Erratum to: Logarithmic coefficients and a coefficient conjecture for univalent functions

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Received: 7 April 2017 / Accepted: 11 April 2017 / Published online: 24 April 2017
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Erratum to: Monatsh Math DOI 10.1007/s00605-017-1024-3

In the proof of Theorem 1 in [1], the present authors stated the expression for $\gamma_n(f_\lambda)$ wherein in the second term of the expression in the bracket a factor n in denominator was missing, although it does not affect the statement of Theorem 1. As a result, it might be useful for the reader to know the remaining part of the proof as it is not obvious. The correct expression for $\gamma_n(f_\lambda)$ should be

$$\gamma_n(f_\lambda) = \frac{1}{2} \left(\frac{1 + \lambda^n}{n} + (-1)^n \frac{\lambda^n}{n(1 + \lambda)^n} \right) \quad \text{for } n \geq 1.$$

Communicated by A. Constantin.

The online version of the original article can be found under doi:[10.1007/s00605-017-1024-3](https://doi.org/10.1007/s00605-017-1024-3).

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According to this, for $n \geq 1$, we obtain

$$\sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \frac{(1+\lambda^n)^2}{n^2} + 2 \frac{(-1)^n}{n^2} \left[\left(\frac{\lambda^2}{1+\lambda} \right)^n + \left(\frac{\lambda}{1+\lambda} \right)^n \right] + \frac{1}{n^2} \left(\frac{\lambda}{1+\lambda} \right)^{2n} \right\}$$

and by a computation, it follows easily that

$$\begin{aligned} \sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 &= \frac{1}{4} \left(\frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right) \\ &\quad + \frac{1}{2} \left[\text{Li}_2 \left(\frac{-\lambda^2}{1+\lambda} \right) + \text{Li}_2 \left(\frac{-\lambda}{1+\lambda} \right) \right] + \frac{1}{4} \text{Li}_2 \left(\frac{\lambda^2}{(1+\lambda)^2} \right) \\ &= \frac{1}{4} \left(\frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right) + \frac{1}{4} A(\lambda) \\ &< \frac{1}{4} \left(\frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right) \quad \text{for } 0 < \lambda \leq 1, \end{aligned}$$

and we complete the proof, provided

$$A(\lambda) := 2 \left[\text{Li}_2 \left(\frac{-\lambda^2}{1+\lambda} \right) + \text{Li}_2 \left(\frac{-\lambda}{1+\lambda} \right) \right] + \text{Li}_2 \left(\frac{\lambda^2}{(1+\lambda)^2} \right) < 0 \quad \text{for } 0 < \lambda \leq 1.$$

We shall now prove that $A(\lambda) < 0$ for $0 < \lambda \leq 1$. Because $\text{Li}_2(z^2) = 2(\text{Li}_2(z) + \text{Li}_2(-z))$, the last claim is equivalent to

$$\frac{A(\lambda)}{2} = 2\text{Li}_2 \left(\frac{-\lambda}{1+\lambda} \right) + \left[\text{Li}_2 \left(\frac{\lambda}{1+\lambda} \right) + \text{Li}_2 \left(\frac{-\lambda^2}{1+\lambda} \right) \right] < 0$$

for $0 < \lambda \leq 1$. According to the integral representation

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = z \int_0^1 \frac{\log(1/t)}{1-tz} dt,$$

we can rewrite $A(\lambda)$ in a convenient form as

$$A(\lambda) = -2\lambda \int_0^1 B(\lambda, t) \log(1/t) dt,$$

where

$$\begin{aligned} B(\lambda, t) &= \frac{2}{1 + \lambda + t\lambda} - \frac{1}{1 + \lambda - t\lambda} + \frac{\lambda}{1 + \lambda + t\lambda^2} \\ &= \frac{(1 + \lambda) - 3t\lambda}{(1 + \lambda)^2 - t^2\lambda^2} + \frac{\lambda}{1 + \lambda + t\lambda^2} \\ &= \frac{N(\lambda, t)}{[(1 + \lambda)^2 - t^2\lambda^2][1 + \lambda + t\lambda^2]} \end{aligned}$$

with

$$N(\lambda, t) = (1 + \lambda)^3 - (3 - \lambda)(1 + \lambda)\lambda t - 4\lambda^3 t^2.$$

Clearly, $B(1, t) > 0$ for $t \in [0, 1)$, and it follows that $A(1) < 0$. On the other hand, since $N(\lambda, t)$ is a decreasing function of t for $t \in [0, 1]$, we obtain that

$$N(\lambda, t) \geq N(\lambda, 1) = (1 + \lambda)^3 - (3 - \lambda)(1 + \lambda) - 4\lambda^3 = 1 - \lambda^3 + \lambda^2(1 - \lambda) > 0$$

for $0 < \lambda < 1$. Consequently, $B(\lambda, t) > 0$ for all $t \in [0, 1]$ and for $0 < \lambda < 1$. This observation shows that $A(\lambda) < 0$ for $0 < \lambda \leq 1$. This proves the claim, and thus, the proof of Theorem 1 is complete. $\square$

The case $\lambda = 1$ has a special place in Remark 1 in [1], and it also requires some comments. Indeed, in view of the above changes we conclude that for the function f_1 we have

$$\begin{aligned} \sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 &= \sum_{n=1}^{\infty} \left(\frac{1}{n} + (-1)^n \frac{1}{n^{2n+1}} \right)^2 \\ &= \frac{\pi^2}{6} + \frac{1}{4} \text{Li}_2 \left(\frac{1}{4} \right) + \text{Li}_2 \left(-\frac{1}{2} \right) \\ &= \frac{\pi^2}{6} + \frac{1}{2} \left[\text{Li}_2 \left(\frac{1}{2} \right) + 3\text{Li}_2 \left(\frac{-1}{2} \right) \right], \end{aligned}$$

where we have used the fact that $\text{Li}_2(z^2) = 2(\text{Li}_2(z) + \text{Li}_2(-z))$. It follows that

$$\sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 < \frac{\pi^2}{6}$$

holds, because

$$\text{Li}_2 \left(\frac{1}{2} \right) + 3\text{Li}_2 \left(\frac{-1}{2} \right) < 0.$$

As a direct approach, it is easy to see that

$$\operatorname{Li}_2(z) + 3\operatorname{Li}_2(-z) = \sum_{n=1}^{\infty} \frac{1}{n^2} (1 + 3(-1)^n) z^n = \sum_{k=1}^{\infty} \frac{z^{2k}}{k^2} - 2 \sum_{k=1}^{\infty} \frac{z^{2k-1}}{(2k-1)^2}$$

and thus, we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 &= \frac{\pi^2}{6} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^k} \left(\frac{1}{k^2} - \frac{1}{(k-1/2)^2} \right) \\ &= \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{4^k} \left(\frac{4k-1}{k^2(2k-1)^2} \right) \end{aligned}$$

and thus, we have the desired inequality.

Reference

1. Obradović, M., Ponnusamy, S., Wirths, K.-J.: Logarithmic coefficients and a coefficient conjecture for univalent functions. *Monatsh. Math.* (2017). doi:[10.1007/s00605-017-1024-3](https://doi.org/10.1007/s00605-017-1024-3)