On Borel almost disjoint families

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Abstract There is a close correspondence between uncountable almost disjoint families of subsets of ω and Aleksandrov–Urysohn compacta (in short, **AU**-compacta)—separable, uncountable compact spaces whose second derived set is a singleton. We shall show in particular, that **AU**-compacta embeddable in the space of first Baire class functions on the Cantor set 2^ω , with the pointwise topology, are exactly the ones determined by almost disjoint families that are Borel sets in 2^ω , and they are also distinguished among **AU**-compacta by the property that the cylindrical σ -algebras of their function spaces are standard measurable spaces. Although the first condition implies the third one for arbitrary separable compact space, it is an open problem, whether the reverse implication is always true.

Keywords Almost disjoint · Borel · $C(K)$ · Cylindrical σ -algebra · Pointwise topology

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1 Introduction

Let $B_1(2^\omega)$ be the space of real-valued first Baire class functions on the Cantor set, equipped with the topology of pointwise convergence. Given a compact set K , we shall denote by $C(K)$ the Banach space of real-valued continuous functions on K . For a Banach space X , we denote by $Cyl(X)$ the cylindrical σ -algebra in X , i.e., the smallest σ -algebra for which all functionals from the dual space X^* are measurable.

One of the main results of this paper is the following.

Theorem 1.1 *For any compact separable space K whose set of accumulation points is the one-point compactification of an uncountable discrete space, the following conditions are equivalent:*

- (i) K embeds in $B_1(2^\omega)$,
- (ii) $(C(K), Cyl(C(K)))$ is a standard Borel space,
- (iii) K is determined by a Borel almost disjoint family of subsets of ω .

Compact spaces K that are the subject of this theorem were considered first by Aleksandrov and Urysohn [1] and we shall call them **AU**-compacta, cf. [12].

There is a close correspondence between **AU**-compacta and uncountable almost disjoint families of subsets of natural numbers ω , explained in Sect. 2.2, and condition (iii) in Theorem 1.1 refers to this correspondence. Moreover, identifying subsets of ω with the characteristic functions, we consider any almost disjoint family on ω as a subset of the Cantor set 2^ω .

Condition (ii) in Theorem 1.1 means that there is a bijection of $C(K)$ onto 2^ω taking the cylindrical sets in $C(K)$ to Borel sets in the Cantor set and vice versa. The compact space K being separable, this is equivalent to the condition that for any countable D dense in K , the restrictions of functions from $C(K)$ to D form a Borel set in the countable product $\mathbb{R}^D$ of the real line, cf. Sect. 2.1.

Using a theorem of Dodos [5] (based on a deep result of Debs [3]) we shall show that the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 holds true for arbitrary compact separable spaces, cf. Theorem 3.2. It is an open problem whether the reverse implication is always true (this is the case for separable linearly ordered compact spaces), cf. [13].

We shall show in Sect. 7 that even for compact (as subsets of 2^ω) almost disjoint families, there are continuum many topological types of **AU**-compacta determined by these families.

However, we have the following

Theorem 1.2 *There are Borel almost disjoint families $\mathcal{A}_\alpha \subset 2^\omega$, $\alpha < \omega_1$, such that for any Borel almost disjoint family $\mathcal{A} \subset 2^\omega$, the compact space determined by $\mathcal{A}$ embeds as a retract in the compact space determined by some $\mathcal{A}_\alpha$.*

A boundedness theorem for a transfinite index introduced in [11], proved in Sect. 6, shows that the uncountable collection $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ in Theorem 1.2 cannot be replaced by any countable collection of Borel almost disjoint families, even if the requirement concerning retractions is dropped.

In Sect. 8 we show that any **AU**-compactum embeddable in $B_1(2^\omega)$ can be embedded in fact in the unit ball of the double dual to a separable Banach space not containing

any copy of ℓ_1 , equipped with the weak* topology (it is not known if this is true for any separable compact subspace of $B_1(2^\omega)$).

The last section of this paper deals with important Johnson–Lindenstrauss spaces. Following [8], one can associate with each AU-compactum K a Banach space $\text{JL}(K)$ —a twisted sum of c_0 and the Hilbert space of uncountable density, cf. [17]. We shall show for the measurable spaces $(\text{JL}(K), \text{Cyl}(\text{JL}(K)))$ a counterpart of Theorem 1.1.

2 Terminology and some background

Our topological terminology follows [6, 10] and the terminology related to the descriptive set theory follows [9, 10]. We denote by $|E|$ the cardinality of a set E .

2.1 The spaces $B_1(S)$ and $C_D(K)$

Given a separable metrizable space S , $B_1(S)$ is the space of real-valued first Baire class functions on S , equipped with the topology of pointwise convergence.

Rosenthal compacta are compact spaces which can be embedded in $B_1(\omega^\omega)$, where ω^ω is the space of the irrationals, cf. [7]. Let us recall an important characterization of separable Rosenthal compacta due to Godefroy, introducing first some notation.

Let K be a separable compact space. For each countable set D dense in K , we consider

$$C_D(K) = \{f|D : f \in C(K)\} \subset \mathbb{R}^D,$$

i.e., the space of restrictions of continuous functions on K to D , which is a subspace of the countable product of the real line. The space $C_D(K)$ can be identified with the topological space $(C(K), \tau_D)$, where τ_D is the topology of pointwise convergence on D .

Now, a separable compact space K is a Rosenthal compactum if, and only if, for any countable set D dense in K , the set $C_D(K)$ is analytic, cf. [7, Theorem 4]. We shall also use a fundamental fact that Rosenthal compacta are Fréchet spaces, cf. [2].

We should mention that there are separable compact subspaces of $B_1(\omega^\omega)$, even linearly ordered ones, which do not embed in $B_1(2^\omega)$, cf. [13, 15].

Following Godefroy [7], one can check that, for a separable compact space K , the measurable space $(C(K), \text{Cyl}(C(K)))$ is standard if and only if, for each countable set D dense in K , $C_D(K)$ is a Borel set in $\mathbb{R}^D$, cf. [13, Section 2].

2.2 The Aleksandrov–Urysohn compacta

As was agreed in Sect. 1, by an AU-compactum we mean an uncountable separable compact space K whose set K' of accumulation points has exactly one non-isolated point. It will be also convenient to use the following description of AU-compacta.

Let D be a countable set and let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of D , i.e., the intersection of any two distinct members of $\mathcal{A}$ is finite.

Let $A \mapsto p_A$ be a one-to-one correspondence between members of $\mathcal{A}$ and points in some fixed set disjoint from D , and let ∞ be a point distinct from points in D and any point p_A . In the set

$$K_{\mathcal{A}} = D \cup \{p_A : A \in \mathcal{A}\} \cup \{\infty\}$$

we introduce a topology declaring that points of D are isolated, basic neighborhoods p_A are of the form $\{p_A\} \cup (A \setminus F)$, where $F \subset A$ is finite, and ∞ is the point at infinity of the locally compact space $D \cup \{p_A : A \in \mathcal{A}\}$. Then $K_{\mathcal{A}}$ is an AU-compactum. Conversely, given an AU-compactum K with the countable set of isolated points D , for any $x \in K'$ distinct from the accumulation point of K' , pick an open compact set V_x in K with $V_x \cap K' = \{x\}$, and set $\mathcal{A} = \{V_x \cap D : x \in K'\}$. The collection $\mathcal{A}$ is almost disjoint and the identity on D extends to a homeomorphism of K onto the compactum $K_{\mathcal{A}}$.

