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Fermion Mass Mixing in Vacuum

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Abstract Renormalization group procedure for effective particles is applied to a theory of fermions that interact only through mass mixing terms in their Hamiltonian. Problems with virtual pair production in vacuum are avoided by using the front form of Hamiltonian dynamics. Masses and states of physical fermions emerge at the end of a calculation that is carried out exactly irrespective of the strength of the mass mixing terms. An a priori infinite set of renormalization group equations for all momentum modes of fermion quantum fields is reduced to just one equation for a two-by-two mass matrix. In distinction from scalars, fermions never become tachyons but appear chirally rotated when the mass mixing interaction term is sufficiently strong.

1 Introduction

Among the issues that need to be dealt with on the way to defining and solving relativistic quantum field theory beyond the perturbative pictures based on free propagation or weak binding of field quanta, one may distinguish the issues of vacuum and renormalization, both intricately related with the struggle to understand relativity and quantum mechanics simultaneously in one theory. Literature concerning these two issues is of enormous size and scope and this article is not to review it. Instead, this article offers a description of how the renormalization group procedure for effective particles (RGPEP) can be applied to an elementary quantum field theory for the relativistic fermions that interact through a mass mixing term of arbitrary strength [1]. The idea is to present how the RGPEP maintains symmetries of special relativity in an interacting quantum field theory and keeps the vacuum state simple.

The example is solved exactly. The vacuum issue is addressed by using the front form (FF) of dynamics instead of the familiar instant form (IF) [2]. Therefore, the vacuum divergences [3] due to fermion-anti-fermion pair creation do not occur. The remaining interactions are included in the RGPEP equations. The equations produce a relativistic spectrum of solutions in agreement with the general rules of quantum representation of symmetries of special relativity [4].

Since the calculation described here concerns basic issues and differs from the IF approach, a lot of basic details are included explicitly. First, the IF approach to the model example is reviewed, introducing useful notation in a familiar context and pointing out difficulties. Then the RGPEP approach is presented in a concise form that indicates new elements in comparison with the IF approach.

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2 Theory of Two Kinds of Free Fermion Fields

The standard form of Lagrangian density for two kinds of free fermion fields ψ and ϕ with mass parameters μ and ν , respectively, is

$$\mathcal{L} = \bar{\psi}(i\partial - \mu)\psi + \bar{\phi}(i\partial - \nu)\phi. \quad (1)$$

The condition of stationary action, $\delta S = 0$, where $S = \int d^4x \mathcal{L}$, implies equations of motion $(i\partial - \mu)\psi = 0$ and $(i\partial - \nu)\phi = 0$. The corresponding canonical Hamiltonian in the IF of dynamics is

$$H = \int d^3x T^{00}, \quad (2)$$

where T^{00} can be identified as the time-time component of the energy-momentum density tensor

$$T^{\rho\sigma} = \frac{\partial \mathcal{L}}{\partial \partial_\rho \psi_\alpha} \partial^\sigma \psi_\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\rho \phi_\alpha} \partial^\sigma \phi_\alpha - g^{\rho\sigma} \mathcal{L}. \quad (3)$$

The integral in Eq. (2) extends over the three-dimensional volume in space-time that is defined by the condition $t = 0$, where t is the time co-ordinate of the inertial observer who defines the theory. As a result of direct evaluation, the IF Hamiltonian for fermion fields according to this observer has the form

$$H = \int d^3x [\psi^\dagger(i\alpha\partial + \beta\mu)\psi + \phi^\dagger(i\alpha\partial + \beta\nu)\phi]. \quad (4)$$

The corresponding FF Hamiltonian is defined differently. This will be explained later.

