Finite groups with some NR -subgroups or $\mathcal{H}$ -subgroups

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Abstract Berkovich investigated the following concept: a subgroup H of a finite group G is called an NR -subgroup (Normal Restriction) if whenever $K \trianglelefteq H$, then $K^G \cap H = K$, where K^G is the normal closure of K in G . Bianchi, Gillio Berta Mauri, Herzog and Verardi proved a characterization of soluble T -groups by means of $\mathcal{H}$ -subgroups: a subgroup H of G is said to be an $\mathcal{H}$ -subgroup of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$. In this article we give new characterizations of finite soluble PST -groups in terms of NR -subgroups or $\mathcal{H}$ -subgroups. We will show that they are different from the ones given by Ballester-Bolínches, Esteban-Romero and Pedraza-Aguilera. Robinson established the structure of minimal non- PST -groups. We give the classification of groups all of whose second maximal subgroups (of even order) are soluble PST -groups.

Keywords NR -subgroups · $\mathcal{H}$ -subgroups · T -groups · PT -groups · PST -groups

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1 Introduction and notation

All groups considered in this paper are finite. We use conventional notions and notations, as in [4, 10, 18]. Throughout this article G stands for a finite group and $\pi(G)$ denotes the set of primes dividing $|G|$. A subgroup H of G is said to be *permutable*

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in G if H permutes with every subgroup of G . A group G is said to be a PT -group (respectively, T -group) if permutability (respectively, normality) is a transitive relation in G . By a result of Ore [4] PT -groups are exactly those groups where all subnormal subgroups are permutable. A subgroup of G is called s -permutable in G if it permutes with all Sylow subgroups of G . A group G is said to be a PST -group if s -permutability is a transitive relation in G . By a result of Kegel ([4], Theorem 1.2.14(3)) PST -groups are exactly those groups where all subnormal subgroups are s -permutable. In the literature there are several characterizations of finite soluble T -groups, PT -groups and PST -groups (see [3–6, 8]).

A subgroup H of G is called a CR -subgroup (Character Restriction) of G if every complex irreducible character of H is a restriction of some irreducible character of G (see [12]). It is well known that if H is a CR -subgroup of G and $K \trianglelefteq H$, then $K^G \cap H = K$. In [7] Berkovich introduced an interesting subgroup embedding property:

Definition 1.1 Let G be a group, $H \leq G$. A triple (G, H, K) is said to be *special* in G , if $K \trianglelefteq H \leq G$ and $H \cap K^G = K$. A subgroup H is said to be an NR -subgroup of G (Normal Restriction) if, whenever K is normal in H , the triple (G, H, K) is special in G .

In [19] Tong-Viet showed that, if every maximal subgroup of G is an NR -subgroup of G , then G is soluble. In [7] Berkovich proved that, if all Sylow subgroups of a group G are NR -subgroups, then G is supersoluble.

For groups $H \leq T \leq G$ we say that H is *strongly closed* in T with respect to G if $H^g \cap T \leq H$ for all $g \in G$. Bianchi, Gillio Berta Mauri, Herzog and Verardi ([8], Theorem 10) proved a characterization of soluble T -groups by means of $\mathcal{H}$ -subgroups: a subgroup H of G is said to be an $\mathcal{H}$ -subgroup of G if $N_G(H) \cap H^g \leq H$ for all $g \in G$. By [9] if H is a p -subgroup of G , then H is an $\mathcal{H}$ -subgroup of G if and only if H is strongly closed in P with respect to G for some Sylow p -subgroup P of G containing H . In [2] Asaad showed that if every maximal subgroup of every Sylow subgroup of G is an $\mathcal{H}$ -subgroup of G , then G is supersoluble.

The basic structure of soluble T -, PT - and PST -groups was established by Gaschütz, Zacher, and Agrawal; this result shows that the classes of all soluble T -, PT -, and PST -groups are closed under taking subgroups and under taking epimorphic images. Since a non-abelian simple group is a T -group the first assertion is not true for the class of all T -groups (PT - and PST -groups). For some other characterizations of these groups see [3–6]. The structure of minimal non- T -groups, minimal non- PT -groups and minimal non- PST -groups was established by Robinson [16, 17]. By ([6], Corollary 5 and Theorem 6) and [17] groups considered in [1, 6, 14] can be used for characterizations of minimal non- PST -groups. We give new characterizations of soluble T -, PT -, PST -groups and minimal non- PST -groups in terms of NR -subgroups or $\mathcal{H}$ -subgroups. We will show the differences between these characterizations and the ones given in [3–6].