The space $2^D = \{u \in \mathbb{R}^D : u(D) \subset \{0, 1\}\}$ is the Cantor set. We shall view points of 2^D interchangeably as function from D to $\{0, 1\}$ or subsets of D , identifying a set $A \subset D$ with its characteristic function χ_A .

Let us notice that if $\mathcal{A}$ is an almost disjoint family of subsets of D which is analytic in 2^D , then $K_{\mathcal{A}}$ is a Rosenthal compactum, cf. Lemma 4.4. It is not clear, however, if every AU-compactum K which is Rosenthal compact is homeomorphic to the compactum $K_{\mathcal{A}}$ for some analytic almost disjoint family $\mathcal{A}$.

We shall also show in Remark 4.7 (answering a query of the referee) that, for any family $\mathcal{A}$, the space $K_{\mathcal{A}}$ embeds in $B_1(C_D(K_{\mathcal{A}}))$.

Compacta $K_{\mathcal{A}}$, for Borel and analytic families $\mathcal{A}$ were investigated in [11, 15]. In Sect. 6, we shall use a transfinite stratification of AU-compacta which are Rosenthal compacta, introduced in [11].

2.3 The Johnson–Lindenstrauss spaces $\mathbf{JL}(K)$

Let K be an AU-compactum and let K' be the set of accumulation points of K .

We denote by $\|\cdot\|_{\infty}$ the supremum norm in $C(K)$, and let $\ell_2(K')$ be the Hilbert space of square-summable functions $u : K' \rightarrow \mathbb{R}$ with the norm

$$\|u\|_2 = \left(\sum_{x \in K'} (u(x))^2 \right)^{\frac{1}{2}}.$$

The Johnson–Lindenstrauss space associated with K is the space

$$\mathbf{JL}(K) = \{f \in C(K) : f|_{K'} \in \ell_2(K')\},$$

equipped with the norm

$$\|f\| = \max(\|f\|_{\infty}, \|f|_{K'}\|_2), \quad f \in \mathbf{JL}(K),$$

cf. [8]. Let us recall that the functions in $\mathbf{JL}(K)$ vanishing on K' form a copy of c_0 such that $\mathbf{JL}(K)/c_0$ can be identified with $\ell_2(K')$. An illuminating account of basic properties of $\mathbf{JL}(K)$ can be found in [17, Example 3].

3 Function spaces on separable compacta in $B_1(2^\omega)$

A basis for this section is the following theorem of Dodos [5].

Theorem 3.1 (Dodos) *Let $D \subset B_1(2^\omega)$ be a countable set whose closure in $B_1(2^\omega)$ is compact and let $\mathcal{C}$ be the subspace of the Cantor set 2^D consisting of subsets of D with exactly one accumulation point in $B_1(2^\omega)$. Then there exists a Borel set $\mathcal{B} \subset \mathcal{C}$ such that each element of $\mathcal{C}$ contains some element of $\mathcal{B}$.*

We shall derive from the Dodos theorem the following result, more general than the implication (i) $\Rightarrow$ (ii) in Theorem 1.1.

Theorem 3.2 *Let K be a separable compact space embeddable in $B_1(2^\omega)$. Then, for any countable set D dense in K , the function space $C_D(K)$ is Borel.*

Proof Let us fix a countable set D dense in K , and let $\mathcal{C}$ and $\mathcal{B}$ be as in Theorem 3.1. Let

$$\mathcal{A} = \mathcal{B} \cup \{\{f\} : f \in D\}. \quad (3.1)$$

The set $\mathcal{A}$ is a Borel subset of 2^D . Given an $A \in \mathcal{C}$ we denote by ϕ_A the unique accumulation point of A . We also define

$$\phi_{\{f\}} = f, \quad \text{for } f \in D. \quad (3.2)$$

Let us notice that $u : D \rightarrow \mathbb{R}$ belongs to $C_D(K)$ if and only if the oscillation of u at any point of K is zero, i.e., for any $f \in K$ and $\varepsilon > 0$, there is a neighborhood V of f in K such that $\text{diam } u(V \cap D) \leq \varepsilon$. Since K is Fréchet, cf. Sect. 2, for any $u \in \mathbb{R}^D$ and $f \in K$ the oscillation of u at f is positive if and only if, for some $n \geq 1$, either there exist $A, B \in \mathcal{B}$ such that $\phi_A = \phi_B = f$, and $|u(g) - u(h)| \geq 1/n$, whenever $g \in A, h \in B$, or $f \in D$ and there exists an $A \in \mathcal{B}$ such that $\phi_A = \phi_{\{f\}} = f$, cf. Eq. (3.2), and $|u(g) - u(f)| \geq 1/n$ for $g \in A$. Therefore, setting

$$\begin{aligned} \mathcal{H}(n) &= \{(u, A, B) \in \mathbb{R}^D \times \mathcal{B} \times \mathcal{A} : \\ &\quad \phi_A = \phi_B \text{ and } |u(g) - u(h)| \geq 1/n \text{ for } g \in A, h \in B\}, \end{aligned} \quad (3.3)$$

we have

$$C_D(K) = \mathbb{R}^D \setminus \bigcup_{n \geq 1} \text{proj}(\mathcal{H}(n)), \quad (3.4)$$

where $\text{proj} : \mathbb{R}^D \times \mathcal{B} \times \mathcal{A} \rightarrow \mathbb{R}^D$ is the projection onto the first factor. We shall show that each

$$\mathcal{H}(n) \text{ is a Borel set in } \mathbb{R}^D \times 2^D \times 2^D. \quad (3.5)$$

To that end, we shall verify first that, cf. Eqs. (3.1), (3.2),

$$\mathcal{D} = \{(A, B, t) \in \mathcal{B} \times \mathcal{A} \times 2^\omega : \phi_A(t) \neq \phi_B(t)\} \text{ is Borel.} \quad (3.6)$$

Indeed, one can check that the mapping $(A, t) \mapsto \phi_A(t)$ is Borel, which implies readily formula (3.6).

Now, for any $A \in \mathcal{B}$, $B \in \mathcal{A}$, the section $\{t \in 2^\omega : (A, B, t) \in \mathcal{D}\}$ of $\mathcal{D}$ at (A, B) is σ -compact, as the functions ϕ_A, ϕ_B are of the first Baire class, cf. Eqs. (3.6) and (3.2). Therefore, the projection of $\mathcal{D}$ parallel to 2^ω is Borel [9, 18.18], and so is the set $\{(A, B) \in \mathcal{B} \times \mathcal{A} : \phi_A = \phi_B\}$. This readily implies (3.5), cf. Eq. (3.3).

From (3.5) we infer that the space $C_D(K)$ is coanalytic, cf. Eq. (3.4), and since $C_D(K)$ is also analytic, cf. [7], it is Borel. $\square$

Remark 3.3 Any metrizable σ -compact space S is a continuous image of the space $\omega \times 2^\omega$, which, in turn, embeds as an open subset in 2^ω . Using this observation, one can easily verify that the space $B_1(S)$ embeds into $B_1(2^\omega)$. Therefore Theorem 3.2 holds true for any separable compact subspace K of $B_1(S)$, where S is metrizable and σ -compact.

4 Simple compacta in $B_1(2^\omega)$ and a proof of Theorem 1.1

We shall show that, among AU-compacta, the compacta $K_{\mathcal{A}}$ associated with Borel families $\mathcal{A}$ are the ones that can be embedded in $B_1(\omega^\omega)$ in a particularly simple way. This will be used in our proof of Theorem 1.1 and also, in the next section, in the proof of Theorem 1.2.

