Moments of volumes of lower-dimensional random simplices are not monotone

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Abstract In a d -dimensional convex body K , for $n \leq d + 1$, random points $X_0, \dots, X_{n-1}$ are chosen according to the uniform distribution in K . Their convex hull is a random $(n - 1)$ -simplex with probability 1. We denote its $(n - 1)$ -dimensional volume by $V_{K[n]}$. The k -th moment of the $(n - 1)$ -dimensional volume of a random $(n - 1)$ -simplex is monotone under set inclusion, if $K \subseteq L$ implies that the k -th moment of $V_{K[n]}$ is not larger than that of $V_{L[n]}$. Extending work of Rademacher (Mathematika 58:77–91, 2012) and Reichenwallner and Reitzner (Mathematika 62:949–958, 2016), it is shown that for $n \leq d$, the moments of $V_{K[n]}$ are not monotone under set inclusion. As a consequence, the nonmonotonicity of the expected surface area of the convex hull of $n \leq d + 1$ uniform random points in a d -dimensional convex body follows.

Keywords Random polytopes · Random simplices · Approximation of convex bodies · Sylvester’s problem · Monotonicity under inclusion

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1 Introduction

For a d -dimensional convex body K and $n \leq d + 1$, we denote the $(n - 1)$ -dimensional volume of the convex hull of n independently and uniformly distributed random points $X_0, \dots, X_{n-1}$ in K by $V_{K[n]}$. Since the n points are not contained in any $(n - 2)$ -

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dimensional hyperplane with probability 1, their convex hull is almost surely an $(n - 1)$ -simplex. In 2006, Meckes [5] asked whether for a pair of convex bodies $K, L \subseteq \mathbb{R}^d$, $K \subseteq L$ would imply

$$\mathbb{E}V_{K[d+1]} \leq \mathbb{E}V_{L[d+1]}. \quad (1)$$

Interest in this question comes from the fact that it would imply a positive solution to the slicing problem, see e.g. [7]. For a more general statement of the conjecture we refer to [9].

In 2012, Rademacher [7] showed that Meckes' conjecture (1) is not true in general, since there exist counterexamples for each dimension $d \geq 4$. His results were generalized to higher moments of $V_{K[d+1]}$ by Reichenwallner and Reitzner [8], and the expected value in dimension three was handled in [4]:

Theorem 1 *Let $k \in \mathbb{N}$ and $d \geq 2$. For two-dimensional convex bodies K and L , $K \subseteq L$ implies that $\mathbb{E}V_{K[d+1]} \leq \mathbb{E}V_{L[d+1]}$ and $\mathbb{E}V_{K[d+1]}^2 \leq \mathbb{E}V_{L[d+1]}^2$. Unless $d = 2$ and $k \in \{1, 2\}$, there exist two d -dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}V_{K[d+1]}^k > \mathbb{E}V_{L[d+1]}^k$.*

Note that the problem is trivial in dimension one. In this paper, we perform another extension by considering convex hulls of $n \leq d + 1$ random points in d -dimensional convex bodies. In order to keep the statement of the result simple, we restrict to $n \leq d$ and refer to the theorem above for the case $n = d + 1$.

Theorem 2 *Let $d, n, k \in \mathbb{N}$ with $d \geq 2$ and $2 \leq n \leq d$. Then there exist two d -dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}V_{K[n]}^k > \mathbb{E}V_{L[n]}^k$.*

Due to Rademacher [7], it is already known that the moments of $V_{K[n]}$ are not monotone under set inclusion in general. However, it might still come as a surprise that for $n \leq d$, there exist counterexamples for *any* moment in *any* dimension, particularly in dimension two, where the first two moments of the area of a random triangle are monotone.

Theorem 1 is shown using an equivalence statement, first stated in Lemma 12 in [7]. In consideration of Proposition 16 ibidem, it is clear that the lemma can be generalized in the following way:

Lemma 1 *For $d, n, k \in \mathbb{N}$ with $2 \leq n \leq d + 1$, monotonicity under inclusion of the map $K \mapsto \mathbb{E}V_{K[n]}^k$, where K ranges over all d -dimensional convex bodies, holds if and only if we have for each convex body $K \subseteq \mathbb{R}^d$ and for each $x \in \text{bd } K$ that $\mathbb{E}V_{K[n]}^k \leq \mathbb{E}V_{K[n],x}^k$.*

Here, $V_{K[n],x}$ denotes the $(n - 1)$ -dimensional volume of a random $(n - 1)$ -simplex, which is the convex hull of a fixed point x and $n - 1$ independently and uniformly distributed random points in K . The lemma allows us to consider one convex body K , rather than a pair of convex bodies, and compute two different moments: the moment of the $(n - 1)$ -dimensional volume of a random $(n - 1)$ -simplex in K as well as the same, but fixing one of the n points to be a point on the boundary of K , denoted by $\text{bd } K$.

For the case $n = d + 1$, Rademacher chooses K to be a d -dimensional halfball and x the midpoint of the base which is a $(d - 1)$ -dimensional ball. This pair forms the counterexample in Theorem 1 to the monotonicity of all moments for $d \geq 4$, and to the monotonicity of the k -th moment for $k \geq 11$ in dimension two and for $k \geq 2$ in dimension three (see [8]; in fact, in dimension two, it does so for $k \geq 8$). Additional counterexamples are given by the triangle, together with the midpoint of one of its edges, for $d = 2$ and $k \geq 3$ in [8] and the tetrahedron, together with the centroid of one of its facets, for $d = 3$ and $k = 1$ in [4].

For our generalization, we use the following lemma, which allows us to create counterexamples inductively:

Lemma 2 *Let $d, n, k \in \mathbb{N}$ with $2 \leq n \leq d + 1$. If there exist two d -dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}V_{K[n]}^k > \mathbb{E}V_{L[n]}^k$, there also exist two $(d + 1)$ -dimensional convex bodies K', L' satisfying $K' \subseteq L'$ and $\mathbb{E}V_{K'[n]}^k > \mathbb{E}V_{L'[n]}^k$.*

We use the counterexamples mentioned above as well as Lemma 2 to get results on the monotonicity of the moments of $V_{K[n]}$, where K ranges over all d -dimensional convex bodies, $d \geq 2$ and $2 \leq n \leq d$. Since, obviously, monotonicity holds in dimension 1, in order to show that the moments of the length of a random chord in d -dimensional convex bodies are not monotone for $d > 1$, we create another counterexample by computing the moments of the length of a random chord inside a rectangular triangle where both points are randomly chosen as well as the same where one point is fixed to be the midpoint of the hypotenuse.

Proposition 1 *For the triangle T_2 with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, and $c_2 = (1/2, 1/2)$ the midpoint of the edge $\{(x, y) : x, y \geq 0, x + y = 1\}$, we have that $\mathbb{E}V_{T_2[2], c_2}^k < \mathbb{E}V_{T_2[2]}^k$ for each $k \in \mathbb{N}$.*

Concerning the case $n = 3, k = 1$, we need to construct a pair of three-dimensional convex bodies, or one three-dimensional convex body and a suitable point on its boundary, leading to a counterexample. Already the computation of the expected volume of a random tetrahedron in dimension three is a hard problem, and the same is supposed to be true for the expected area of a random triangle in dimension three.

The only three-dimensional convex sets where the expected volume of a random full-dimensional simplex is known are the ball [6], the cube [12] and the tetrahedron [1], and the ball is the only one for which the expected area of a random triangle inside is known. Since numerical simulations suggest that the ball cannot form a counterexample in our concerns, and neither can the cube, we consider random triangles in a tetrahedron, which nothing is known about. A suitable choice for x should be the center c of one of the facets, but we have to take into account that the considered moments depend on the shape of T as well as of the facet which c lies in. Additionally, already the determination of the expected volume of a random tetrahedron in a tetrahedron T , done by Buchta and Reitzner [1], was extremely hard, and it seems that a computation of $\mathbb{E}V_{T[3]}$ and $\mathbb{E}V_{T[3], c}$ is out of reach. Nevertheless we will prove the following proposition.