Recall that a subgroup H of a group G is called a *second maximal subgroup* or a *2-maximal subgroup* of G , if H is a maximal subgroup of some maximal subgroup of G . Second maximal subgroups were introduced by Huppert in [11] in which it was proved that a group is supersoluble if every its 2-maximal subgroup is normal. We give

the classification of groups all of whose second maximal subgroups (of even order) are soluble **PST**-groups.

2 Preliminaries

Theorem 2.1 [4] *Let L be the nilpotent residual of a group G . Then the following assertions hold.*

1. (Agrawal) G is a soluble **PST**-group if and only if L is an abelian Hall subgroup of odd order of G in which G acts by conjugation as a group of power automorphisms.
2. (Zacher) G is a soluble **PT**-group if and only if G is a soluble **PST**-group with Iwasawa Sylow subgroups.
3. (Gaschütz) G is a soluble **T**-group if and only if G is a soluble **PST**-group with Dedekind Sylow subgroups.

Lemma 2.2 *Let p be a prime. Let N be a normal p' -subgroup of a group G and P be any p -subgroup of G . Then P is an NR -subgroup of G if and only if PN/N is an NR -subgroup of G/N .*

Proof Assume first that P is an NR -subgroup of G . Let M/N be a normal subgroup of PN/N . Then there exists a normal subgroup L of P such that $M = LN$. Since $M^G \cap PN = L^G N \cap PN = (L^G N \cap P)N = (L^G \cap P)N = LN = M$, we have that $(M/N)^{G/N} \cap (PN/N) = (L^G N \cap PN)/N = M/N$. Therefore PN/N is an NR -subgroup of G/N .

Conversely, assume that PN/N is an NR -subgroup of G/N . Let L be a normal subgroup of P . Then $LN/N \trianglelefteq PN/N$ and $(LN/N)^{G/N} \cap (PN/N) = LN/N$. Therefore $LN = (L^G N) \cap (PN) = (L^G N \cap P)N = (L^G \cap P)N$. Hence $L = L^G \cap P$. It follows that P is an NR -subgroup of G . This ends the proof.

Lemma 2.3 *If G is a q -nilpotent group for a prime q and Q is a Sylow q -subgroup of G , then Q is an NR -subgroup of G .*

Proof By hypothesis $G = O_{q'}(G)Q$. Let $H \trianglelefteq Q$. Then $O_{q'}(G)H \trianglelefteq G$ and, consequently, $H^G \cap Q \leq O_{q'}(G)H \cap Q = H$. This ends the proof.

Lemma 2.4 [7] *If all Sylow subgroups of G are NR -subgroups of G , then G is supersoluble.*

Lemma 2.5 [8] *A subgroup H of a group G is normal in G if and only if H is subnormal in G and is an $\mathcal{H}$ -subgroup of G .*

Lemma 2.6 *Let N be a normal subgroup of a group G . Assume that p is a prime dividing the order of G and P is a Sylow p -subgroup of N . If H is a subgroup of N such that (G, P, H) is special in G , then H is an $\mathcal{H}$ -subgroup of G .*

Proof Let S be a Sylow p -subgroup of G containing P . Then $H^S \cap S = H^S \cap N \cap S = H^S \cap P \leq H^G \cap P = H$ for every $g \in G$. This ends the proof.

Lemma 2.7 [2] *If every maximal subgroup of every Sylow subgroup of G is an $\mathcal{H}$ -subgroup of G , then G is supersoluble.*

By Lemma 2.6 it follows that Lemma 2.7 implies Lemma 2.4.

Lemma 2.8 ([13], Theorem B) *Let G be a non-soluble group. Assume that soluble subgroups of G are either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:*

1. $PSL(2, 2^f)$, where f is a positive integer such that $2^f - 1$ is a prime;
2. $PSL(2, q)$, where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod{8}$;
3. $SL(2, q)$, where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod{8}$.

3 Soluble T -, PT - and PST -groups

Definition 3.1 A group G is called an NR -group if every Sylow subgroup of G is an NR -subgroup of G . A group G is called an $\mathcal{H}$ -group if every maximal subgroup of every Sylow subgroup of G is an $\mathcal{H}$ -subgroup of G .

By Lemma 2.4 every NR -group is supersoluble. By Lemma 2.7 every $\mathcal{H}$ -group is supersoluble. The following example shows that there exists a supersoluble group which is neither an NR -group nor an $\mathcal{H}$ -group.