Definition 4.1 Let S be a separable metrizable space. We say that a compactum $K \subset B_1(S)$ is simple if K is separable, elements of K are characteristic functions of closed sets in S , and whenever $f \in K$ is non-isolated in K , $|f^{-1}(1)| \leq 1$.

In particular, a simple uncountable compactum in $B_1(S)$ is an AU-compactum.

Remark 4.2 If $K \subset B_1(S)$ is a simple compactum and S is absolutely Borel, then K is homeomorphic to a simple compactum in $B_1(\omega^\omega)$.

Indeed, let $\varphi: F \rightarrow S$ be a continuous injective map onto S defined on a closed subset F of the irrationals ω^ω , cf. [9, 13.7]. We define $\Phi: K \rightarrow B_1(\omega^\omega)$ letting $\Phi(f)(t) = f(\varphi(t))$, whenever $t \in F$ and $\Phi(f)(t) = 0$ if $t \notin F$. The mapping Φ is an embedding and the compactum $\Phi(K)$ is simple.

This remark can be strengthened as follows, cf. [13, Proposition 1].

Lemma 4.3 Let S be absolutely Borel. Then any simple compactum in $B_1(S)$ can be embedded in $B_1(2^\omega)$.

Proof Let $\mathbb{Q}_n = \{t \in 2^\omega : |t^{-1}(1)| \leq n\}$ and $\mathbb{P} = 2^\omega \setminus \bigcup_n \mathbb{Q}_n$. Since $\mathbb{P}$ is homeomorphic to ω^ω , by Remark 4.2 it is enough to show that any simple compactum $K \subset B_1(\mathbb{P})$ can be embedded in $B_1(2^\omega)$. To that end, let extend each function $f \in K$ to a function $\tilde{f} \in B_1(2^\omega)$ in the following way. We shall list (without repetitions) isolated points

in K as $f_1, f_2, \dots$ and let $\tilde{f}_n$ be the characteristic function of the set $\overline{f_n^{-1}(1)} \setminus \mathbb{Q}_n$, the closure being taken in 2^ω . If $f \in K$ is non-isolated, we extend f to 2^ω letting $\tilde{f} = 0$ for $t \notin \mathbb{P}$ (recall that $|f^{-1}(1)| \leq 1$).

One readily checks that the map $f \mapsto \tilde{f}$ is an embedding of K into $B_1(2^\omega)$. $\square$

The next lemma provides the implication (iii) $\Rightarrow$ (i) in Theorem 1.1.

Lemma 4.4 *Let $\mathcal{A} \subset 2^\omega$ be an almost disjoint family. Then the compactum $K_{\mathcal{A}}$ associated with $\mathcal{A}$ is homeomorphic to a simple compactum in $B_1(\mathcal{S})$, where $\mathcal{S} = \mathcal{A} \cup \{\{n\} : n \in \omega\} \subset 2^\omega$. Consequently, if $\mathcal{A}$ is Borel then $K_{\mathcal{A}}$ can be embedded in $B_1(2^\omega)$, and if $\mathcal{A}$ is analytic then $K_{\mathcal{A}}$ is Rosenthal compact.*

Proof We shall define an embedding φ of $K_{\mathcal{A}} = \omega \cup \{p_A : A \in \mathcal{A}\} \cup \{\infty\}$, cf. Sect. 2, in $B_1(\mathcal{S})$ as follows. If $n \in \omega$, $\varphi(n)$ is the characteristic function of the closed-and-open set $\{A \in \mathcal{S} : n \in A\}$ in $\mathcal{S}$. If $A \in \mathcal{A}$, $\varphi(p_A)$ is the characteristic function of the singleton $\{A\}$, and finally, $\varphi(\infty)$ is the zero function on $\mathcal{S}$.

Let us check that $\varphi : K_{\mathcal{A}} \rightarrow B_1(\mathcal{S})$ is a continuous injection. Indeed, φ is injective and for any $A \in \mathcal{A}$ and $n \in \omega$, the sets $\varphi^{-1}(\{f \in B_1(\mathcal{S}) : f(A) = 1\}) = A \cup \{p_A\}$ and $\varphi^{-1}(\{f \in B_1(\mathcal{S}) : f(\{n\}) = 1\}) = \{n\}$ are closed-and-open in $K_{\mathcal{A}}$. It remains to observe that, since $\varphi(K_{\mathcal{A}})$ consists of characteristic functions, the sets $\{f \in B_1(\mathcal{S}) : f(X) = 1\} \cap \varphi(K_{\mathcal{A}})$, $X \in \mathcal{S}$, and their complements in $\varphi(K_{\mathcal{A}})$ form a subbase in $\varphi(K_{\mathcal{A}})$.

If the family $\mathcal{A}$ is Borel in 2^ω , so is the space $\mathcal{S}$, therefore the second assertion of the lemma follows from Lemma 4.3. $\square$

We shall now address the implication (ii) $\Rightarrow$ (iii) in Theorem 1.1.

Lemma 4.5 *Let K be an AU-compactum and let D be the set of isolated points of K . If the function space $C_D(K)$ is Borel then K is homeomorphic to a compactum $K_{\mathcal{A}}$ associated with a Borel almost disjoint family $\mathcal{A} \subset 2^\omega$.*

Proof Let $K' = K \setminus D$ be the set of accumulation points of K and let p be the unique non-isolated point of the space K' . For any point $x \in K' \setminus \{p\}$, there is a sequence $d_n \in D$ converging to x . Therefore the mapping $\varphi_x : C_D(K) \rightarrow \mathbb{R}$, given by $\varphi_x(f|D) = f(x) = \lim_n f(d_n)$, $f|D \in C_D(K)$, is of the first Baire class. Since p is the limit of the sequence of the points from $K' \setminus \{p\}$, the same argument shows that the mapping $f|D \mapsto f(p)$ is of the second Baire class on $C_D(K)$. Therefore, the set $\{f|D \in C_D(K) \cap 2^D : f(p) = 0\}$ is Borel in the Cantor set 2^D . It follows that

$$\mathcal{S} = \{A \subset D : \overline{A} \text{ is an infinite closed-and-open set in } K \text{ missing } p\}, \quad (4.1)$$

where the closure is taken in K , is a Borel set in 2^D . Let us check that

$$\mathcal{M} = \{A \in \mathcal{S} : |\overline{A} \cap K'| = 1\} \text{ is Borel in } 2^D. \quad (4.2)$$

Indeed, the set

$$\mathcal{H} = \{(A, B, C) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S} : B \cup C \subset A, B \cap C = \emptyset\} \quad (4.3)$$

is Borel in $2^D \times 2^D \times 2^D$. Moreover, for any $A \in \mathcal{S}$, since $\overline{A} \cap K'$ is finite, there are only countably many elements of $\mathcal{S}$ contained in A , cf. Eq. (4.1), and hence, for any $A \in \mathcal{S}$, the section $\mathcal{H}(A) = \{(B, C) \in \mathcal{S} \times \mathcal{S} : (A, B, C) \in \mathcal{H}\}$ is countable. In effect,

$$\mathcal{M} = \mathcal{S} \setminus \text{proj } \mathcal{H} \text{ is Borel,} \quad (4.4)$$

where $\text{proj} : 2^D \times 2^D \times 2^D \rightarrow 2^D$ is the projection onto the first factor, cf. Eqs. (4.2), (4.3), and [9, 18.10].