Proposition 2 *For the tetrahedron T_3 with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, and $c_3 = (1/3, 1/3, 1/3)$ the centroid of the facet $\{(x, y, z) : x, y, z \geq 0, x + y + z = 1\}$, we have that $\mathbb{E}V_{T_3[3], c_3} < \mathbb{E}V_{T_3[3]}$.*

A combination of this result and a direct computation of $\mathbb{E}V_{T_3[3]}^2$ and $\mathbb{E}V_{T_3[3],c_3}^2$ with Rademacher's Lemma 1 yields Theorem 2. Proposition 2 is obtained by a combination of methods from stochastic geometry with results from approximation theory. In the background, we use a result about the precise approximation of the square root function by suitable polynomials.

A next step might naturally be given by a generalization of Theorems 1 and 2 to intrinsic volumes, among them the surface area. While a derivation of results on its higher moments seems to be rather involved, it appears that the nonmonotonicity of the expected surface area of a random simplex, eventually not of full dimension, is a direct consequence of Theorem 2. For a d -dimensional convex body K and $2 \leq n \leq d+1$, let $S_{K[n]}$ denote the surface area of the convex hull of n points that are chosen according to uniform distribution in K . Since the (relative) boundary of an $(n-1)$ -simplex is given by the union of n facets and the expected value is additive, the following corollary holds true:

Corollary 1 *Let $d, n \in \mathbb{N}$ with $d \geq 2$ and $2 \leq n \leq d+1$. Then there exist two d -dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}S_{K[n]} > \mathbb{E}S_{L[n]}$.*

More generally, it shall be mentioned that the expected value of the sum of the m -dimensional volumes of all m -faces of the convex hull of $n \geq m+2$ uniform random points in a convex body K of dimension $d \geq n-1$ is not monotone under set inclusion.

Finally, we consider a similar problem: Assume that n points are uniformly distributed on the boundary of a convex body K and let $\mathcal{V}_{K[n]}$ be the $(n-1)$ -dimensional volume of their convex hull. Then we can use a technique similar to that in the background of Lemma 2 to show that, in general, monotonicity under set inclusion does not hold for the moments of $(n-1)$ -dimensional volumes of this type of random $(n-1)$ -simplices either. In detail, we show the following:

Theorem 3 *Let $d, n, k \in \mathbb{N}$ with $d \geq 3$ and $2 \leq n \leq d$. Unless $d = 3, n = 3$ and $k \in \{1, 2\}$, there exist two d -dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}\mathcal{V}_{K[n]}^k > \mathbb{E}\mathcal{V}_{L[n]}^k$.*

This paper is organized in the following way. We prove Lemma 2 in Sect. 2. In Sect. 3, we compute the moments of the lengths of two types of random chords inside a triangle as described above, and thereafter we give bounds for the expected areas of random triangles in a tetrahedron. These results will be used in Sect. 5 for the proof of Theorem 2. Finally, Theorem 3 will be proved in Sect. 6.

As a general reference for the tools and results we need in the following, we refer to the book on Stochastic and Integral Geometry by Schneider and Weil [10]. More recent surveys on random polytopes are due to Hug [3] and Reitzner [9].

2 Proof of Lemma 2

Let $d, n, k \in \mathbb{N}$ with $2 \leq n \leq d+1$ and assume that there exist two d -dimensional convex bodies K and L satisfying $K \subseteq L$ and $\mathbb{E}V_{K[n]}^k > \mathbb{E}V_{L[n]}^k$. Our goal is to

construct two $(d + 1)$ -dimensional convex bodies K', L' satisfying similar conditions. For $\varepsilon > 0$, we define two $(d + 1)$ -dimensional convex bodies by

$$K_\varepsilon := K \times [0, \varepsilon], \quad L_\varepsilon := L \times [0, \varepsilon].$$

Then, $K_\varepsilon \subseteq L_\varepsilon$ for all ε , and if $\mathbb{E}V_{K_\varepsilon[n]}^k$ and $\mathbb{E}V_{L_\varepsilon[n]}^k$ converge to $\mathbb{E}V_{K[n]}^k$ and $\mathbb{E}V_{L[n]}^k$, resp., for ε tending to zero, there exists an $\varepsilon_0 > 0$ such that $\mathbb{E}V_{K_{\varepsilon_0}[n]}^k > \mathbb{E}V_{L_{\varepsilon_0}[n]}^k$.

It remains to proof the continuity of the map $\varepsilon \mapsto \mathbb{E}V_{K_\varepsilon[n]}^k$ in $\varepsilon = 0$. We express the $(n - 1)$ -dimensional volume of an $(n - 1)$ -simplex with vertices $x_0, \dots, x_{n-1}$ using a matrix that contains the vectors spanning the $(n - 1)$ -simplex in its columns:

$$\begin{aligned} \text{vol}_{n-1} \text{conv}(x_0, x_1, \dots, x_{n-1}) \\ = \frac{1}{(n-1)!} \sqrt{\det((x_1 - x_0, \dots, x_{n-1} - x_0)^T (x_1 - x_0, \dots, x_{n-1} - x_0))}. \end{aligned}$$

Then we have for $\varepsilon > 0$:

$$\begin{aligned} \mathbb{E}V_{K_\varepsilon[n]}^k &= \frac{1}{\text{vol } K_\varepsilon^n} \int_{K_\varepsilon^n} \text{vol}_{n-1} \text{conv}(x_0, \dots, x_{n-1})^k d(x_0, \dots, x_{n-1}) \\ &= \frac{1}{\varepsilon^n \text{vol}_d K^n} \frac{1}{((n-1)!)^k} \int_{K^n} \int_{[0, \varepsilon]^n} \det B^{k/2} d(z_0, \dots, z_{n-1}) d(x_0, \dots, x_{n-1}), \end{aligned}$$

where $B = (b_{ij})_{1 \leq i, j \leq n-1}$ with

$$b_{ij} = \sum_{m=1}^d \left(x_i^{(m)} - x_0^{(m)} \right) \left(x_j^{(m)} - x_0^{(m)} \right) + (z_i - z_0)(z_j - z_0),$$

writing $x_i = (x_i^{(1)}, \dots, x_i^{(d)}, z_i)$. Now, fix the first d coordinates of the points $x_0, \dots, x_{n-1}$ and let $\zeta := \min_{z_i} \det B$ and $Z := \max_{z_i} \det B$, where the minimum and the maximum are taken over all possible choices of $(z_0, \dots, z_{n-1}) \in [0, \varepsilon]^n$. The extrema exist since the determinant is continuous and the interval $[0, \varepsilon]$ is compact. We have:

$$\begin{aligned} \frac{1}{\text{vol}_d K^n} \frac{1}{((n-1)!)^k} \int_{K^n} \zeta^{k/2} d(x_0, \dots, x_{n-1}) &\leq \mathbb{E}V_{K_\varepsilon[n]}^k \\ &\leq \frac{1}{\text{vol}_d K^n} \frac{1}{((n-1)!)^k} \int_{K^n} Z^{k/2} d(x_0, \dots, x_{n-1}). \end{aligned}$$

If ε tends to zero, the coordinates z_i do so as well, and all summands of ζ and Z containing a factor z_i vanish in the limit. Therefore, we get that

$$\mathbb{E}V_{K[n]}^k \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E}V_{K_\varepsilon[n]}^k \leq \mathbb{E}V_{K[n]}^k,$$

which implies that $\lim_{\varepsilon \rightarrow 0} \mathbb{E}V_{K_\varepsilon[n]}^k = \mathbb{E}V_{K[n]}^k$. $\square$

3 Random chords in a triangle

Essential for our investigations of the case $n = 2$ are the moments of two different types of random distances in a given triangle T . In detail, we need the moments of the length of a random chord in a right-angled triangle, which arises as the segment between two uniformly chosen random points in this triangle, as well as the moments of the distance of a random point in this triangle from the midpoint of its hypotenuse, which is equivalent to the length of a random chord where one point is fixed to be this midpoint.