Example 3.1 Let p be an odd prime, $P = \langle a \rangle \times \langle b \rangle$ be an elementary abelian group of order p^2 . Let x be the automorphism of P of order 2 given by $a^x = a$, $b^x = b^{-1}$. Let $G = P \rtimes \langle x \rangle$ be the corresponding semidirect product. Then G is supersoluble. But G is neither an NR -group nor an $\mathcal{H}$ -group, since $\langle ab \rangle^G \cap P = P$ and $(\langle ab \rangle)^x \cap N_G(\langle ab \rangle) = \langle ab^{-1} \rangle$.

By Lemma 2.6 every NR -group is an $\mathcal{H}$ -group. The following example shows that the converse does not hold.

Example 3.2 Let p be an odd prime and let $A = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c \rangle$ be an extraspecial group of order p^3 and exponent p . Let $B = \langle x \rangle$ be a cyclic group of order p and $P = A \times B$. Let y be the automorphism of P of order 2 given by $a^y = a^{-1}$, $b^y = b^{-1}$, $x^y = x^{-1}$. Let $G = P \rtimes \langle y \rangle$ be the corresponding semidirect product. Then every maximal subgroup of P is normal in G , so is an $\mathcal{H}$ -subgroup of G . But $H = \langle xc \rangle$ is normal in P , $H^G = \langle x, c \rangle$, $H^G \cap P = H^G$, so G is not an NR -group.

The following example shows that the class of all NR -groups (the class of all $\mathcal{H}$ -groups) is not closed under taking subgroups.

Example 3.3 Let p be an odd prime and let $P = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c \rangle$ be an extraspecial group of order p^3 and exponent p . Let x be the automorphism of P of order 2 given by $a^x = a^{-1}$, $b^x = b^{-1}$. Let $G = P \rtimes \langle x \rangle$ be the corresponding semidirect product. It is easily seen that G is an NR -group and G is an $\mathcal{H}$ -group. Let $H = \langle b, c, x \rangle$. Then $\langle bc \rangle \triangleleft \langle b, c \rangle$, but $(\langle bc \rangle)^H \cap \langle b, c \rangle = \langle b, c \rangle$, so H is not an NR -group. Furthermore since $\langle b, c \rangle$ is a Sylow p -subgroup of H and $\langle bc \rangle^x \cap \langle b, c \rangle = \langle b^{-1}c \rangle$ we have that H is not an $\mathcal{H}$ -group.

Example 3.3 also shows that there exist NR -groups ($\mathcal{H}$ -groups) which are not soluble PST -groups.

Following [6, 20] we say that G is a T_0 -group (respectively, a PT_0 -group, a PST_0 -group) if $G/\Phi(G)$ is a T -group (respectively, a PT -group, a PST -group). The following result was proved in [6].

Theorem 3.1 *Let G be a group. The following statements are equivalent:*

1. G is a soluble PST -group;
2. every subgroup of G is a T_0 -group;
3. every subgroup of G is a PT_0 -group;
4. every subgroup of G is a PST_0 -group.

The following examples show the differences between the classes of T_0 -, PT_0 -, PST_0 -groups and the classes of NR -, $\mathcal{H}$ -groups.

Example 3.4 Every non-abelian simple group is a T -group, a PT -group and a PST -group. Clearly by Lemmas 2.4 and 2.7 it is neither an NR -group nor an $\mathcal{H}$ -group, since it is not supersoluble.

Example 3.5 The group from Example 3.2 is a T_0 -group, a PT_0 -group, and PST_0 -group. It is supersoluble, but it is not an NR -group.

We do not know if there exists a soluble T_0 -group (a PT_0 -group, a PST_0 -group) that is not an $\mathcal{H}$ -group. Now we use NR -groups and $\mathcal{H}$ -groups to characterize soluble PST -groups.

Theorem 3.2 *Let G be a group. The following conditions are equivalent:*

1. G is a soluble PST -group;
2. G and its subgroups are NR -groups;
3. G and its subgroups are $\mathcal{H}$ -groups.

Proof (1) $\Rightarrow$ (2) We claim that every soluble PST -group is an NR -group. Assume that G is a soluble PST -group. Let L be the nilpotent residual of G . By Theorem 2.1 L is an abelian Hall subgroup of odd order of G in which G acts by conjugation as a group of power automorphisms. Let $p \in \pi(L)$ and $G_p \in \text{Syl}_p(G)$. Since every subgroup of G_p is normal in G , it follows that G_p is an NR -subgroup of G . Let $q \in \pi(G) \setminus \pi(L)$, $G_q \in \text{Syl}_q(G)$. Since G is q -nilpotent, G_q is an NR -subgroup of G by Lemma 2.3. Since the class of all soluble PST -groups is closed under taking subgroups, it follows that G and its subgroups are NR -groups.