Now, let us consider in $\mathcal{M}$ the equivalence relation $=_*$, where $A =_* B$ means that the symmetric difference of A and B is finite, and let $[A] = \{B \in \mathcal{M} : A =_* B\}$ be the equivalence class of $A \in \mathcal{M}$. Let us notice that for any $A \in \mathcal{M}$

$$[A] \text{ is a countable } G_\delta\text{-set in } \mathcal{M}. \quad (4.5)$$

Indeed, by Eqs. (4.1) and (4.2), for any $A, B \in \mathcal{M}$, $A =_* B$ means exactly that $A \cap B$ is infinite, and hence $\mathcal{M} \setminus [A]$ is a countable union of closed sets $\{B \in \mathcal{M} : B \cap A = F\}$, where $F \subset A$ is finite.

Similar arguments show also that the relation

$$\{(A, B) \in \mathcal{M} \times \mathcal{M} : A =_* B\} \text{ is Borel in } 2^D \times 2^D. \quad (4.6)$$

By Kechris [9, 18.10], the saturation of any Borel set in $\mathcal{M}$ with respect to $=_*$ is Borel, and the fact that equivalence classes are G_δ -sets in $\mathcal{M}$, cf. Eq. (4.5), provides a Borel selector for this relation, cf. [16, 5.9.2].

In effect, we get $\mathcal{A} \subset \mathcal{M}$ such that

$$\mathcal{A} \subset 2^D \text{ is Borel and } |\mathcal{A} \cap [A]| = 1 \text{ for } A \in \mathcal{M}. \quad (4.7)$$

The Borel family $\mathcal{A}$ is almost disjoint and it remains to make sure that the compactum $K_{\mathcal{A}} = \omega \cup \{p_A : A \in \mathcal{A}\} \cup \{\infty\}$ associated with $\mathcal{A}$, cf. Sect. 2, is homeomorphic to K .

This, however, follows readily from (4.2) and (4.7): identifying the isolated points of $K_{\mathcal{A}}$ and K , sending each p_A to the unique point in $\overline{A} \cap K'$ and ∞ to p , one defines a homeomorphism of $K_{\mathcal{A}}$ onto K . $\square$

Since the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 was already established in Theorem 3.2, summarizing the results of this, and the preceding section, we obtain a Proof of Theorem 1.1.

Remark 4.6 Our reasoning shows also that one can add to the equivalent conditions in Theorem 1.1 yet another condition:

(iv) K is homeomorphic to a simple compactum in $B_1(\omega^\omega)$.

Remark 4.7 We shall show that any AU-compactum K embeds in $B_1(C_D(K))$, confirming a suggestion made by the referee.

Let

$$\mathcal{E} = \{f \in C_D(K) : f(D) \subset \{0, 1\}, f(\infty) = 0\},$$

∞ being the only non-isolated point of K' .

Identifying points in K with corresponding evaluation maps on $\mathcal{E}$, one embeds K into $B_1(\mathcal{E})$. To embed K into $B_1(C_D(K))$, it is enough to find a closed set $\mathcal{F} \subset C_D(K)$ and a continuous surjection $\varphi: \mathcal{F} \rightarrow \mathcal{E}$. Indeed, given such φ , one can define an embedding $\Phi: B_1(\mathcal{E}) \rightarrow B_1(C_D(K))$, assigning to $u \in B_1(\mathcal{E})$ the function $\Phi(u)$ which coincides with $u \circ \varphi$ on $\mathcal{F}$ and is zero outside of $\mathcal{F}$.

Now, the beginning of the Proof of Lemma 4.5 shows that $\mathcal{E}$ is a Borel set in $C_D(K) \subset \mathbb{R}^D$. Let $\mathcal{E}^*$ be a Borel set in $\mathbb{R}^D$ whose trace on $C_D(K)$ is $\mathcal{E}$, and let $\mathcal{G} \subset \mathbb{R}^D \times \omega^\omega$ be a closed set projecting onto $\mathcal{E}^*$. Then $\mathcal{F}' = \mathcal{G} \cap (C_D(K) \times \omega^\omega)$ is a closed set in $C_D(K) \times \omega^\omega$ and let $\varphi: \mathcal{F}' \rightarrow \mathcal{E}$ be the restriction to $\mathcal{F}'$ of the projection.

One can complete the reasoning noticing that $C_D(K) \times \omega^\omega$ embeds in $C_D(K)$ as a closed subspace. To this end, one should use the well-known fact that ω^ω can be embedded as a closed subset into a subspace c_0 of $\mathbb{R}^\omega$, consisting of sequences converging to zero, and the fact $C_D(K)$ is homeomorphic to $C_D(K) \times c_0$, cf. the proof of Proposition 5.1 in [4].

5 Proof of Theorem 1.2

We shall use a representation of AU-compacta associated with Borel almost disjoint families, described in Sect. 4.

Let T be an absolutely Borel space. Given a simple compactum $K \subset B_1(T)$, cf. Definition 4.1, the collection $\mathcal{C} = \{f^{-1}(1): f \text{ is an isolated point in } K\}$ has the following properties:

$$\text{elements of } \mathcal{C} \text{ are closed in } T, \quad (5.1)$$

$$\text{if } \mathcal{E} \subset \mathcal{C} \text{ is infinite, then } \left| \bigcap \mathcal{E} \right| \leq 1. \quad (5.2)$$

Conversely, let $\mathcal{C}$ be a countable family of subsets of T satisfying (1) and (2), let χ_C be the characteristic function of $C \in \mathcal{C}$, and let

$$K(\mathcal{C}) = \overline{\{\chi_C: C \in \mathcal{C}\}}, \quad (5.3)$$

where the closure is considered in $\mathbb{R}^T$. The compactum $K(\mathcal{C})$ is separable and, by Eq. (5.2), each accumulation point in $K(\mathcal{C})$ is either the characteristic function of a singleton, or else it is the zero function. Therefore, cf. Definition 4.1,

$$K(\mathcal{C}) \subset B_1(T) \text{ is a simple compactum.} \quad (5.4)$$

Passing to a construction of families described in Theorem 1.2, let us consider the space $\mathcal{K}(2^\omega)$ of all closed sets in 2^ω , equipped with the Vietoris topology and let us fix a continuous surjection

$$\varphi = (\varphi_1, \varphi_2, \dots): 2^\omega \rightarrow \mathcal{K}(2^\omega) \times \mathcal{K}(2^\omega) \times \dots. \quad (5.5)$$

We shall denote by $2^{<\omega}$ the set of all functions $\sigma: \{0, 1, \dots, n-1\} \rightarrow \{0, 1\}$, $n \in \omega$, given $\sigma \in 2^{<\omega}$ we denote by $|\sigma|$ the cardinality of the domain of σ and $\sigma \prec s$, where

$s \in 2^{<\omega} \cup 2^\omega$, means that s extends σ . We set, for $\sigma \in 2^{<\omega}$,

$$C(\sigma) = \{(s, t) \in 2^\omega \times 2^\omega : \sigma \prec s \text{ and } t \in \varphi_{|\sigma|}(s)\}. \quad (5.6)$$

Since the function $\varphi_{|\sigma|} : 2^\omega \rightarrow \mathcal{K}(2^\omega)$ is continuous, the set $C(\sigma)$ is closed in $2^\omega \times 2^\omega$.