3.1 Moments of the distance of a random point from a point on the boundary

We first prove a lemma that will be useful for the computation of the moments of the distance of a random point from a fixed point on the boundary, since we can dissect a triangle by a line through this fixed point.

Lemma 3 *Let T be a triangle with vertices A , B and C . Denote the lengths of the edges opposite to A , B and C by a , b and c , resp., and the angles in these corners by α , β and γ , resp. Then it holds for the moments of the distance of a random point in T from A :*

$$\mathbb{E}V_{T[2],A}^k = \frac{2(c \sin \beta)^{k+1}}{(k+2)a} \left(\cos \beta {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \beta^2 \right) - \cos(\alpha + \beta) {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos(\alpha + \beta)^2 \right) \right).$$

Here, the function

$${}_2F_1(a_1, a_2; b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{x^k}{k!}$$

with $(a)_k = a(a+1) \cdots (a+k-1)$ is a hypergeometric function.

Proof Since the expected length of a random chord in a triangle does not depend on the position of the latter, we build our coordinate system ensuring that A is identical to the origin o and the edge c lies on the x -axis. Then:

$$\mathbb{E}V_{T[2],o}^k = \frac{1}{\text{vol } T} \int_T ||x||^k dx.$$

Note that $\text{vol } T = ac \sin \beta/2$. Using polar coordinates, we get that

$$\mathbb{E}V_{T[2],o}^k = \frac{2}{ac \sin \beta} \int_0^\alpha \int_0^{l(\varphi)} r^{k+1} dr d\varphi = \frac{1}{k+2} \frac{2}{ac \sin \beta} \int_0^\alpha l(\varphi)^{k+2} d\varphi,$$

where $l(\varphi)$ denotes the length of the intersection of the triangle with the line whose angle to the x -axis is φ . By the sine rules, $l(\varphi) = c \sin \beta / \sin(\beta + \varphi)$, and

$$\mathbb{E} V_{T[2],o}^k = \frac{2(c \sin \beta)^{k+1}}{(k+2)a} \int_0^\alpha \frac{1}{\sin(\beta + \varphi)^{k+2}} d\varphi.$$

With

$$\int \frac{1}{\sin \varphi^{k+2}} d\varphi = -\cos \varphi {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \varphi^2\right),$$

we get that

$$\begin{aligned} \mathbb{E} V_{T[2],o}^k &= \frac{2(c \sin \beta)^{k+1}}{(k+2)a} \left(\cos \beta {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \beta^2\right) \right. \\ &\quad \left. - \cos(\alpha + \beta) {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos(\alpha + \beta)^2\right) \right). \end{aligned}$$

□

Given this lemma, we can compute the moments of the distance of a random point from a point on the boundary of a triangle.

Proposition 3 *Let T be a triangle with vertices A , B and C . Denote the lengths of the edges opposite to A , B and C by a , b and c , resp., and the angles in these corners by α , β and γ , resp. Furthermore, let D be a point lying on the edge between A and B and c_1 the distance of D from A . Then it holds for the moments of the distance of a random point in T from D :*

$$\begin{aligned} \mathbb{E} V_{T[2],D}^k &= \frac{2}{(k+2)bc \sin \alpha} \left((c_1 \sin \alpha)^{k+2} \cos \alpha {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \alpha^2\right) \right. \\ &\quad - (c_1 \sin \alpha)^{k+2} \cos(\alpha + \delta) {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos(\alpha + \delta)^2\right) \\ &\quad + ((c - c_1) \sin \beta)^{k+2} \cos \beta {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \beta^2\right) \\ &\quad \left. + ((c - c_1) \sin \beta)^{k+2} \cos(\beta - \delta) {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos(\beta - \delta)^2\right) \right), \end{aligned}$$

where

$$\delta = \arcsin\left(b \sin \alpha / \sqrt{c_1^2 + b^2 - 2c_1b \cos \alpha}\right).$$

Proof We split T into two parts by the line segment between C and D . Denote the length of this line segment by d and the angle between AD and DC by δ . Then, by cosine and sine rule,

$$d = \sqrt{c_1^2 + b^2 - 2c_1b \cos \alpha} \quad \text{and} \quad \delta = \arcsin(b \sin \alpha / d).$$

If we denote the triangle ACD by T_1 and the triangle BCD by T_2 , we have:

$$\mathbb{E}V_{T[2],D}^k = \frac{\text{vol } T_1}{\text{vol } T} \mathbb{E}V_{T_1[2],D}^k + \frac{\text{vol } T_2}{\text{vol } T} \mathbb{E}V_{T_2[2],D}^k.$$

The previous lemma yields the result. $\square$

In order to find a counterexample to the monotonicity of the moments of the length of random chords, it suffices to consider one specific triangle. We choose a right-angled, isosceles one and apply Proposition 3 to this triangle.

Corollary 2 *For the triangle T_2 with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$, and $c_2 = (1/2, 1/2)$, we have:*

$$\mathbb{E}V_{T_2[2],c_2}^k = \frac{2^{1/2-k}}{k+2} {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \frac{1}{2}\right).$$

We give some values of $\mathbb{E}V_{T_2[2],c_2}^k$ for this specific triangle:

$$\begin{aligned} \mathbb{E}V_{T_2[2],c_2} &= \frac{1}{6\sqrt{2}}(2 + \sqrt{2} \operatorname{arcsinh} 1) = 0.3825 \dots, \\ \mathbb{E}V_{T_2[2],c_2}^2 &= \frac{1}{6} = 0.1666 \dots, \\ \mathbb{E}V_{T_2[2],c_2}^3 &= \frac{1}{160\sqrt{2}}(14 + 3\sqrt{2} \operatorname{arcsinh} 1) = 0.0783 \dots, \\ \mathbb{E}V_{T_2[2],c_2}^4 &= \frac{7}{180} = 0.0388 \dots \end{aligned}$$

3.2 Moments of the length of a random chord

In this subsection, we complete our investigations of the triangle by a computation of the moments of the length of a random chord in a triangle, where both endpoints of the chord are uniformly distributed points in the triangle.

Proposition 4 *Let T be a triangle with vertices E_1 , E_2 and E_3 . Denote the length of the edge opposite to E_i by e_i and the angle in this corner by η_i , $i = 1, 2, 3$. Then it holds for the moments of the distance of two random point in T :*

$$\begin{aligned} \mathbb{E}V_{T[2]}^k &= \frac{8}{(k+2)(k+3)(k+4)} \frac{1}{(e_1 e_3 \sin \eta_2)^2} \sum_{i \neq j} \sin \eta_i^{k+3} e_j^{k+4} \\ &\quad \times \left[\cos \eta_i \cos \eta_j {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_j^2\right) \right] \end{aligned}$$

$$\begin{aligned}
 & + \cos \eta_i^2 {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_i^2 \right) \\
 & + \frac{\sin \eta_i}{k+2} \left(\frac{1}{\sin \eta_j^{k+2}} - \frac{1}{\sin \eta_i^{k+2}} \right) \Bigg].
 \end{aligned}$$

