On the Bertini theorem in arbitrary characteristic

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Abstract We give a simple direct proof of the Kleiman Bertini theorem in arbitrary characteristic. We also give a simple proof of Serre splitting theorem.

Keywords Bertini theorem · Serre splitting theorem

Mathematics Subject Classification (1991) 14 A

1 Introduction

Kleiman in the paper [4] proved Bertini theorem in arbitrary characteristic (Corollary 12 in [4]). Kleiman deduced this theorem from his general transversality theorem: if X is G -homogenous algebraic variety and Z, W are smooth varieties over X , then the fiber product of Z and a general translate gY over X is smooth. This theorem is valid in characteristic zero, however Kleiman shows that his result still works in positive characteristic, if we additionally assume that the action of the group G is sufficiently good (this assumption is satisfied for the action of the group $PGL(n)$ on the projective space $\mathbb{P}^n$) and that Z, W are unramified over X .

The aim of this note is to give a simple, direct proof of the Bertini Theorem in arbitrary characteristic. We also partially generalize our method to locally free sheafs of higher ranks (mainly for $\text{char } k = 0$). In particular we give a simple proof of the

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fact that if a variety X is smooth and $\mathcal{F}$ is a locally free sheaf on X , with sufficiently many sections, then a generic section of $\mathcal{F}$ is transversal to X . As a Corollary we give a proof of the Atiyah–Serre Splitting Theorem (compare with [6], [1] and [5]).

2 Notations and definitions

We assume for simplicity, that the base field k is algebraically closed. Let X be algebraic variety and $\mathcal{F}$ be a locally free sheaf on X of rank r . Take a section $s \in \Gamma(X, \mathcal{F})$. We describe a scheme of zeroes of $s^{-1}(0)$ in the following way: locally we can assume that $X = U$ is an affine variety and $\mathcal{F} = \mathcal{O}_U^r$ is trivial. Hence $s = (s_1, \dots, s_r)$ where $s_i \in \Gamma(U, \mathcal{O}_U) = k[U]$. Then $s^{-1}(0)$ is given by the ideal $(s_1, \dots, s_r) \subset k[U]$. We say that the section s is transversal to X if either $s^{-1}(0)$ is smooth and it has dimension $\dim X - r$ or $s^{-1}(0) = \emptyset$.

Let X, Y be smooth varieties and $f : X \rightarrow Y$ be a morphism. We say that f is unramified, if f separates infinitely near points of X , i.e., the mapping $d_x f : T_x X \rightarrow T_{f(x)} Y$ is a monomorphism for every closed point $x \in X$ (see [3], p. 15).

If X is an affine variety we will denote by $k[X]$ the ring $\Gamma(X, \mathcal{O}_X)$. If M is a $k[X]$ module then by $M^\sim$ we denote the sheafification of M —see [2], Definition on the page 110.

3 Main result

Theorem 3.1 (Bertini theorem in arbitrary characteristic) *Let X be a smooth algebraic variety of dimension d and let $\mathcal{F}$ be an invertible sheaf on X . Assume that $\mathcal{F}$ is generated by global sections $s_1, \dots, s_r \in \Gamma(X, \mathcal{F})$. Assume that the morphism Φ from X to the projective space $\mathbb{P}^{r-1}$ given by $s_1, \dots, s_r$ is unramified. Then there is a Zariski open non-empty subset $U \subset k^r$ such that for every $c = (c_1, \dots, c_r) \in U$ the zero set of the section $s = \sum_{i=1}^r c_i s_i$ is smooth.*

Proof Since X is quasi-compact we can assume that X is affine and the sheaf $\mathcal{F}$ is trivial. Hence we can identify $\mathcal{F}$ with $\mathcal{O}_X$ and now $s_i : X \rightarrow k$ are regular functions. Additionally we can assume that $s_r \equiv 1$. Indeed, since $s_1, \dots, s_r$ generates $\mathcal{F}$ we have that open (affine!) subsets $U_i := X \setminus \{s_i = 0\}$ cover X . Consequently we can assume that $s_r \neq 0$ in X . Take $s'_i = s_i/s_r$, $i = 1, \dots, r$. If we prove our theorem for s'_i , then we automatically prove it also for originals s_i . Indeed, note that for a fixed $c = (c_1, \dots, c_r) \in k^r$ the ideals $(\sum_{i=1}^r c_i s_i)$ and $(\sum_{i=1}^r c_i s'_i)$ are equal in $k[X]$. Moreover, we can assume that on X we have global local coordinates $x_1, \dots, x_d$, i.e., the mapping $(x_1, \dots, x_d) : X \rightarrow k^d$ is étale.