Let $\mathbb{P}$ be the set of functions in 2^ω with infinite support ($\mathbb{P}$ is homeomorphic to ω^ω) and let

$$S = \{s \in 2^\omega : \text{for each infinite } N \subset \omega, |\mathbb{P} \cap \bigcap \{\varphi_n(s) : n \in N\}| \leq 1\}. \quad (5.7)$$

Let us notice that

$$S \text{ is coanalytic.} \quad (5.8)$$

Indeed, the set

$$H = \{(s, t_1, t_2, u) \in 2^\omega \times \mathbb{P} \times \mathbb{P} \times \mathbb{P} : \\ t_1 \neq t_2 \text{ and, for any } n \text{ with } u(n) = 1, t_i \in \varphi_n(s) \text{ for } i = 1, 2\} \quad (5.9)$$

is Borel, and $2^\omega \setminus S = \text{proj } H$, where proj is the projection of $2^\omega \times \mathbb{P} \times \mathbb{P} \times \mathbb{P}$ onto the first factor.

Decomposing S into constituents, cf. [9, 10], we have

$$S = \bigcup_{\alpha < \omega_1} S_\alpha, \quad S_\alpha \text{ is a Borel set in } 2^\omega. \quad (5.10)$$

Let us fix $\alpha < \omega_1$, and let us consider the countable collection, cf. Eq. (5.6),

$$\mathcal{C}_\alpha = \{C(\sigma) \cap T_\alpha : \sigma \in 2^{<\omega}\}, \quad T_\alpha = S_\alpha \times \mathbb{P}. \quad (5.11)$$

Members of $\mathcal{C}_\alpha$ are closed in the absolutely Borel space T_α . Let us check that $\mathcal{C}_\alpha$ satisfies condition (5.2). Let $\mathcal{E} \subset \mathcal{C}_\alpha$ be an infinite collection with $L = \bigcap \mathcal{E} \neq \emptyset$. One can pick $\sigma_1 \prec \sigma_2 \prec \dots$ with $|\sigma_1| < |\sigma_2| < \dots$ such that $C(\sigma_i) \cap T_\alpha \in \mathcal{E}$, cf. Eqs. (5.6), (5.11). Let $s \in 2^\omega$ be the unique extension of all σ_i . Since $\bigcap \mathcal{E} \neq \emptyset$, $L = \{s\} \times J$ and $J \subset \varphi_{|\sigma_i|}(s)$ for $i = 1, 2, \dots$. Since $s \in S$, by Eq. (5.7), we infer that J is a singleton, and so is L .

Having checked that a countable collection $\mathcal{C}_\alpha$ satisfies (5.1) and (5.2), we shall consider the corresponding simple compactum defined in (5.3), cf. Eq. (5.4),

$$K_\alpha = K(\mathcal{C}_\alpha) \subset B_1(T_\alpha). \quad (5.12)$$

We shall show that for any uncountable Borel almost disjoint family $\mathcal{A} \subset 2^\omega$, the associated compactum $K_{\mathcal{A}}$ embeds in some K_α as a retract.

To that end, let us replace $K_{\mathcal{A}}$ by its homeomorphic copy $K \subset B_1(\mathbb{P})$ which is a simple compactum, cf. Lemma 4.4. Let $d_1, d_2, \dots$ be isolated points of K , enumerated without repetitions, and let $C_n = d_n^{-1}(1)$. Since φ defined in (5.5) is a surjection, there

is $s \in 2^\omega$ such that $C_n = \varphi_n(s) \cap \mathbb{P}$ for $n = 1, 2, \dots$. Since the compactum K is simple, by the remark opening this section we have $s \in S$, cf. Eq. (5.7). Let us fix an α with $s \in S_\alpha$, cf. Eq. (5.10), let σ_n be the restriction of s to $\{0, \dots, n-1\}$, and let f_n be the characteristic function of the set $C(\sigma_n) \cap T_\alpha$, cf. Eqs. (5.6) and (5.11).

We have $f_n(s, t) = d_n(t)$ for any $t \in \mathbb{P}$, and if $u \neq s$ and $t \in \mathbb{P}$, $f_n(u, t) = 0$ for all but finitely many n . It follows that the correspondence $d_n \mapsto f_n$ extends to a homeomorphism of K onto the closure $L = \overline{\{f_n : n \in \omega\}} \subset K_\alpha$.

Moreover, denoting by $\chi_\emptyset$ the zero function on T_α , we see that $L \setminus \{\chi_\emptyset\}$ is open in $K_\alpha \setminus \{\chi_\emptyset\}$, as the functions in $L \setminus \{\chi_\emptyset\}$ do not vanish on $\{s\} \times \mathbb{P}$, while any function from $K_\alpha \setminus L$ is zero on $\{s\} \times \mathbb{P}$. Therefore, sending points from $K_\alpha \setminus L$ to $\chi_\emptyset$ and not moving the points in L , one defines a retraction of K_α onto L .

Now, to complete the proof it is enough to use Remark 4.6 to get, for each $\alpha < \omega_1$, a Borel almost disjoint family $\mathcal{A}_\alpha \subset 2^\omega$ such that the compactum $K_{\mathcal{A}_\alpha}$ associated with this family is homeomorphic to the simple compactum K_α . $\square$

6 A boundedness theorem

We shall show in this section that any collection of Borel almost disjoint families satisfying the assertion of Theorem 1.2 (even without the condition concerning retractions) must be uncountable.

We shall derive this result from a certain boundedness theorem for the following transfinite index $\eta(K)$ associated to any separable Rosenthal compact space K in [11]:

$$\eta(K) = \begin{cases} \min\{\alpha : \text{there exists a countable dense } D \subset K \text{ such that} \\ C_D(K) \text{ is a Borel set of the class } \alpha \text{ in } \mathbb{R}^D\} & \text{if such } D \text{ exist} \\ \omega_1 & \text{otherwise} \end{cases}$$

The result stated at the beginning of this section is an immediate consequence of the following two theorems.

Theorem 6.1 [11, Theorem 4.1] *For every countable ordinal α , there exists a Borel almost disjoint family $\mathcal{B}_\alpha$ such that $\eta(K_{\mathcal{B}_\alpha}) \geq \alpha$.*

Theorem 6.2 *For any Borel almost disjoint family $\mathcal{A}$, the indices $\eta(L)$ of separable compact spaces L that can be embedded in the compact space determined by $\mathcal{A}$, are bounded below ω_1 .*

For a Proof of Theorem 6.2 we will need the following auxiliary fact.

Lemma 6.3 *Let K be a separable scattered compact space. If, for some countable dense set $D \subset K$, the space $C_D(K) \cap 2^D$ is a Borel subset of 2^D of additive class α , then K is Rosenthal compact and $\eta(K) \leq 1 + \alpha + 1$.*

Proof Let $CO_D(K)$ be the family of all intersections $U \cap D$, where U is a closed-and-open subset of K . Obviously, $C_D(K) \cap 2^D$ is the family of all characteristic functions of elements of $CO_D(K)$. Modifying slightly an argument from the proof of Theorem 4.1 from [11, Sec. 4.2] one can easily verify that, for a bounded function

$f : D \rightarrow \mathbb{R}$, we have

$$f \in C_D(K) \Leftrightarrow (\forall p, q \in \mathbb{Q} \ p < q) [\exists r \in \mathbb{Q} \cap (p, q) \ f^{-1}((r, \infty)) \in CO_D(K)].$$

Then, the argument from [11, Sec. 4.2] shows that $C_D(K)$ is a Borel set of the class $\leq 1 + \alpha + 1$. Now, the fact that K is Rosenthal compact follows from [4, Corollary 2.4] and clearly $\eta(K) \leq 1 + \alpha + 1$. $\square$

Recall that, for subsets A, B of ω , we write $A \subseteq_* B$ if $A \setminus B$ is finite, and $A =_* B$ if $A \subseteq_* B$ and $B \subseteq_* A$. For a set X , by $[X]^{<\omega}$ ($[X]^\omega$) we denote the family of all finite subsets of X (all subsets of X of the cardinality ω). We also denote by $<$ the order on $P(\omega)$ corresponding to the lexicographic order on 2^ω .