Proof As before, the area of the triangle is given by $e_1 e_3 \sin \eta_2 / 2$. Using the affine Blaschke-Petkantschin formula—see e.g. [10]—we integrate over all lines H intersecting the triangle:

$$\begin{aligned}
 \mathbb{E} V_{T[2]}^k &= \frac{1}{\text{vol } T^2} \int_{T^2} \|x_0 - x_1\|^k d(x_0, x_1) \\
 &= \frac{4}{(e_1 e_3 \sin \eta_2)^2} \int_{A(2,1)} \int_{(H \cap T)^2} \|x_0 - x_1\|^k d(x_0, x_1) dH,
 \end{aligned}$$

where $A(2, 1)$ denotes the affine Grassmannian of lines in $\mathbb{R}^2$. We represent a line H by its unit normal vector $u \in S_1$ and its distance $t > 0$ from the origin and we therefore denote the line by

$$H_{t,u} = \{x \in \mathbb{R}^2 : \langle x, u \rangle = t\}.$$

We choose the normalization of the Haar measure dH in such a way that $dH = dt du$, where dt and du correspond to Lebesgue measures in $\mathbb{R}$ and S_1 . We write the appearing integral as an expectation to get

$$\begin{aligned}
 \mathbb{E} V_{T[2]}^k &= \frac{4}{(e_1 e_3 \sin \eta_2)^2} \int_{S_1} \int_0^\infty \int_{(H_{t,u} \cap T)^2} \|x_0 - x_1\|^{k+1} d(x_0, x_1) dt du \\
 &= \frac{4}{(e_1 e_3 \sin \eta_2)^2} \int_{S_1} \int_0^\infty \text{vol}(H_{t,u} \cap T)^2 \mathbb{E} V_{H_{t,u} \cap T}^{k+1} dt du.
 \end{aligned}$$

The $(k+1)$ -st moment of the distance of two random points in an interval I of length l has already been computed by Solomon [11], amongst others, to be given by

$$\mathbb{E} V_I^{k+1} = \frac{2l^{k+1}}{(k+2)(k+3)}.$$

Hence, we obtain

$$\mathbb{E} V_{T[2]}^k = \frac{8}{(k+2)(k+3)} \frac{1}{(e_1 e_3 \sin \eta_2)^2} \int_{S_1} \int_0^\infty \text{vol}(H_{t,u} \cap T)^{k+3} dt du.$$

A line $H_{t,u}$ that intersects the triangle T a.s. meets exactly two edges of T . It splits T into a triangle and a quadrangle. We say that $H_{t,u}$ cuts off a vertex E_i from T if E_i is contained in the triangular part. Furthermore, we write

$$\mathcal{I}^{(i)} = \int_{S_1} \int_0^\infty \mathbb{1}(H_{t,u} \text{ cuts off } E_i \text{ from } T) \operatorname{vol}(H_{t,u} \cap T)^{k+3} dt du$$

for $i = 1, 2, 3$, which gives

$$\mathbb{E} V_{T[2]}^k = \frac{8}{(k+2)(k+3)} \frac{1}{(e_1 e_3 \sin \eta_2)^2} (\mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)}).$$

We state the following lemma, which will be proved right after the end of the proof of this proposition:

Lemma 4 *It holds for $i = 1, 2, 3$:*

$$\begin{aligned} \mathcal{I}^{(i)} = & \frac{\sin \eta_i^{k+3}}{k+4} \left(e_j^{k+4} \left[\cos \eta_i \cos \eta_j {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_j^2 \right) \right. \right. \\ & + \cos \eta_i^2 {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_i^2 \right) \\ & + \frac{\sin \eta_i}{k+2} \left(\frac{1}{\sin \eta_j^{k+2}} - \frac{1}{\sin \eta_i^{k+2}} \right) \Bigg] \\ & + e_l^{k+4} \left[\cos \eta_i \cos \eta_l {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_l^2 \right) \right. \\ & + \cos \eta_i^2 {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_i^2 \right) \\ & + \frac{\sin \eta_i}{k+2} \left(\frac{1}{\sin \eta_l^{k+2}} - \frac{1}{\sin \eta_i^{k+2}} \right) \Bigg] \Bigg). \end{aligned}$$

This lemma immediately yields the result. $\square$

Proof of Lemma 4 Let $i \in \{1, 2, 3\}$. We substitute $u = (\cos \varphi, \sin \varphi)$ such that the line $\varphi = 0$ contains the edge between E_i and E_j for $j \neq i$ fixed, and by the well-known fact that $du = d\varphi$ and with $H_{t,\varphi} = \{(x, y) \in \mathbb{R}^2 : x \cos \varphi + y \sin \varphi = t\}$, we get the integral

$$\mathcal{I}^{(i)} = \int_{-\pi}^{\pi} \int_0^\infty \mathbb{1}(H_{t,\varphi} \text{ cuts off } E_i \text{ from } T) \operatorname{vol}(H_{t,\varphi} \cap T)^{k+3} dt d\varphi.$$

Let $l \in \{1, 2, 3\} \setminus \{i, j\}$. A line $H_{t,\varphi}$ cuts off E_i from T if and only if $\varphi \in [\eta_i - \pi/2, \pi/2]$ and

- (i) $0 \leq t \leq e_j \cos(\eta_1 - \varphi)$, if $\varphi \leq \frac{\pi}{2} - \eta_j$,
 (ii) $0 \leq t \leq e_l \cos \varphi$, if $\varphi \geq \frac{\pi}{2} - \eta_j$.

We use the notation $l(t, \varphi) = \text{vol}(H_{t, \varphi} \cap T)$ for the length of the intersection of T with a line $H_{t, \varphi}$. Basic trigonometric considerations show that

$$l(t, \varphi) = t(\tan(\eta_i - \varphi) + \tan \varphi) = t \frac{\sin \eta_i}{\cos \varphi \cos(\eta_i - \varphi)}.$$

This leads to the expression

$$\begin{aligned} \mathcal{I}^{(i)} &= \int_{\eta_i - \pi/2}^{\pi/2 - \eta_j} \left(\frac{\sin \eta_i}{\cos \varphi \cos(\eta_i - \varphi)} \right)^{k+3} \int_0^{e_j \cos(\eta_i - \varphi)} t^{k+3} dt d\varphi \\ &+ \int_{\pi/2 - \eta_j}^{\pi/2} \left(\frac{\sin \eta_i}{\cos \varphi \cos(\eta_i - \varphi)} \right)^{k+3} \int_0^{e_l \cos \varphi} t^{k+3} dt d\varphi \\ &= \frac{\sin \eta_i^{k+3}}{k+4} \left(e_j^{k+4} \int_{\eta_i - \pi/2}^{\pi/2 - \eta_j} \frac{\cos(\eta_i - \varphi)}{\cos \varphi^{k+3}} d\varphi + e_l^{k+4} \int_{\pi/2 - \eta_j}^{\pi/2} \frac{\cos \varphi}{\cos(\eta_i - \varphi)^{k+3}} d\varphi \right). \end{aligned}$$

With a substitution $\psi = \eta_i - \varphi$ and with $\cos(\eta_i - \varphi) = \cos \eta_i \cos \varphi + \sin \eta_i \sin \varphi$, we can compute the outer integral:

$$\begin{aligned} \mathcal{I}^{(i)} &= \frac{\sin \eta_i^{k+3}}{k+4} \left(e_j^{k+4} \left[\cos \eta_i \cos \eta_j {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_j^2 \right) \right. \right. \\ &+ \cos \eta_i^2 {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_i^2 \right) \\ &+ \left. \frac{\sin \eta_i}{k+2} \left(\frac{1}{\sin \eta_j^{k+2}} - \frac{1}{\sin \eta_i^{k+2}} \right) \right] \\ &+ e_l^{k+4} \left[\cos \eta_i \cos \eta_l {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_l^2 \right) \right. \\ &+ \cos \eta_i^2 {}_2F_1 \left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \cos \eta_i^2 \right) \\ &+ \left. \left. \frac{\sin \eta_i}{k+2} \left(\frac{1}{\sin \eta_l^{k+2}} - \frac{1}{\sin \eta_i^{k+2}} \right) \right] \right). \end{aligned}$$

□

Again, we apply this proposition to the right-angled, isosceles triangle.

