilde{\mathcal{D}}} f_{\sigma} = (M_{\tilde{\mathcal{D}}} f)_{\sigma}$, $(f_{\sigma})_{\text{loc}, (0,2)^n}^{\#} = (f_{\text{loc}, (0,2)^n}^{\#})_{\sigma}$, and $(0,2)^n = \bigcup_{\sigma \in G} \sigma((0,1]^n)$, it suffices to prove that

$$\|M_{\tilde{\mathcal{D}}} f\|_{L^p((0,1]^n)} \leq C_p \|f_{\text{loc}, (0,2)^n}^{\#}\|_{L^p((0,2)^n)} \quad (4.8)$$

for $f \in L^1((0, 2)^n)$, $\int_{(0, 2)^n} f = 0$. Repeating the proof of Theorem 2.1 with the use of Proposition 4.5 we arrive at

$$\begin{aligned} \|M_{\mathcal{S}} f\|_{L^p((0, 1]^n)} &\leq C \|Sf\|_{L^p((0, 1]^n)} + C \|f_{\text{loc}, (0, 2)^n}^{\#}\|_{L^p((0, 1]^n)} \\ &\leq C' |f|_{Q_1} + C' \|f_{\text{loc}, (0, 2)^n}^{\#}\|_{L^p((0, 1]^n)}. \end{aligned} \quad (4.9)$$

Recall that the integral of f is zero. Hence, applying (4.6), we obtain the desired inequality (4.8). $\square$

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References

1. Duoandikoetxea, J.: Fourier Analysis. American Mathematical Society, Providence (2001)
2. Fefferman, C., Stein, E.M.: H^p spaces of several variables. Acta Math. **129**, 137–195 (1972)
3. Grafakos, L.: Modern Fourier Analysis. Graduate Texts in Mathematics, vol. 250, 3rd edn. Springer, New York (2014)
4. Stein, E.M.: Harmonic Analysis Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)