(2) $\Rightarrow$ (3) This allows by Lemma 2.6.

(3) $\Rightarrow$ (1) Assume that G is a group of minimal order satisfying that G and its subgroups are $\mathcal{H}$ -groups but G is not a PST -group. Then G is a minimal non- PST -group and so $G = P \rtimes Q$, where $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$, $\pi(G) = \{p, q\}$ and Q is cyclic.

We claim that every subgroup H of P is normalized by G_q for any $G_q \in \text{Syl}_q(G)$. Let $H = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = P$ be a series between H and

P such that $|H_i : H_{i+1}| = p$ for $i = 0, \dots, n-1$. Applying induction on n we may assume that $n > 0$. Since $G_q \leq N_G(H_{n-1})$ and so $G_q H_{n-1}$ is an $\mathcal{H}$ -group we have $G_q \leq N_G(H_n)$ by Lemma 2.5.

Let G_q be any Sylow q -subgroup of G and $G_q = \langle x \rangle$. Since $G_q \leq N_G(H)$ for all $H \leq P$, it follows that x induces the power automorphism on P . By ([4], Lemma 1.3.4) P is abelian, since x induces a nontrivial automorphism on P and has order prime to p , so by ([15], 13.4.3) the automorphism x is fixed-point-free. Hence P is not a 2-group. Therefore by Theorem 2.1 (1) G is a soluble **PST**-group, a contradiction. This ends the proof.

By Theorems 3.2 and 2.1 we obtain the following corollaries.

Theorem 3.3 *Let G be a group. The following conditions are equivalent:*

1. G is a soluble **PT**-group;
2. G and its subgroups are **NR**-groups with Iwasawa Sylow subgroups;
3. G and its subgroups are $\mathcal{H}$ -groups with Iwasawa Sylow subgroups.

Theorem 3.4 *Let G be a group. The following conditions are equivalent:*

1. G is a soluble **T**-group;
2. G and its subgroups are **NR**-groups with Dedekind Sylow subgroups;
3. G and its subgroups are $\mathcal{H}$ -groups with Dedekind Sylow subgroups.

Corollary 3.5 *Let G be a group. The following conditions are equivalent:*

1. G is a minimal non-**PST**-group;
2. G is a minimal non-**NR**-group;
3. G is a minimal non- $\mathcal{H}$ -group.

The above characterizations of **PST**-groups are also different from the ones from ([4], Theorem 2.3.8) or ([5], Theorem 13) since for every group any Sylow subgroup is an **NR**-group but the groups from Examples 3.4, 3.5 are not soluble **PST**-groups. The following example shows that there exists a group which is not a soluble **PST**-group but every subnormal subgroup of it is an **NR**-group (see for example Corollary 2 from [3]).

Example 3.6 Let G be the group from Example 3.3. Then G and its p -subgroups are all subnormal subgroups of G . Clearly they are **NR**-groups but G is not a soluble **PST**-group.

4 Non-abelian groups all of whose second maximal subgroups (of even order) are soluble **PST** -groups

Theorem 4.1 *If all proper subgroups of even order of a group G are **NR**-groups, then G is either 2-nilpotent or minimal non-nilpotent. In particular, G is soluble.*

Proof Let H be a proper subgroup of G . Then either H has an odd order or H is supersoluble as **NR**-group. In any case H is 2-nilpotent. Then G is either 2-nilpotent or minimal non-2-nilpotent (in fact, non-nilpotent by ([10], IV.5.4)). In particular by ([10], IV.5.4), the definition of 2-nilpotent groups and Feit–Thompson Theorem on solubility of groups of odd order it follows that G is soluble. This ends the proof.