Corollary 3 For the triangle T_2 with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$, we have:

$$\mathbb{E}V_{T_2[2]}^k = \frac{2^{(5-k)/2} + 2^{7/2}}{(k+2)(k+3)(k+4)} {}_2F_1\left(\frac{1}{2}, \frac{k+3}{2}; \frac{3}{2}; \frac{1}{2}\right).$$

As before, we give some values of $\mathbb{E}V_{T_2[2]}^k$ for this triangle:

$$\begin{aligned}\mathbb{E}V_{T_2[2]} &= \frac{1 + 2\sqrt{2}}{30} (2 + \sqrt{2} \operatorname{arcsinh} 1) = 0.4142 \dots, \\ \mathbb{E}V_{T_2[2]}^2 &= \frac{2}{9} = 0.2222 \dots, \\ \mathbb{E}V_{T_2[2]}^3 &= \frac{1 + 4\sqrt{2}}{840} (14 + 3\sqrt{2} \operatorname{arcsinh} 1) = 0.1405 \dots, \\ \mathbb{E}V_{T_2[2]}^4 &= \frac{1}{10} = 0.1.\end{aligned}$$

4 Random triangles in a tetrahedron

In this section, our aim is to show that for a specific tetrahedron T and the center c of one of its facets, we have

$$\mathbb{E}V_{T[3]} > \mathbb{E}V_{T[3],c}. \quad (2)$$

Therefore, we want to find polynomials $P(x) = \sum_{i=0}^n a_i x^i$ and $Q(x) = \sum_{i=0}^l b_i x^i$ such that $P(x) \leq |x|$ and $Q(x) \geq |x|$ for relevant ranges of values $x \in \mathbb{R}$ and consequently

$$\mathbb{E}V_{T[3]} \geq \mathbb{E}P(V_{T[3]}) > \mathbb{E}Q(V_{T[3],c}) \geq \mathbb{E}V_{T[3],c},$$

which suffices in order to show that the pair (T, c) is a counterexample to monotonicity of the expected area of a random triangle in dimension three. Simulations suggest that

$$\mathbb{E}V_{T[3]} = 0.0592\dots \text{ and } \mathbb{E}V_{T[3],c} = 0.0466\dots \quad (3)$$

and conjecture (2) should hold true.

4.1 A lower bound for the expected area of a random triangle in a tetrahedron

Let T be a tetrahedron. For random points $X_0, X_1, X_2 \in T$, we write $X_i = (x_i, y_i, z_i)$. The volume of the triangle with vertices X_0, X_1 and X_2 is given by

$$\begin{aligned}V_{T[3]} &= \frac{1}{2} \sqrt{\det \begin{pmatrix} x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x_0 & y_2 - y_0 & z_2 - z_0 \end{pmatrix} \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \\ z_1 - z_0 & z_2 - z_0 \end{pmatrix}} \\ &= \frac{1}{2} D(x_0, \dots, z_2)^{1/2},\end{aligned}$$

and this is the square root of a polynomial of degree four in the coordinates of X_0 , X_1 and X_2 . For even moments of $V_{T[3]}$, we get rid of the square root:

$$\mathbb{E} V_{T[3]}^{2k} = \text{vol } T^{-3} 2^{-2k} \int_T \int_T \int_T D(x_0, \dots, z_2)^k d(x_0, y_0, z_0) d(x_1, y_1, z_1) d(x_2, y_2, z_2). \quad (4)$$

Note that the expectation $\mathbb{E} V_{T[3]}$ depends on the shape of the tetrahedron T . We consider the specific tetrahedron

$$T_3 = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x + y + z \leq 1\},$$

i.e., that with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Expanding the determinant, the polynomial D can be written as

$$D(x_0, \dots, z_2) = \sum_{i=1}^{63} s_i \quad \text{with} \quad s_i = \varepsilon_i 2^{c_i} x_0^{e_{i0}} \cdots z_2^{e_{i8}},$$

where $\varepsilon_i \in \{-1, 1\}$, $c_i \in \{0, 1\}$ and $e_{ij} \geq 0$, $j = 0, \dots, 8$, are explicitly given constants and $\sum_j e_{ij} = 4$ for $i = 1, \dots, 63$. By the Multinomial Theorem, and using the multinomial coefficient

$$\binom{k}{k_1, \dots, k_{63}} = \frac{k!}{k_1! \cdots k_{63}!},$$

the k -th power of it can be rewritten as

$$\begin{aligned} D(x_0, \dots, z_2)^k &= \sum_{\sum_1^{63} k_i = k} \binom{k}{k_1, \dots, k_{63}} \prod_{i=1}^{63} s_i^{k_i} \\ &= \sum_{\sum_1^{63} k_i = k} (-1)^{k'} 2^{k''} \binom{k}{k_1, \dots, k_{63}} \prod_{i=0}^2 x_i^{l_i} y_i^{m_i} z_i^{n_i}. \end{aligned} \quad (5)$$

For completeness, we list the exponents appearing in (5):

$$\begin{aligned} k' &= k_{19} + \cdots + k_{37} + k_{40} + k_{44} + k_{47} + k_{51} + k_{54} + k_{55} + k_{59} + k_{63}, \\ k'' &= k_{19} + \cdots + k_{63}, \\ l_0 &= 2k_1 + \cdots + 2k_4 + 2k_{19} + 2k_{20} + k_{27} + k_{29} + k_{33} + k_{35} + k_{37} + \cdots + k_{48}, \\ m_0 &= 2k_5 + 2k_9 + 2k_{13} + 2k_{14} + k_{21} + k_{23} + 2k_{25} + 2k_{26} + k_{34} + k_{36} + \cdots + k_{38} \\ &\quad + k_{43} + k_{44} + k_{49} + k_{50} + k_{55} + \cdots + k_{60}, \\ n_0 &= 2k_7 + 2k_{11} + 2k_{15} + 2k_{17} + k_{22} + k_{24} + k_{28} + k_{30} + 2k_{31} + 2k_{32} + k_{40} + k_{41} \\ &\quad + k_{46} + k_{47} + k_{52} + k_{53} + k_{55} + k_{56} + k_{58} + k_{59} + k_{61} + k_{62}, \end{aligned}$$

$$\begin{aligned}
l_1 &= 2k_5 + \cdots + 2k_8 + 2k_{21} + 2k_{22} + k_{25} + k_{29} + k_{31} + k_{35} + k_{37} + \cdots + k_{42} \\
&\quad + k_{49} + \cdots + k_{54}, \\
m_1 &= 2k_1 + 2k_{10} + 2k_{15} + 2k_{16} + k_{19} + k_{23} + 2k_{27} + 2k_{28} + k_{32} + k_{36} + k_{37} + k_{39} \\
&\quad + k_{43} + k_{45} + k_{49} + k_{51} + k_{55} + \cdots + k_{57} + k_{61} + \cdots + k_{63}, \\
n_1 &= 2k_3 + 2k_{12} + 2k_{13} + 2k_{18} + k_{20} + k_{24} + k_{26} + k_{30} + 2k_{33} + 2k_{34} + k_{40} + k_{42} \\
&\quad + k_{46} + k_{48} + k_{52} + k_{54} + k_{55} + k_{57} + k_{58} + k_{60} + \cdots + k_{63}, \\
l_2 &= 2k_9 + 2k_{10} + 2k_{11} + 2k_{12} + 2k_{23} + 2k_{24} + k_{25} + k_{27} + k_{31} + k_{33} + k_{43} \\
&\quad + \cdots + k_{54}, \\
m_2 &= 2k_2 + 2k_6 + 2k_{17} + 2k_{18} + k_{19} + k_{21} + 2k_{29} + 2k_{30} + k_{32} + k_{34} + k_{38} + k_{39} \\
&\quad + k_{44} + k_{45} + k_{50} + k_{51} + k_{58} + \cdots + k_{63}, \\
n_2 &= 2k_4 + 2k_8 + 2k_{14} + 2k_{16} + k_{20} + k_{22} + k_{26} + k_{28} + 2k_{35} + 2k_{36} + k_{41} + k_{42} \\
&\quad + k_{47} + k_{48} + k_{53} + k_{54} + k_{56} + k_{57} + k_{59} + k_{60} + k_{62} + k_{63}.
\end{aligned}$$