Proof of Theorem 6.2 For $n = 1, 2, \dots$ we define

$$\mathcal{B}_n = \left\{ (A_1, \dots, A_n, B) \in \mathcal{A}^n \times [\omega]^\omega : A_1 < \dots < A_n, B \subseteq_* \bigcup_{i=1}^n A_i \right\}. \quad (6.1)$$

Letting $\pi_n : \mathcal{A}^n \times [\omega]^\omega \rightarrow [\omega]^\omega$ be the projection onto last axis, we put

$$\mathcal{C}_1 = \mathcal{B}_1, \quad \mathcal{C}_{n+1} = \mathcal{B}_{n+1} \setminus \pi_{n+1}^{-1}(\pi_n(\mathcal{B}_n)) \quad (6.2)$$

$$p_n = \pi_n|_{\mathcal{C}_n}, \quad \mathcal{D}_n = p_n(\mathcal{C}_n), \quad (6.3)$$

for $n = 1, 2, \dots$. The set $\mathcal{B}_n$ is Borel. Since $\mathcal{C}_{n+1} = \mathcal{B}_{n+1} \setminus \bigcup_{i=1}^n \pi_{n+1}^{-1}(\mathcal{D}_i)$ and the map p_n is injective, one can show by induction that $\mathcal{C}_n$ and $\mathcal{D}_n$ are also Borel and the inverse function $p_n^{-1} : \mathcal{D}_n \rightarrow \mathcal{C}_n$ is a Borel map. Let β be the upper bound for Borel classes of $\mathcal{C}_n, \mathcal{D}_n$ and $p_n^{-1}, n = 1, 2, \dots$. Let $r_n : \mathcal{D}_n \rightarrow P(\omega)$ be defined by

$$r_n(B) = A_1 \cup \dots \cup A_n, \quad (A_1, \dots, A_n, B) \in \mathcal{C}_n. \quad (6.4)$$

Notice that the sets $A_1, \dots, A_n$ in (6.4) are uniquely determined by $B \in \mathcal{D}_n$. Since $r_n(B)$ is the union of the first n coordinates of $p_n^{-1}(B)$, the map r_n is also of the Borel class $\leq \beta$.

Let $h : L \rightarrow K_{\mathcal{A}}$ be a homeomorphic embedding of an AU-compactum L into $K_{\mathcal{A}}$. Denote the image $h(L \setminus L')$ of the set of isolated points of L by T . It may happen that $T \setminus \omega \neq \emptyset$, but clearly $\infty \notin T$. By Lemma 6.3 it is enough to show that the space $C_T(h(L)) \cap 2^T$ is Borel of additive class $\leq \beta + 2$.

For $n = 1, 2, \dots$ we put

$$\mathcal{E}_n = \{D \in \mathcal{D}_n : D \subset T, r_n(D) \cap T =_* D\}. \quad (6.5)$$

Let $s_n : \mathcal{D}_n \rightarrow P(\omega) \times P(\omega)$ be defined by

$$s_n(D) = (r_n(D) \cap T, D). \quad (6.6)$$

The map s_n is of the Borel class $\leq \beta$. Let

$$\Delta^* = \{(A, B) \in P(T \cap \omega) \times P(T \cap \omega) : A =_* B\}. \quad (6.7)$$

Since Δ^* is an F_σ -subset of $P(\omega) \times P(\omega)$, the set $\mathcal{E}_n = s_n^{-1}(\Delta^*)$ is Borel of the class $\leq \beta + 1$, and the union $\mathcal{E} = \bigcup_{n=1}^\infty \mathcal{E}_n$ is Borel of additive class $\leq \beta + 2$. Observe that

$$\begin{aligned} \mathcal{E} = \{E \in [T \cap \omega]^\omega : (\exists A_1, \dots, A_n \in \mathcal{A}) \\ A_1 \cup \dots \cup A_n \cap T =_* E, n = 1, 2, \dots\}. \end{aligned} \quad (6.8)$$

Let $\mathcal{F} = \mathcal{E} \cup [T \cap \omega]^{<\omega}$. One can easily verify that $\mathcal{F}$ is the family of all sets of the form $U \cap T \cap \omega$, where U is a closed-and-open subset of $K_{\mathcal{A}}$ (equivalently, a closed-and-open subset of $h(L)$) such that $\infty \notin U$. It is also clear that, for such set U , the intersection $U \cap (T \setminus \omega)$ is finite. Therefore, we can describe the family $\mathcal{G}$ of all traces of closed-and-open subsets of $h(L)$ on T in the following way

$$\mathcal{G} = \bigcup_{S \in [T \setminus \omega]^{<\omega}} \{F \cup S : F \in \mathcal{F}\} \cup \{T \setminus (F \cup S) : F \in \mathcal{F}\}. \quad (6.9)$$

Since the space $C_T(h(L)) \cap 2^T$ can be identified with the family $\mathcal{G}$, the above formula shows that $C_T(h(L)) \cap 2^T$ is a Borel subset of 2^T of additive class $\leq \beta + 2$. $\square$

In view of Proposition 6.2 it is natural to state the following

Problem 6.4 Let K be a separable compact subspace of $B_1(2^\omega)$. Does there exist a countable ordinal α such that $\eta(L) \leq \alpha$ for every separable compact $L \subset K$?

7 Continuum many topological types of $K_{\mathcal{A}}$ determined by compact families $\mathcal{A}$

Example 7.1 There exists a collection $\{\mathcal{A}_\alpha : \alpha < 2^\omega\}$ of compact almost disjoint families such that the spaces $K_{\mathcal{A}_\alpha}$ and $K_{\mathcal{A}_\beta}$ are not homeomorphic for $\alpha \neq \beta$.

To that end we will construct a collection of almost disjoint families $\mathcal{B}$ of subsets of the Cantor tree $T = 2^{<\omega}$, and we shall use the notation $K_{\mathcal{B}} = T \cup \{p_B : B \in \mathcal{B}\} \cup \{\infty\}$ introduced in Sect. 2.2.

Given $x \in 2^\omega$, by B_x we denote the branch $\{x|n : n \in \omega\}$ in T determined by x . For a real number $t \geq 2$, let $\mathbb{N}_t = \{n_i^t = [t^i] : i \in \omega\}$, where $[x]$ is the integer part of a real number x . Let $A_t = \{x \in 2^\omega : x_n = 0 \text{ for } n \notin \mathbb{N}_t\}$ and $\mathcal{A}_t = \{B_x : x \in A_t\}$. Clearly, the family $\mathcal{A}_t$ is almost disjoint and compact (the mapping $x \mapsto B_x$ is a homeomorphism of the compact set A_t onto $\mathcal{A}_t$).

We will prove the following.