Theorem 4.2 *Let G be a group of even order. Assume that every proper subgroup of even order of G is an NR -group. Then either G is an NR -group or $|\pi(G)| \leq 3$.*

Proof Assume that $|\pi(G)| > 3$. By Theorem 4.1 G is soluble and we can consider $\{G_r \mid r \in \pi(G)\}$, a Sylow basis of G , where $G_r \in \text{Syl}_r(G)$ for each $r \in \pi(G)$. Let q be the largest prime in $\pi(G)$. For every $p \in \pi(G)$, $p \neq 2, q$, by Lemma 2.4 we have that G_q is a normal NR -subgroup of the NR -group $G_q G_p G_2$. Then every normal subgroup of G_q is normal in $G_q G_p G_2$ for every $p \neq 2, q$, and so it is normal in G . This implies that G_q is an NR -subgroup of G . Moreover $G/G_q \cong G_{q'} < G$, where $G_{q'}$ is a Hall q' -subgroup of G . Hence $G_{q'}$ is an NR -group and so G/G_q is an NR -group. Therefore G is also an NR -group by Lemma 2.2, since G_q is a normal NR -subgroup of G . This ends the proof.

In the next results we will use Dickson's Theorem ([18], 3.6.25–3.6.26) and some other observations on groups $PSL(2, q)$ and $SL(2, q)$ ([18], §1.9 and §3.6).

Lemma 4.3 *Assume that G is one of the following groups:*

1. $PSL(2, p)$, where p is a prime such that $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
2. $PSL(2, 2^p)$, where p is a prime such that $2^p - 1$ is a prime;
3. $PSL(2, 3^p)$, where p is an odd prime, $3^p \equiv 3 \pmod{8}$ and $(3^p - 1)/2$ is a prime;
4. $SL(2, p)$, where p is a prime such that $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$.

Then every second maximal subgroup of G is a soluble T -group (in particular it is a soluble PST -group).

Proof (1)–(3) By Dickson's Theorem a maximal subgroup of G is one of the following groups:

- (a) a dihedral group of order $2(r \pm 1)/d$, where $d = (r - 1, 2)$ and $r = p$ or 2^p or 3^p .
- (b) a Frobenius group N with elementary abelian kernel of order r and a cyclic complement D of order $(r - 1)/d$, where $d = (r - 1, 2)$ and $r = p$ or 2^p or 3^p ; for the structure of N see ([18], p. 393) (in fact, in this notation $N \cong H/Z(L)$).
- (c) A_4 .

If M is a group of type (a), then either $M \cong \langle a, b \mid a^2 = b^n = 1, b^a = b^{-1} \rangle$ or $M \cong \langle a, b \mid a^2 = b^{2n} = 1, b^a = b^{-1} \rangle$, where n is an odd number. By Theorem 2.1 M is a soluble T -group. If r is a prime, then N of type (b) is a T -group. Assume that $r = 2^p$ or 3^p . From the structure of N of type (b) it is a minimal non-abelian group. Clearly, A_4 is also a minimal non-abelian group. This ends the proof.

(4) Clearly, G contains a unique element of order 2, $|Z(G)| = 2$ and $G/Z(G) \cong PSL(2, p)$. Hence $\Phi(G) = Z(G)$ and $Z(G) < M$ for every maximal subgroup M of G (see also ([18], Corollary, p. 80)). Therefore by ([18], 3.6.17, 3.6.25–3.6.26) a maximal subgroup M of G is one of the following groups:

- (a) $Q_8 \rtimes \langle x \rangle$ where x is an element of order 3, which acts on Q_8 permuting the three maximal subgroups of Q_8 ;

- (b) $Q \triangleleft M$, $Q \in \text{Syl}_p(G)$, $|Q| = p$ and M/Q is a cyclic group whose order is relatively prime to p ;
 (c) $M = \langle x, y \mid x^n = y^2, y^{-1}xy = x^{-1} \rangle$ and $M/Z(G)$ is a dihedral group of order $p \pm 1$.

The group of type (a) is a minimal non- T -group (see also [16]), so every maximal subgroup of it is a soluble T -group. Clearly the groups of types (b)–(c) are soluble T -groups (the Sylow 2-subgroups of the group of type (c) are either cyclic of order 4 or quaternion groups of order 8). This ends the proof.

Lemma 4.4 *Assume that G is one of the following groups:*

1. $\text{PSL}(2, 3^f)$, where f is an odd prime and $3^f \equiv 3 \pmod{8}$;
2. $\text{SL}(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime.

*Then every second maximal subgroup of G of even order is a soluble T -group (in particular it is a soluble **PST**-group).*

Proof 1. By Dickson's Theorem a maximal subgroup of G is one of the following groups:

- (a) a dihedral group of order $3^f \pm 1$;
- (b) a Frobenius group N with elementary abelian kernel of order 3^f and a cyclic complement D of order $(3^f - 1)/2$; for the structure of N see ([18], p. 393) (in fact, in this notation $N \cong H/Z(L)$);
- (c) A_4 .