Integration of the monomials over the tetrahedron T_3 gives with the substitution $z = t$, $y = s(1 - t)$, $x = r(1 - s)(1 - t)$ that

$$\begin{aligned}
\int_{T_3} x^{l_i} y^{m_i} z^{n_i} d(x, y, z) &= \int_0^1 r^{l_i} dr \int_0^1 s^{m_i} (1 - s)^{l_i+1} ds \int_0^1 t^{n_i} (1 - t)^{l_i+m_i+2} dt \\
&= \frac{1}{l_i + 1} B(m_i + 1, l_i + 2) B(n_i + 1, l_i + m_i + 3) \\
&= \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!}.
\end{aligned} \tag{6}$$

Combining this with Eqs. (4) and (5) gives

$$\mathbb{E}V_{T_3[3]}^{2k} = 6^3 2^{-2k} \sum_{\sum_{i=1}^{63} k_i = k} (-1)^{k'} 2^{k''} \binom{k}{k_1, \dots, k_{63}} \prod_{i=0}^2 \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!}.$$

We list the first five even moments of the area of a random triangle in our tetrahedron T_3 :

$$\begin{aligned}
\mathbb{E}V_{T_3[3]}^2 &= \frac{9}{1\,600} = 5.625 \cdot 10^{-3}, \\
\mathbb{E}V_{T_3[3]}^4 &= \frac{27}{196\,000} \approx 1.37755 \cdot 10^{-4}, \\
\mathbb{E}V_{T_3[3]}^6 &= \frac{3\,161}{379\,330\,560} \approx 8.3331 \cdot 10^{-6}, \\
\mathbb{E}V_{T_3[3]}^8 &= \frac{93\,957}{106\,247\,680\,000} \approx 8.8432 \cdot 10^{-7},
\end{aligned}$$

$$\mathbb{E}V_{T_3[3]}^{10} = \frac{209\,022\,679}{1\,551\,386\,124\,288\,000} \approx 1.34733 \cdot 10^{-7}.$$

We have seen that the $(2k)$ -th moment of $V_{T_3[3]}$ can be computed, with fast increasing complexity in k . Also note that the area of a triangle in T_3 is not larger than $\sqrt{3}/2$, which is the area of the facet $\{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x + y + z = 1\}$. Hence, we want to approximate the square root function in the interval $[0, 3/4]$ by a polynomial

$$P(x) = \sum_{i=0}^n a_i x^i$$

of degree n for some $n \in \mathbb{N}$ such that $P(x) \leq \sqrt{x}$ for all $x \in [0, \frac{3}{4}]$. This way, we can approximate $\mathbb{E}V_{T_3[3]}$ from below by the polynomial P :

$$\mathbb{E}V_{T_3[3]} \geq \mathbb{E}P(V_{T_3[3]}).$$

Moreover, the best polynomial for fixed $n \in \mathbb{N}$ can be found via the linear optimization problem

$$\max_P \mathbb{E}P(V_{T_3[3]}) = \max_{a_i} \sum_{i=0}^n a_i \mathbb{E}P(V_{T_3[3]})^{2i} \quad \text{s.t.} \quad P(x^2) \leq x, \quad x \in \left[0, \frac{\sqrt{3}}{2}\right].$$

This constraint is infinite dimensional, but we get an upper bound on $\mathbb{E}P(V_{T_3[3]})$ via the finite dimensional linear program

$$\begin{aligned} \max_P \mathbb{E}P(V_{T_3[3]}) &= \max_{a_i} \sum_{i=0}^n a_i \mathbb{E}V_{T_3[3]}^{2i} \quad \text{s.t.} \quad P(x_\ell^2) \leq x_\ell, \quad x_\ell \in \left[0, \frac{\sqrt{3}}{2}\right], \\ &\ell = 0, \dots, L. \end{aligned}$$

For $n = 6$ and $L = 200$ equidistant points $x_\ell \in [0, \frac{\sqrt{3}}{2}]$, we numerically compute via Matlab and the optimization toolbox CVX [2] that

$$\mathbb{E}P(V_{T_3[3]}) < 0.04647,$$

and in consideration of the simulations in (3), this yields that we do not get a sufficiently precise estimate using only $n = 6$ even moments.

We solve the above linear program for $n = 7$ and $L = 1\,000$, get the squares of three interpolation nodes with the square root function in $(0, \frac{\sqrt{3}}{2})$ and rationalize these points to

$$\{x_1, x_2, x_3\} = \left\{ \frac{2}{19}, \frac{4}{15}, \frac{8}{17} \right\}.$$

Additionally, we take the points $x_0 = 0$ and $x_4 = \frac{47}{54} > \frac{\sqrt{3}}{2}$ as interpolation nodes of our desired polynomial. We want to use these five points to get a polynomial of degree 7. Consider a variation of Lemma 2 in [4] with two conditions

$$P(x_j^2) = x_j \quad \text{and} \quad P'(x_j^2) = \frac{1}{2x_j}$$

for $j = 1, 2, 3$ and the single conditions $P(x_0^2) = P(0) = 0$ and $P(x_4^2) = x_4$. Then we have with $f(x) = \sqrt{x}$ that

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x(x-x_4^2) \prod_{j=1}^3 (x-x_j^2)^2 \geq 0$$

for some $\xi \in [0, x_4^2]$, since $f^{(n+1)}(x) < 0$, $x \geq 0$ and $x \leq x_4$ for all $x \in [0, \frac{3}{4}]$.

We solve the interpolation problem and finally get a polynomial $P_{\text{cert}}(x) = \sum_{i=1}^7 a_i x^i$ of degree 7 with explicitly given rational coefficients $a_1, \dots, a_7$ and the property $P_{\text{cert}} \leq \sqrt{x}$, and this polynomial yields with usage of the even moments computed above that

$$\begin{aligned} \mathbb{E} V_{T_3[3]} &\geq \mathbb{E} P_{\text{cert}}(V_{T_3[3]}) \\ &= \frac{91\,096\,443\,868\,688\,279\,627\,145\,529\,306\,423\,486\,222\,767\,653\,883\,418\,444\,589}{1\,940\,591\,908\,276\,502\,542\,424\,150\,723\,573\,750\,536\,040\,094\,315\,231\,313\,920\,000} \\ &> 0.046942. \end{aligned} \quad (7)$$

4.2 An upper bound for the expected area of a random triangle in a tetrahedron where one point is fixed

Now we consider the convex hull of two random points X_1, X_2 and one fixed point in a tetrahedron T . We write again $X_i = (x_i, y_i, z_i)$ and choose the centroid $c = (x_c, y_c, z_c)$ of one of the facets of T as third point. The volume of the triangle with vertices X_1, X_2 and c is given by

$$\begin{aligned} V_{T[3],c} &= \frac{1}{2} \sqrt{\det \left(\begin{pmatrix} x_1 - x_c & y_1 - y_c & z_1 - z_c \\ x_2 - x_c & y_2 - y_c & z_2 - z_c \end{pmatrix} \begin{pmatrix} x_1 - x_c & x_2 - x_c \\ y_1 - y_c & y_2 - y_c \\ z_1 - z_c & z_2 - z_c \end{pmatrix} \right)} \\ &= \frac{1}{2} D(x_0, \dots, z_c)^{1/2}, \end{aligned}$$

and this is again the square root of a polynomial of degree four in the coordinates of X_1, X_2 and c . As before, we have:

$$\mathbb{E} V_{T[3],c}^{2k} = \text{vol } T^{-2} 2^{-2k} \int_T \int_T D(x_1, \dots, z_c)^k d(x_1, y_1, z_1) d(x_2, y_2, z_2). \quad (8)$$

The expectation $\mathbb{E} V_{T[3],c}$ particularly depends on the facet whose centroid we choose to be the fixed vertex. We take again the tetrahedron T_3 as defined in Sect. 4.1 and choose $c_3 = (1/3, 1/3, 1/3)$, the centroid of the facet $\{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x + y + z = 1\}$, to be the fixed vertex of the random triangle.