Let

$$V = \left\{ (c, x) \in k^r \times X : c_r + \sum_{i=1}^{r-1} c_i s_i(x) = 0 \right\}.$$

The variety V is smooth. Indeed, let $(g_1, \dots, g_r)$ be the set of generators of the ideal $I(V) \subset k[X][c_1, \dots, c_r]$ and let $J(V)(c, x) = [\frac{\partial g_i}{\partial z_j}(c, x)]$, where $z =$

$(c_1, \dots, c_r, x_1, \dots, x_d)$ is a set of global local coordinates on $k^r \times X$. Let us note that a polynomial $h = \sum_{i=1}^r c_i s_i(x)$ does belong to the ideal $I(V)$. Now we see that $\text{rank } J(V) \geq 1$, because partial derivatives of h with respect to c_i form a matrix $[s_i(x)]_{1 \leq i \leq r}$, which has a rank 1. Hence $\dim T_{(c,x)} V \leq d + r - 1$. On the other hand $\dim V = d + r - 1$ and consequently we have the equality $\dim T_{(c,x)} V = d + r - n = \dim V$.

Consider the projection:

$$q : V \in (c, x) \rightarrow c \in k^r.$$

We show that for generic $c \in k^r$ the mapping q is smooth on the set $q^{-1}(U_c)$, where U_c is suitable neighborhood of c .

Indeed, let us compute the tangent space at (c, x) . It is given by the equation

$$dc_r + \sum_{i=1}^{r-1} s_i(x) dc_i + \sum_j \left(\sum_i^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} \right) dx_j.$$

Let us note that $\text{rank} \left[\frac{\partial s_i(x)}{\partial x_j} \right]_{1 \leq i \leq r-1, 1 \leq j \leq d} = d$. Indeed the mapping $\Phi : X \ni x \rightarrow (s_1(x), \dots, s_{r-1}(x)) \in k^{r-1}$ is unramified.

Put $L_c := \{x \in X : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \dots, d\}$. Let us note that the mapping q is not a submersion in a neighborhood of $q^{-1}(c)$ exactly if $L_c \neq \emptyset$. We show that the set $S := \{c \in k^r : L_c \neq \emptyset\}$ is a constructible subset of k^r of dimension less than r .

Indeed, let $W = \{(c', x) \in k^{r-1} \times X : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \dots, d\}$. Let $\pi : W \ni (c', x) \rightarrow x \in X$. Let us note that the fiber of π is a linear subspace of k^{r-1} of dimension $r-1-d$ (as a kernel of a suitable linear mapping). Hence $\dim W \leq r-1$.

Let $W' = \{(c, x) \in V : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \dots, d\}$. We have a surjective mapping

$$s : W \ni (c', x) \rightarrow \left(\left(c', - \sum_{i=1}^{r-1} c'_i s_i(x) \right), x \right) \in W'.$$

Since $S = \rho(W')$, where $\rho : W' \ni (c, x) \rightarrow c \in k^r$ and $\dim W = \dim s(W) = \dim W'$ we have $\dim cl(S) \leq r-1$.

This means that the projection $q : V \rightarrow k^r$ is a submersion outside a proper algebraic subset $\bar{S} \subset k^r$. In particular the zero set of a generic section $s = \sum_{i=1}^r c_i s_i$ is smooth. $\square$

Using a similar method we can generalize this result to higher dimension. As a Corollary we obtain Atiyah-Serre Splitting Theorem.

Theorem 3.2 *Let X be an algebraic variety of dimension d and let $\mathcal{F}$ be a locally-free sheaf on X of rank n . Assume that $\mathcal{F}$ is generated by global sections $s_1, \dots, s_r \in \Gamma(X, \mathcal{F})$. Then there is a Zariski open non-empty subset $U \subset k^r$ such that for every*

$c = (c_1, \dots, c_r) \in U$ the zero set of the section $s = \sum_{i=1}^r c_i s_i$ is either empty or it has dimension $d - n$. Moreover, if $\text{char } k = 0$ and X is smooth, then a generic section $s = \sum_{i=1}^r c_i s_i$ is transversal to X .