As in the Proof of Lemma 4.3 we get that the groups of type (a) are soluble T -groups, the groups of type (b) are of odd order and A_4 is a minimal non-abelian group. Therefore G is a group all of whose second maximal subgroups of even order are soluble T -groups. This ends the proof.

2. Clearly, G contains a unique element of order 2, $|Z(G)| = 2$ and $G/Z(G) \cong \text{PSL}(2, 3^f)$. Hence $\Phi(G) = Z(G)$ and $Z(G) < M$ for every maximal subgroup M of G (see also ([18], Corollary, p. 80)). Therefore by ([18], 3.6.17, 3.6.25, 3.6.26) a maximal subgroup M of G is one of the following groups:

- (a) $Q_8 \rtimes \langle x \rangle$ where x is an element of order 3, which acts on Q_8 permuting the three maximal subgroups of Q_8 ;
- (b) $M = \langle x, y \mid x^n = y^2, y^{-1}xy = x^{-1} \rangle$ and $M/Z(G)$ is a dihedral group of order $3^f \pm 1$;
- (c) $N \times Z(G)$, where N is a group of type (b) from the proof of (1).

The group of type (a) is a minimal non- T -group (so every maximal subgroup of it is a soluble T -group), the group of type (b) is a soluble T -group (the Sylow 2-subgroups of it are either cyclic of order 4 or quaternion groups of order 8). Let M be the group of type (c) and let L be a maximal subgroup of M such that $Z(G) < L$. From the structure of N it is easily seen that L is abelian. This ends the proof.

Theorem 4.5 *Let G be a non-abelian simple group all of whose second maximal subgroups of even order are soluble **PST**-groups. Then G is one of the following groups:*

1. $\text{PSL}(2, 2^f)$, where f is a prime such that $2^f - 1$ is a prime;

2. $PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $PSL(2, 3^f)$, where f is an odd prime and $3^f \equiv 3 \pmod{8}$.

Proof By Lemmas 4.3–4.4 every group of type (1)–(3) is a group all of whose second maximal subgroups of even order are soluble **PST**-groups.

We will show that there are no other groups satisfying these conditions. Let G be a non-abelian simple group all of whose second maximal subgroups of even order are soluble **PST**-groups. Let M be an arbitrary maximal subgroup of G . Since soluble **PST**-groups are NR -groups, by Theorem 4.1 M is either 2-nilpotent or minimal non-nilpotent, in particular M is soluble by Theorem 4.1 and Feit–Thompson Theorem. Then G is a minimal simple group (see Thompson’s classification of minimal simple groups in ([10], Bemerkung II.7.5, p. 190)) and also G is one of the simple groups in Lemma 2.8. Since, if f is odd, then $3^f \equiv 3 \pmod{8}$, this ends the proof.

Theorem 4.6 *Let G be a non-abelian simple group all of whose second maximal subgroups are soluble **PST**-groups. Then G is one of the following groups:*

1. $PSL(2, 2^f)$, where f is a prime such that $2^f - 1$ is a prime;
2. $PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $p \equiv 5 \pmod{8}$;
3. $PSL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime.

Proof By Lemma 4.3 every group of type (1)–(3) is a group all of whose second maximal subgroups are soluble **PST**-groups. Let G be a group all of whose second maximal subgroups are soluble **PST**-groups. By Theorem 4.5 we should only show that $G \not\cong PSL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is composite. If not, then G possesses a Frobenius group N with kernel P of order 3^f and a cyclic complement D of order $(3^f - 1)/2$. For the structure of N see ([18], p. 393) (in fact, in this notation $N \cong H/Z(L)$). Moreover P is an elementary abelian 3-group. Since $(3^f - 1)/2$ is composite, it follows that N possesses a proper subgroup $\langle x \rangle P$, where $\langle x \rangle$ is a proper subgroup of D of prime order. By hypothesis $\langle x \rangle P$ is a soluble **PST**-group. Then x acts on P as a power automorphism. Let $y \in P$. Then $\langle y \rangle \triangleleft \langle x \rangle P$. Hence $\langle x \rangle \langle y \rangle = \langle x \rangle \times \langle y \rangle$ and so $x \in C_N(y)$. Since $C_N(y) \leq P$ by ([18], 3.6.4(i)), we get a contradiction. This ends the proof.