Expanding the determinant and inserting the coordinates of c_3 , the polynomial D can be displayed as

$$D(x_1, \dots, z_{c_3}) = \frac{1}{9} \sum_{i=1}^{54} s_i \quad \text{with} \quad s_i = \varepsilon_i 2^{c_i} 3^{d_i} x_1^{e_{i0}} \cdots z_2^{e_{i5}},$$

where $\varepsilon_i \in \{-1, 1\}$, $c_i, d_i \in \{0, 1, 2\}$ and $e_{ij} \geq 0$, $j = 0, \dots, 5$, are explicitly given constants and $\sum_j e_{ij} \leq 4$ for $i = 1, \dots, 54$. Again with usage of the Multinomial Theorem, the series representation of the k -th power of it is given by

$$\begin{aligned} D(x_1, \dots, z_{c_3})^k &= 3^{-2k} \sum_{\sum_1^{54} k_i = k} \binom{k}{k_1, \dots, k_{54}} \prod_{i=1}^{54} s_i^{k_i} \\ &= 3^{-2k} \sum_{\sum_1^{54} k_i = k} (-1)^{k'} 2^{k''} 3^{k'''} \binom{k}{k_1, \dots, k_{54}} \prod_{i=1}^2 x_i^{l_i} y_i^{m_i} z_i^{n_i}, \quad (9) \end{aligned}$$

where we use abbreviations analogously as in the previous proof:

$$\begin{aligned} k' &= k_7 + \cdots + k_{10} + k_{12} + k_{15} + k_{17} + k_{18} + k_{21} + \cdots + k_{33} + k_{52} + \cdots + k_{54}, \\ k'' &= k_1 + \cdots + k_6 + 2k_7 + \cdots + 2k_9 + k_{10} + \cdots + k_{45} + k_{52} + \cdots + k_{54}, \\ k''' &= k_{22} + \cdots + k_{45} + 2k_{46} + \cdots + 2k_{54}, \\ l_1 &= 2k_1 + k_7 + k_{10} + \cdots + k_{13} + 2k_{22} + 2k_{23} + k_{28} + k_{32} + k_{34} + \cdots + k_{39} \\ &\quad + 2k_{46} + 2k_{47} + k_{52} + k_{53}, \\ m_1 &= 2k_3 + k_8 + k_{10} + k_{14} + k_{18} + k_{19} + k_{24} + 2k_{26} + 2k_{27} + k_{33} + k_{34} \\ &\quad + k_{38} + k_{40} + k_{42} + \cdots + k_{44} + 2k_{48} + 2k_{50} + k_{52} + k_{54}, \\ n_1 &= 2k_5 + k_9 + k_{12} + k_{16} + k_{18} + k_{20} + k_{25} + k_{29} + 2k_{30} + 2k_{31} + k_{36} \\ &\quad + k_{39} + k_{41} + k_{42} + k_{44} + k_{45} + 2k_{49} + 2k_{51} + k_{53} + k_{54}, \\ l_2 &= 2k_2 + k_7 + k_{14} + \cdots + k_{17} + 2k_{24} + 2k_{25} + k_{26} + k_{30} + k_{34} \\ &\quad + \cdots + k_{37} + k_{40} + k_{41} + 2k_{48} + 2k_{49} + k_{52} + k_{53}, \\ m_2 &= 2k_4 + k_8 + k_{11} + k_{15} + k_{20} + \cdots + k_{22} + 2k_{28} + 2k_{29} + k_{31} + k_{35} \\ &\quad + k_{38} + k_{40} + k_{42} + k_{43} + k_{45} + 2k_{46} + 2k_{51} + k_{52} + k_{54}, \\ n_2 &= 2k_6 + k_9 + k_{13} + k_{17} + k_{19} + k_{21} + k_{23} + k_{27} + 2k_{32} + 2k_{33} + k_{37} \\ &\quad + k_{39} + k_{41} + k_{43} + k_{44} + k_{45} + 2k_{47} + 2k_{50} + k_{53} + k_{54}. \end{aligned}$$

Now, Eqs. (6), (8) and (9) give

$$\mathbb{E} V_{T_3[3], c_3}^{2k} = 6^{2-2k} \sum_{\sum_1^{54} k_i = k} (-1)^{k'} 2^{k''} 3^{k'''} \binom{k}{k_1, \dots, k_{54}} \prod_{i=0}^2 \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!}.$$

Again, we list the first five even moments of the area of a random triangle in our tetrahedron T_3 , where one point lies in c_3 :

$$\begin{aligned}\mathbb{E}V_{T_3[3],c_3}^2 &= \frac{7}{2400} \approx 2.91667 \cdot 10^{-3}, \\ \mathbb{E}V_{T_3[3],c_3}^4 &= \frac{11}{529200} \approx 2.07861 \cdot 10^{-5}, \\ \mathbb{E}V_{T_3[3],c_3}^6 &= \frac{2839}{10973491200} \approx 2.58714 \cdot 10^{-7}, \\ \mathbb{E}V_{T_3[3],c_3}^8 &= \frac{29419}{622402704000} \approx 4.72668 \cdot 10^{-9}, \\ \mathbb{E}V_{T_3[3],c_3}^{10} &= \frac{4134139}{363523012905600} \approx 1.13724 \cdot 10^{-10}.\end{aligned}$$

Note that the area of a triangle in T_3 , where one vertex is fixed to lie in c_3 , is not larger than $\sqrt{3}/6$. Hence, we want to approximate the square root function in the interval $[0, 1/12]$ by a polynomial

$$Q(x) = \sum_{i=0}^l b_i x^i$$

of degree $l \in \mathbb{N}$ such that $Q(x) > \sqrt{x}$ for all $x \in [0, 1/12]$. Then we have that

$$\mathbb{E}V_{T_3[3],c_3} \leq \mathbb{E}Q(V_{T_3[3],c_3}).$$

We use Lemma 2 in [4], which yields that for $m \in \mathbb{N}$, $l = 2m + 1$ and $0 < x_0 < \dots < x_m$, the system of equations

$$Q(x_j^2) = x_j \text{ and } Q'(x_j^2) = \frac{1}{2x_j} \quad \text{for } j = 0, \dots, m$$

determines uniquely a polynomial $Q(x) = \sum_{i=0}^l b_i x^i$ with the property $Q(x) \geq \sqrt{x}$ for all $x \in \mathbb{R}$. We get a lower bound on $\mathbb{E}Q(V_{T_3[3],c_3})$ via the finite dimensional linear program

$$\begin{aligned}\min_Q \mathbb{E}Q(V_{T_3[3],c_3}) &= \min_{b_i} \sum_{i=0}^l b_i \mathbb{E}V_{T_3[3],c_3}^{2i} \quad \text{s.t.} \quad Q(x_\ell^2) \geq x_\ell, \quad x_\ell \in \left[0, \frac{\sqrt{3}}{6}\right], \\ &\quad \ell = 0, \dots, L.\end{aligned}$$

For $l = 14$ and $L = 200$ equidistant points $x_\ell \in [0, \frac{\sqrt{3}}{6}]$, we numerically compute via Matlab and CVX that

$$\mathbb{E}Q(V_{T_3[3],c_3}) > 0.04699,$$

and this approximation is not sufficient with consideration of (7).