Proposition 7.2 For every $2 \leq t < s$, the AU -compacta $K_{\mathcal{A}_s}$ and $K_{\mathcal{A}_t}$ are not homeomorphic.

Proof Assume, towards a contradiction, that $\varphi: K_{\mathcal{A}_s} \rightarrow K_{\mathcal{A}_t}$ is a homeomorphism. Then, it is obvious that $\varphi(T) = T$, $\varphi(\infty) = \infty$, and there exists a bijection $\psi: A_s \rightarrow A_t$ such that $\varphi(p_{B_x}) = p_{B_{\psi(x)}}$ for $x \in A_s$. By continuity of φ , for every $x \in A_s$, there exists a $k_x \in \omega$ such that

$$\varphi(\{x|n: n \geq k_x\}) \subset B_{\psi(x)}. \quad (7.1)$$

Let $C_k = \{y \in A_t: k_{\psi^{-1}(y)} = k\}$. Applying the Baire Category theorem, one can find k such that the set C_k is dense in some open set in A_t . For a sequence $\sigma \in 2^i$, $i \in \omega$, we denote the basic clopen set $\{y \in A_t: y_{n_j^t} = \sigma_j, j < i\}$ in A_t by U_σ . Take $i \in \omega$ and $\sigma \in 2^i$ such that the set C_k is dense in U_σ .

One can easily verify that for some $j \geq 1$ we have

$$n_{i+j}^t < n_{j-1}^s - k. \quad (7.2)$$

Indeed, since $(s/t)^j$ tends to ∞ , we can find $j \geq 1$ such that

$$\left(\frac{s}{t}\right)^j > s(t^i + k + 1) > s\left(t^i + \frac{k+1}{t^j}\right)$$

therefore $s^{j-1} > t^{i+j} + k + 1$ and it remains to recall the definition of the numbers n_l^s, n_l^t :

$$n_{i+j}^t = [t^{i+j}] \leq t^{i+j} < s^{j-1} - k - 1 < [s^{j-1}] - k = n_{j-1}^s - k.$$

For a sequence $\tau \in 2^j$, let $\sigma^\wedge \tau$ denote the sequence $(\sigma_0, \dots, \sigma_{i-1}, \tau_0, \dots, \tau_{j-1}) \in 2^{i+j}$. Since C_k is dense in U_σ , for every $\tau \in 2^j$, we can pick a point $y_\tau \in C_k \cap U_{\sigma^\wedge \tau}$. For distinct τ, τ' we have $y_\tau | n_{i+j}^t \neq y_{\tau'} | n_{i+j}^t$, hence

$$|B_{y_\tau} \cap B_{y_{\tau'}}| < n_{i+j}^t \quad \text{for } \tau, \tau' \in 2^j, \tau \neq \tau'. \quad (7.3)$$

Observe that the cardinality of the set $\{x|n_{j-1}^s: x \in A_s\}$ equals 2^{j-1} and therefore there exist $\tau, \tau' \in 2^j$, $\tau \neq \tau'$ such that $\psi^{-1}(y_\tau) | n_{j-1}^s = \psi^{-1}(y_{\tau'}) | n_{j-1}^s$. It follows that $|B_{\psi^{-1}(y_\tau)} \cap B_{\psi^{-1}(y_{\tau'})}| \geq n_{j-1}^s$. Therefore, by inequality (7.2), for the set $A = (B_{\psi^{-1}(y_\tau)} \cap B_{\psi^{-1}(y_{\tau'})}) \setminus \bigcup_{i < k} 2^i$ we have $|A| \geq n_{j-1}^s - k > n_{i+j}^t$. On the other hand, for every $\rho \in A$, condition (7.1) implies that $\varphi(\rho) \in B_{y_\tau} \cap B_{y_{\tau'}}$, a contradiction with inequality (7.3). $\square$

8 Embeddings in $B_1(2^\omega)$ related to the Odell–Rosenthal theorem

Important examples of separable Rosenthal compact spaces are provided by the following theorem of Odell and Rosenthal [14], where $(B_{X^{**}}, w^*)$ is the unit ball of the second dual X^{**} of a Banach space X , equipped with the weak* topology:

Theorem 8.1 (Odell–Rosenthal) *A separable Banach space X does not contain an isomorphic copy of ℓ_1 , if and only if, $(B_{X^{**}}, w^*)$ is a (separable) Rosenthal compact space.*

Actually, in this case the restriction of an $x^{**} \in B_{X^{**}}$ to B_{X^*} gives an embedding of $(B_{X^{**}}, w^*)$ into $B_1((B_{X^*}, w^*))$, hence $(B_{X^{**}}, w^*)$ embeds into $B_1(2^\omega)$. Observe that $(B_{X^{**}}, w^*)$ has a dense subset B_X of elements continuous on (B_{X^*}, w^*) , therefore $\eta((B_{X^{**}}, w^*)) = 2$, see [11, Thm. 2.3] and, cf. Sect. 6.

It is not clear which separable compact subspaces of $B_1(2^\omega)$ can be embedded into $(B_{X^{**}}, w^*)$ for some separable Banach space X not containing any isomorphic copy of ℓ_1 . The following result provides a useful criterium to that effect.

Theorem 8.2 [11] *Let $L \subset B_1(2^\omega)$ be a compact set with $L \cap C(2^\omega)$ dense in L . Then there exists a separable Banach space X , such that X does not contain any isomorphic copy of ℓ_1 , and L embeds into $(B_{X^{**}}, w^*)$.*

This result was proved in [11, Sec. 6.2] for the some special family of subspaces L of $B_1(2^\omega)$, but the proof used only the properties of L described in the assumptions of the above theorem. Let us notice that the assumptions of Theorem 8.2 yield the separability of L and this theorem suggests the following problem.

Problem 8.3 Let $K \subset B_1(2^\omega)$ be a separable compact set. Does there exist a compact space $L \subset B_1(2^\omega)$ such that $L \cap C(2^\omega)$ is dense in L and K embeds into L ?

We shall show that this is the case for Aleksandrov–Urysohn compacta.

Theorem 8.4 *Let $\mathcal{A}$ be a Borel almost disjoint family of subsets of ω . There exists a compact subspace L of $B_1(2^\omega)$ such that the intersection $L \cap C(2^\omega)$ is dense in L and the space $K_{\mathcal{A}}$ embeds in L .*

Proof As shown in the proofs of Lemma 4.4 and Remark 4.2, there is a closed subset F of ω^ω such that $K_{\mathcal{A}}$ is homeomorphic to a simple compactum K in $B_1(F)$ whose isolated points are characteristic functions of closed-and-open sets in F . Let T be the set of all isolated points of K . Take any embedding φ of F into 2^ω and let ψ be the embedding of F into $2^\omega \times 2^T$ defined by

$$\psi(x) = (\varphi(x), (f(x))_{f \in T}) \quad \text{for } x \in F. \quad (8.1)$$

The image $\psi(F)$ is a G_δ -subset of $2^\omega \times 2^T$, and therefore we can write $2^\omega \times 2^T = \bigcup_{n \in \omega} A_n$, where A_n are closed in $2^\omega \times 2^T$. Let $\{f_n : n \in \omega\}$ be an enumeration without repetitions of T . For every $n \in \omega$, we define the following open subset of $2^\omega \times 2^T$:

$$G_n = \{(x, y) \in 2^\omega \times 2^T : y(f_n) = 1\} \setminus A_n. \quad (8.2)$$

Let g_n be the characteristic function of G_n . Observe that g_n is an extension of the function $f_n \circ \psi^{-1} : \psi(F) \rightarrow \{0, 1\}$ over $2^\omega \times 2^T$. Therefore, one can easily verify that the closure M of $\{g_n : n \in \omega\}$ in $B_1(2^\omega \times 2^T)$ is homeomorphic to K (cf. the Proof of Lemma 4.4) and, for accumulation points g of M , we have $|g^{-1}(1)| \leq 1$.