Theorem 4.7 *Let G be a group all of whose second maximal subgroups of even order are soluble **PST**-groups. Then G is either a soluble group or one of the following groups:*

1. $PSL(2, 2^f)$, where f is a prime such that $2^f - 1$ is a prime;
2. $PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $PSL(2, 3^f)$, where f is an odd prime and $3^f \equiv 3 \pmod{8}$;
4. $SL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime;
5. $SL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;

Proof By Lemmas 4.3, 4.4 every group of type (1)–(5) is a group all of whose second maximal subgroups of even order are soluble **PST**-groups. We will show that there are no other groups satisfying these conditions. Let G be a group all of whose second maximal subgroups of even order are soluble **PST**-groups. Assume that G is a non-soluble group. As in the Proof of Theorem 4.5 maximal subgroups of G are either 2-nilpotent or minimal non-nilpotent. Therefore G is one of the groups from Theorem 2.8.

We make the following claims:

- (i) $G \not\cong PSL(2, p^f)$, where $p > 3$, $f > 1$ and $p^f \equiv 3$ or $5 \pmod{8}$.
If not, by Dickson's Theorem $PSL(2, p^f)$ contains a non-soluble proper subgroup $PSL(2, p)$, a contradiction.
- (ii) $G \not\cong PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$.
If not, by Dickson's Theorem G contains a non-soluble proper subgroup isomorphic to A_5 , a contradiction.
- (iii) $G \not\cong PSL(2, 3^f)$, where f is even or composite and $3^f \equiv 3$ or $5 \pmod{8}$.
If not, since $3^2 \equiv 1 \pmod{8}$, f is odd and $3^f \equiv 3 \pmod{8}$. Assume that f is composite and let k be a prime dividing f . By Dickson's Theorem $PSL(2, 3^f)$ contains a non-soluble proper subgroup $PSL(2, 3^k)$, a contradiction.
- (iv) $G \not\cong PSL(2, 3^f)$, where f is an odd prime, $3^f \equiv 5 \pmod{8}$.
Since f is odd, it follows that $3^f \equiv 3 \pmod{8}$.
- (v) $G \not\cong SL(2, p^f)$, where $p > 3$, $f > 1$ and $p^f \equiv 3$ or $5 \pmod{8}$.
If not, we have $G/Z(G) \cong PSL(2, p^f)$ and $PSL(2, p^f)$ contains a non-soluble proper subgroup $PSL(2, p)$, a contradiction.
- (vi) $G \not\cong SL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$.
If not, we have $G/Z(G) \cong PSL(2, p)$. The rest of the proof is similar to that of (ii) and (v).
- (vii) $G \not\cong SL(2, 3^f)$, where f is even or composite and $3^f \equiv 3$ or $5 \pmod{8}$.
If not, we have $G/Z(G) \cong PSL(2, 3^f)$. The rest of the proof is similar to that of (iii) and (v).
- (viii) $G \not\cong SL(2, 3^f)$, where f is an odd prime, $3^f \equiv 5 \pmod{8}$.
Since f is odd, it follows that $3^f \equiv 3 \pmod{8}$.
- (ix) $G \not\cong SL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is composite.
If not, by ([18], p. 393) G contains a proper subgroup L such that $L = H \times Z(G)$, where H is a Frobenius group with kernel P of order 3^f and a cyclic complement D of order $(3^f - 1)/2$. Moreover P is an elementary abelian 3-group. Since $(3^f - 1)/2$ is not a prime, it follows that H possesses a proper subgroup $\langle x \rangle P$, where $\langle x \rangle$ is a proper subgroup of D of prime order. Since G contains a unique element of order 2, $|Z(G)| = 2$ and $G/Z(G) \cong PSL(2, 3^f)$, we have that $\Phi(G) = Z(G)$ and $Z(G) < M$ for every maximal subgroup M of G (see also ([18], Corollary, p. 80)). Therefore $\langle x \rangle P \times Z(G)$ is contained in a second maximal subgroup of even order of G . But $\langle x \rangle P$ is not a **PST**-group (we can proceed as in the Proof of Theorem 4.6), a contradiction.

- (x) $G \not\cong PSL(2, 2^f)$, where f is composite and $2^f - 1$ is a prime.

If not, let p be a prime dividing f . By Dickson's Theorem $PSL(2, 2^f)$ contains a non-soluble proper subgroup $PSL(2, 2^p)$, a contradiction.

From (i)–(x) we obtain the groups of types (1)–(5). This ends the proof.

Theorem 4.8 *Let G be a group all of whose second maximal subgroups are soluble PST -groups. Then G is either a soluble group or one of the following groups:*

1. $PSL(2, 2^f)$, where f is a prime such that $2^f - 1$ is a prime;
2. $PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $PSL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime;
4. $SL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$.