Proof Since X is quasi-compact we can assume that X is affine and the sheaf $\mathcal{F}$ is trivial, i.e., we can identify $\mathcal{F}$ with $\mathcal{O}_X^n$. Moreover we can assume that on X we have a global local coordinates $(x_1, \dots, x_d)$. Hence $s_i = (s_{i1}, \dots, s_{in}) : X \rightarrow k^n$ are regular mappings. Let

$$V = \left\{ (c, x) \in k^r \times X : \sum_{i=1}^r c_i s_i(x) = 0 \right\}.$$

Let us note that $\dim V = r + d - n$, where $d = \dim X$. Indeed, we have a surjection $\pi : V \ni (c, x) \rightarrow x \in X$. Any fiber of π is a linear subspace of k^r of dimension $r - n$ (it is a kernel of surjective linear mapping $F_x : k^r \ni c \rightarrow \sum_{i=1}^r c_i s_i(x) \in \mathbf{E}_x^n$, where $\mathbf{E}^n = X \times k^n$ is a trivial vector bundle of rank n). Consequently $\dim V = \dim X + r - n = d + r - n$. Now consider the second projection:

$$q : V \ni (c, x) \rightarrow c \in k^r.$$

If it is dominated then the generic fiber has dimension $\dim V - \dim k^r = d + r - n - r = d - n$, otherwise the generic fiber is empty.

Moreover, if X is smooth over k , then V is smooth. Indeed, let $(g_1, \dots, g_r)$ be the set of generators of the ideal $I(V) \subset k[X][c_1, \dots, c_r]$ and let $J(V)(c, x) = [\frac{\partial g_i}{\partial z_j}(c, x)]$, where $z = (c_1, \dots, c_r, x_1, \dots, x_d)$ is a set of global local coordinates on $k^r \times X$. Let us note that polynomials $h_j = \sum_{i=1}^r c_i s_{ij}(x)$ does belong to the ideal $I(V)$. Now we see that $\text{rank } J(V) \geq n$, because partial derivatives of h_j with respect to c_i form a matrix $[s_{ij}(x)]_{1 \leq i \leq r, 1 \leq j \leq n}$, which has a rank n . Hence $\dim T_{(c,x)} V \leq d + r - n$. On the other hand $\dim V = d + r - n$ and consequently we have the equality $\dim T_{(c,x)} V = d + r - n = \dim V$. If additionally $\text{char } k = 0$, then the generic fiber is also smooth (generic smoothness—see [2], Corollary 10.7, p. 272). $\square$

Corollary 3.3 (Atiyah–Serre Splitting Theorem) *Let X be an algebraic variety of dimension d and let $\mathcal{F}$ be a locally free sheaf on X of rank n . Assume that $\mathcal{F}$ is generated by global sections $s_1, \dots, s_r \in \Gamma(X, \mathcal{F})$. If $n > d$ then $\mathcal{F}$ contains a trivial subsheaf $\mathcal{A} \subset \mathcal{F}$ of rank $n - d$.*

Proof Indeed, by induction we can find $n - d$ linearly independent sections $t_j = \sum_{i=1}^r c_{j,i} s_i$ (note that the quotient sheaf of $\mathcal{F}$ is generated by the same sections (or rather their classes) as $\mathcal{F}$). $\square$

In particular if we assume that X is affine we have the Serre result in a classical form:

Corollary 3.4 (Serre splitting theorem) *Let X be an affine algebraic variety of dimension d and let $\mathcal{F}$ be a locally free sheaf on X of rank n . If $n > d$ then $\mathcal{F}$ contains a locally free subsheaf $\mathcal{F}'$ of rank d such that*

$$\mathcal{F} = \mathcal{F}' \oplus \mathcal{O}_X^{n-d}.$$

Proof From the previous statement we know that $\mathcal{O}_X^{n-d} \subset \mathcal{F}$. Take $\mathcal{F}' = \mathcal{F}/\mathcal{O}_X^{n-d}$. It is easy to see that $\mathcal{F}'$ is also locally free sheaf of rank d . Moreover we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X^{n-d} \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0.$$

This gives the following exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{O}_X^{n-d}) \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}') \rightarrow H^1(X, \mathcal{O}_X^{n-d}) = 0.$$

Since $k[X]$ modules $\Gamma(X, \mathcal{O}_X^{n-d})$, $\Gamma(X, \mathcal{F})$, $\Gamma(X, \mathcal{F}')$ are projective (it is true locally hence also globally!) we have

$$\Gamma(X, \mathcal{F}) = \Gamma(X, \mathcal{F}') \oplus \Gamma(X, \mathcal{O}_X^{n-d}).$$

Finally

$$\Gamma(X, \mathcal{F})^\sim = \Gamma(X, \mathcal{F}')^\sim \oplus \Gamma(X, \mathcal{O}_X^{n-d})^\sim,$$

and we have

$$\mathcal{F} = \mathcal{F}' \oplus \mathcal{O}_X^{n-d}$$

(see [2], Prop. 5.1, p. 110). □

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