For $l = 15$ and $L = 1000$, we solve the above linear program, compute the interpolation nodes with the square root function numerically, and rationalize these points to

$$\{x_0, \dots, x_7\} = \left\{ \frac{1}{45}, \frac{1}{17}, \frac{1}{11}, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{3}{13}, \frac{7}{27} \right\}.$$

Using these points, we solve the interpolation problem from Lemma 2 in [4] and finally get a polynomial $Q_{\text{cert}}(x) = \sum_{i=0}^{15} b_i x^i$ of degree 15 with explicitly given rational coefficients $b_0, \dots, b_{15}$ and the property $Q_{\text{cert}} \geq \sqrt{x}$. Inserting the even moments computed above, we see that

$$\begin{aligned} \mathbb{E}V_{T_3[3], c_3} &\leq \mathbb{E}Q_{\text{cert}}(V_{T_3[3], c_3}) \\ &= \frac{1\,364\,371\,357\,215\,003\,705\,813\,514\,810\,174\,100\,904\,243\,878\,770\,680\,297\,601\,217\,509\,788\,018\,385\,106\,185\,895\,467\,950\,649\,976\,445\,591\,316\,797\,579\,615\,557\,279\,107\,985\,989}{29\,065\,551\,851\,580\,796\,169\,800\,375\,743\,177\,957\,971\,396\,211\,070\,941\,311\,273\,271\,339\,321\,255\,098\,209\,543\,350\,425\,288\,830\,624\,996\,871\,957\,546\,778\,230\,784\,000\,000\,000\,000} \\ &< 0.046942. \end{aligned}$$

5 Proof of Theorem 2

5.1 Some basic considerations

In the first part of the proof, we use the results on the monotonicity of the moments of volumes of random full-dimensional simplices, which have already been stated in Theorem 1. It follows with Lemma 2 that, for $d \geq 2$ and $2 \leq n \leq d$, monotonicity of the map $K \rightarrow \mathbb{E}V_{K[n]}^k$ for K ranging over all d -dimensional convex bodies K does not hold unless $n = 2$ and $k \in \mathbb{N}$ or $n = 3$ and $k \in \{1, 2\}$.

5.2 Proof of Proposition 1

Now we give a counterexample for $d \geq 2$ and $n = 2$. Here we prove that random chords in a specific right-angled triangle are suitable for our purposes by giving the explicit values.

We consider the ratio $r(k) := \mathbb{E}V_{T_2[2], c_2}^k / \mathbb{E}V_{T_2[2]}^k$ for the triangle T_2 with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ and $c_2 = (1/2, 1/2)$ the midpoint of its hypotenuse. It holds:

$$r(k) = \frac{(k+3)(k+4)}{2^{k+3} + 2^{k/2+2}}.$$

The ratio $r(k)$ is clearly monotonely decreasing, and $r(1)$ is smaller than 1. Therefore, $r(k)$ is smaller than 1 for each $k \in \mathbb{N}$ and Proposition 1 holds.

5.3 Proof of Proposition 2

We can settle the case $n = 3$, $k = 2$ by comparing the second moments of the areas of our two different types of random triangles in a tetrahedron directly. We computed in Sect. 4:

$$\mathbb{E}V_{T_3[3],c_3}^2 = \frac{7}{2400} = 0.002916... < 0.005625 = \frac{9}{1600} = \mathbb{E}V_{T_3[3]}^2.$$

Consequently, with Lemmata 1 and 2, we have shown that the map $K \mapsto \mathbb{E}V_{K[3]}^2$ is not monotone if K ranges over all d -dimensional convex bodies for $d \geq 3$.

In the same section, we have shown that

$$\mathbb{E}V_{T_3[3],c_3} < 0.046942 < \mathbb{E}V_{T_3[3]}.$$

Hence, we also have a counterexample for $k = 1$, $n = 3$ and $d \geq 3$, and the proof of Proposition 2 as well as that of Theorem 2 are completed.

6 Proof of Theorem 3

Let $d \geq 3$, $2 \leq n \leq d$ and $k \in \mathbb{N}$. If $d > 3$, $n \neq 3$ or $k > 2$, there exists a pair of convex bodies $K, L \subseteq \mathbb{R}^{d-1}$ such that $K \subset L$, but $\mathbb{E}V_{K[n]}^k > \mathbb{E}V_{L[n]}^k$. As in the proof of Lemma 2, for $\varepsilon > 0$, we define two d -dimensional convex bodies by

$$K_\varepsilon := K \times [0, \varepsilon], \quad L_\varepsilon := L \times [0, \varepsilon].$$

In the sequel, we will show that $\mathbb{E}\mathcal{V}_{K_\varepsilon[n]}^k$ and $\mathbb{E}\mathcal{V}_{L_\varepsilon[n]}^k$ converge to $\mathbb{E}V_{K[n]}^k$ and $\mathbb{E}V_{L[n]}^k$, resp., for ε tending to zero. Therefore, there exists an $\varepsilon_0 > 0$ such that $\mathbb{E}\mathcal{V}_{K_{\varepsilon_0}[n]}^k > \mathbb{E}\mathcal{V}_{L_{\varepsilon_0}[n]}^k$, but $K_{\varepsilon_0} \subseteq L_{\varepsilon_0}$.

Let $X_0, \dots, X_{n-1}$ be random points in $\text{bd } K_\varepsilon$ and

$$N_0 = \#\{X_i \in K \times \{0, \varepsilon\}\}, \quad N_1 = \#\{X_i \in \text{bd } K \times [0, \varepsilon]\}.$$

Then $N_0 + N_1 = n$. It holds for $0 \leq i \leq n-1$:

$$\begin{aligned} \mathbb{P}(X_i \in K \times \{0, \varepsilon\}) &= \frac{2 \text{vol}_{d-1} K}{2 \text{vol}_{d-1} K + S(K) \varepsilon}, \\ \mathbb{P}(X_i \in \text{bd } K \times [0, \varepsilon]) &= \frac{S(K) \varepsilon}{2 \text{vol}_{d-1} K + S(K) \varepsilon}, \end{aligned}$$

if $S(K)$ is the surface area of the boundary of K . Therefore, we have that

$$\begin{aligned} \mathbb{E}\mathcal{V}_{K_\varepsilon[n]}^k &= \mathbb{P}(N_1 = 0) \mathbb{E}\mathcal{V}_{K \times \{0, \varepsilon\}[n]}^k + \mathbb{P}(N_1 > 0) \mathbb{E}_{N_1 > 0} \mathcal{V}_{K_\varepsilon[n]}^k \\ &= \frac{2^n \text{vol}_{d-1} K^n}{(2 \text{vol}_{d-1} K + S(K) \varepsilon)^n} \mathbb{E}\mathcal{V}_{K \times \{0, \varepsilon\}[n]}^k \\ &\quad + \left(1 - \frac{2^n \text{vol}_{d-1} K^n}{(2 \text{vol}_{d-1} K + S(K) \varepsilon)^n}\right) \mathbb{E}_{N_1 > 0} \mathcal{V}_{K_\varepsilon[n]}^k. \end{aligned} \quad (10)$$

If ε tends to zero, $\mathbb{P}(N_1 > 0)$ and hence the second summand in (10) tend to zero. Now, an argument similar to that in the proof of Lemma 2 yields that $\mathbb{E}\mathcal{V}_{K \times \{0, \varepsilon\}[n]}^k$

converges to $\mathbb{E}V_{K[n]}^k$, and the same is true for K replaced by L , which completes the proof.

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