Proof By Lemma 4.3 every group of type (1)–(4) is a group all of whose second maximal subgroups are soluble PST -groups. By Theorem 4.7 we should only consider the following groups:

- (i) $G \cong PSL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is composite;
- (ii) $G \cong SL(2, 3^f)$, where f is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime.

In the first case we proceed as in the Proof of Theorem 4.6.

In the second case, by ([18], p. 383) G contains a proper subgroup L such that $L = H \times Z(G)$, where H is a Frobenius group with kernel P of order 3^f and a cyclic complement D of order $(3^f - 1)/2$. Moreover P is an elementary abelian 3-group. Since G contains a unique element of order 2, $|Z(G)| = 2$ and $G/Z(G) \cong PSL(2, 3^f)$, it follows that $\Phi(G) = Z(G)$ and $Z(G) < M$ for every maximal subgroup M of G (see also ([18], Corollary, p. 80)). Therefore H is contained in the second maximal subgroup of G . From the structure of H it is easily seen that H is not a soluble PST -group (we can proceed as in the Proof of Theorem 4.6), a contradiction. This ends the proof.

By the Proofs of Theorems 4.5–4.8 it is easy to see that we obtain the same results considering groups of all whose second maximal subgroups (of even order) are soluble T -groups or PT -groups.

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References

1. Asaad, M., Heliel, A.A.: Finite groups in which normality is a transitive relation. Arch. Math. (Basel) **76**(5), 321–325 (2001)
2. Asaad, M.: On p -nilpotence and supersolvability of finite groups. Comm. Algebra **34**, 4217–4224 (2006)

3. Ballester-Bolinches, A., Esteban-Romero, R.: Sylow permutable subnormal subgroups of finite groups. II. *Bull. Austral. Math. Soc.* **64**(3), 479–486 (2001)
4. Ballester-Bolinches, A., Esteban-Romero, R., Asaad, M.: *Products of finite groups*. Walter de Gruyter GmbH & Co, KG, Berlin (2010)
5. Ballester-Bolinches, A., Esteban-Romero, R., Li, Y.: On self-normalizing subgroups of finite groups. *J. Group Theory* **13**(1), 143–149 (2010)
6. Ballester-Bolinches, A., Esteban-Romero, R., Pedraza-Aguilera, M.C.: On a class of p -soluble groups. *Algebra Colloq.* **12**(2), 263–267 (2005)
7. Berkovich, Y.: Subgroups with the character restriction property and related topics. *Houston J. Math.* **24**, 631–638 (1998)
8. Bianchi, M., Gillio Berta Mauri, A., Herzog, M., Verardi, L.: On finite solvable groups in which normality is a transitive relation. *J. Group Theory* **3**, 147–156 (2000)
9. Goldschmidt, D.: Strongly closed 2-subgroups of finite groups. *Ann. Math.* **102**(2), 475–489 (1975)
10. Huppert, B.: *Endliche Gruppen I*. Springer, Berlin (1967)
11. Huppert, B.: Normalteiler und maximal Untergruppen endlicher gruppen. *Math. Z.* **60**, 409–434 (1954)
12. Isaacs, I.M.: Subgroups with the character restriction property. *J. Algebra* **100**, 403–420 (1986)
13. Li, S., Zhao, Y.: Some finite nonsolvable groups characterized by their solvable subgroups. *Acta Math. Sinica (N.S.)* **4**, 5–13 (1988)
14. Ramadan, M.: Finite groups in which permutability is a transitive relation on their Frattini factor groups. *Acta Math. Hungar* **114**(3), 187–193 (2007)
15. Robinson, D.J.S.: *A course in the theory of groups*. Springer, New York (1996)
16. Robinson, D.J.S.: Groups which are minimal with respect to normality being transitive. *Pac. J. Math.* **31**, 777–789 (1969)
17. Robinson, D.J.S.: Minimality and Sylow-permutability in locally finite groups. *Ukrainian Math. J.* **54**(6), 1038–1049 (2002)
18. Suzuki, M.: *Group theory I*. Springer, Berlin (1982)
19. Tong-Viet, H.P.: Groups with normal restriction property. *Arch. Math. (Basel)* **93**, 199–203 (2009)
20. van der Waall, R.W., Fransman, A.: On products of groups for which normality is a transitive relation on their Frattini factor groups. *Quaestiones Math.* **19**(1–2), 59–82 (1996)