For each $n \in \omega$, if g_n is continuous, we define $g_n^k = g_n$ for $k \in \omega$. Otherwise, we take a sequence $(g_n^k)_{k \in \omega}$ of continuous functions from $2^\omega \times 2^T$ into $\{0, 1\}$ converging

pointwise to g_n and such that $g_n^k \leq g_n$ for $k \in \omega$. Let L be the closure of $\{g_n^k : k, n \in \omega\}$ in $2^{2^\omega \times 2^T}$. We will show that $L \subset B_1(2^\omega \times 2^T)$.

It is enough to verify that if h is an accumulation point of L and $|h^{-1}(1)| \geq 2$, then $h = g_n$ for some $n \in \omega$. Fix such h and $x, y \in h^{-1}(1)$, $x \neq y$, and consider the closed-and-open neighborhood $U = \{f \in 2^{2^\omega \times 2^T} : f(x) = f(y) = 1\}$ of h in $2^{2^\omega \times 2^T}$. Since no function in U is an accumulation point of M , there is a finite set $S \subset \omega$ such that $g_n \notin U$ for $n \in \omega \setminus S$. Since $g_n^k \leq g_n$, we also have $g_n^k \notin U$ for $n \in \omega \setminus S$, $k \in \omega$. Therefore $h = g_n$ for some $n \in S$. $\square$

9 Johnson–Lindenstrauss spaces with standard cylindrical σ -algebras

In this section we prove a counterpart of Theorem 1.1 for twisted sums of c_0 and the Hilbert space of uncountable density, defined by Johnson and Lindenstrauss [8], cf. Sect. 2.3.

Theorem 9.1 *For the Johnson–Lindenstrauss space $\text{JL}(K)$ associated with an AU -compactum K , the following are equivalent:*

- (i) *The measurable space $(\text{JL}(K), \text{Cyl}(\text{JL}(K)))$ is standard,*
- (ii) *K is determined by a Borel almost disjoint family $\mathcal{A}$ in 2^ω .*

Proof We adopt the notation from Sects. 2.1, 2.2, and 2.3.

To see that (i) implies (ii), let us consider the set D of isolated points in K and the injective restriction map $f \mapsto f|D$ on $\text{JL}(K)$. Then $\{f|D : f \in C(K), f(K) \subset \{0, 1\} \text{ and } f(\infty) = 0\}$ is a closed subset of the Borel set $\{f|D : f \in \text{JL}(K)\}$, and hence it is Borel in $\mathbb{R}^D$. By Lemma 6.3, we conclude that $C_D(K)$ is Borel in $\mathbb{R}^D$, and Lemma 4.5 provides (ii).

In the other direction, let us assume that $K = K_{\mathcal{A}}$ is determined by a Borel almost disjoint family $\mathcal{A} \subset 2^D$, cf. Sect. 2.2, and let

$$\text{JL}_D(K) = \{f|D : f \in \text{JL}(K)\}. \quad (9.1)$$

We shall show that the set

$$\text{JL}_D(K) \text{ is Borel in } \mathbb{R}^D. \quad (9.2)$$

To that end, let us consider

$$C = \{f|D : f \in C(K), f(\infty) = 0\} \subset C_D(K). \quad (9.3)$$

By Theorem 1.1,

$$C \text{ is Borel in } \mathbb{R}^D. \quad (9.4)$$

Let us fix $D = D_0 \supset D_1 \supset D_2 \cdots$ such that $D \setminus D_i$ is finite, $\cap_i D_i = \emptyset$, and let us consider, for any triple m, n, p of natural numbers, Borel sets

$$M_{m,n,p} = \bigcup_i \bigcap_{(d_1, \dots, d_p) \in D_i^p} \left\{ (u, A_1, \dots, A_p) \in C \times \mathcal{A}^p : (\exists j \, d_j \notin A_j) \text{ or } \left(\sum_{j \leq p} (u(d_j))^2 \geq n \text{ and } |u(d_j)| \geq \frac{1}{m} \text{ for } j = 1, \dots, p \right) \right\}. \quad (9.5)$$

We shall check that also each set

$$\text{proj}_C M_{m,n,p} \text{ is Borel in } \mathbb{R}^D, \quad (9.6)$$

proj_C being the projection onto the first coordinate.

This will follow from the Lusin theorem and (9.5), once we make sure that the restriction

$$\text{proj}_C|_{M_{m,n,p}} \text{ is countable-to-one.} \quad (9.7)$$

Let $u \in C$ and $u = f|D$, $f \in C(K)$. Since f vanishes at ∞ , cf. Eq. (9.3), the set $\mathcal{A}(f) = \{A \in \mathcal{A} : f(p_A) \neq 0\}$ is countable. Now, if $(u, A_1, \dots, A_p) \in M_{m,n,p}$, for any $j \leq p$ and all but finitely many $d \in A_j$, $|u(d)| \geq 1/m$ and in effect, $A_j \in \mathcal{A}(f)$. This yields (9.7).

Since, cf. Eqs. (9.1) and (9.3),

$$\mathbf{JL}_D(K) = C \setminus \bigcap_n \bigcup_{m,p} \text{proj}_C M_{m,n,p}, \quad (9.8)$$

we get (9.2) from (9.4) and (9.6).

Let us consider, for each $d \in D$, the evaluation functional $e_d \in \mathbf{JL}(K)^*$,

$$e_d(f) = f(d), \quad f \in \mathbf{JL}(K), \quad (9.9)$$

and let $\text{Cyl}_D(\mathbf{JL}(K))$ be the σ -algebra in $\mathbf{JL}(K)$ generated by the functionals e_d . The restriction map $f \mapsto f|D$ is an isomorphism between the measurable space $(\mathbf{JL}(K), \text{Cyl}_D(\mathbf{JL}(K)))$ and the space $\mathbf{JL}_D(K)$ equipped with the σ -algebra of Borel sets, and therefore, by Eq. (9.2),

$$(\mathbf{JL}(K), \text{Cyl}_D(\mathbf{JL}(K))) \text{ is standard.} \quad (9.10)$$

It is enough to make sure that

$$\text{Cyl}_D(\mathbf{JL}(K)) = \text{Cyl}(\mathbf{JL}(K)). \quad (9.11)$$

For any $A \in \mathcal{A}$, the evaluation functional $e_A(f) = f(p_A)$, $f \in \mathbf{JL}(K)$, is the pointwise limit $e_A = \lim_{d \in A} e_d$, cf. Eq. (9.8) and hence e_A is $\text{Cyl}_D(\mathbf{JL}(K))$ -measurable. Since any functional $\varphi \in \mathbf{JL}(K)^*$ can be represented as $\varphi = \sum_{d \in D} x(d)e_d + \sum_{A \in \mathcal{A}} y(A)e_A$, where $x : D \rightarrow \mathbb{R}$ is summable and $y : \mathcal{A} \rightarrow \mathbb{R}$ is square-summable, cf. [17], the functional φ is $\text{Cyl}_D(\mathbf{JL}(K))$ -measurable. This demonstrates (9.11) and ends the proof. $\square$

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