3 IF Canonical Quantum Theory for Two Kinds of Free Fermions

In order to obtain a canonical quantum theory of fermions according to the IF rules [5,6] one writes the fields ψ and ϕ as functions of $\mathbf{x}$ at $t = 0$ in terms of their Fourier components,

$$\psi(\mathbf{x}) = \sum_{\mu ps}^f [u_{\mu ps} b_{\mu ps} e^{i\mathbf{p}\mathbf{x}} + v_{\mu ps} d_{\mu ps}^\dagger e^{-i\mathbf{p}\mathbf{x}}], \quad (5)$$

$$\phi(\mathbf{x}) = \sum_{\nu ps}^f [u_{\nu ps} b_{\nu ps} e^{i\mathbf{p}\mathbf{x}} + v_{\nu ps} d_{\nu ps}^\dagger e^{-i\mathbf{p}\mathbf{x}}], \quad (6)$$

where the integration over momentum $\mathbf{p}$ and sum over spins is denoted by the symbol

$$\sum_{\mu ps}^f = \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3 2E_{\mu p}} \quad (7)$$

and the energy is $E_{\mu p} = \sqrt{\mu^2 + \mathbf{p}^2}$. Similar definitions apply to the field ϕ with mass ν in place of μ . The coefficients u and v in the Fourier expansions are spinors of freely moving fermions.

It is important for the discussion that follows to realize that these spinors are obtained by boosting spinors [7] corresponding to fermions at rest, $u_{\mu 0s}$ and $v_{\mu 0s}$ normalized to $\sqrt{2\mu}$ and $\sqrt{2\nu}$, respectively. The explicit formulae are

$$u_{\mu ps} = B(\mu, \mathbf{p}) u_{\mu 0s} \quad \text{and} \quad v_{\mu ps} = B(\mu, \mathbf{p}) v_{\mu 0s}, \quad (8)$$

where the matrix B represents the boosts,

$$B(\mu, \mathbf{p}) = \frac{1}{\sqrt{2\mu(E_{\mu p} + \mu)}} (\not{p}\beta + \mu). \quad (9)$$

Similar definitions hold for fermions with mass ν . This observation is important because the boosting requires knowledge of the fermion mass and energy. The energy is treated in the boost matrices B as if the fermions were

free. Therefore, one may expect difficulties to occur when a theory includes the interactions that influence the value of mass and alter the expression for energy. The difficulties should be expected due to any form of energy that significantly differs from the free one. The usage of spinors of freely moving fermions for construction of strongly interacting quantum field operators is unlikely to be legitimate.

In the absence of interactions, having the Fourier expansions in place, one turns the fields ψ and ϕ into quantum field operators $\hat{\psi}$ and $\hat{\phi}$ by introducing the anti-commutation relations [8–10]

$$\{\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\} = \{\hat{\phi}(\mathbf{x}), \hat{\phi}^\dagger(\mathbf{x}')\} = \delta^3(\mathbf{x} - \mathbf{x}'). \quad (10)$$

These are satisfied as a result of imposing anti-commutation relations on the coefficients b and d in the Fourier expansions, turning them into creation and annihilation operators. Namely, one sets

$$\{b_{\mu ps}, b_{\mu p' s'}^\dagger\} = \{d_{\mu ps}, d_{\mu p' s'}^\dagger\} = 2E_{\mu p}(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'}, \quad (11)$$

for fermions of mass μ , and similar relations for fermions of mass ν . Normal-ordered quantum Hamiltonian,

$$\hat{H}_0 = \int d^3x : \left[\hat{\psi}^\dagger(i\boldsymbol{\alpha}\partial + \beta\mu)\hat{\psi} + \hat{\phi}^\dagger(i\boldsymbol{\alpha}\partial + \beta\nu)\hat{\phi} \right] : \quad (12)$$

is then obtained in the physically right form for free fermions, i.e.,

$$\hat{H}_0 = \sum_{\mu ps}^\dagger E_{\mu p} \left(b_{\mu ps}^\dagger b_{\mu ps} + d_{\mu ps}^\dagger d_{\mu ps} \right) + \sum_{\nu ps}^\dagger E_{\nu p} \left(b_{\nu ps}^\dagger b_{\nu ps} + d_{\nu ps}^\dagger d_{\nu ps} \right). \quad (13)$$

4 IF Canonical Quantum Theory for Fermions Interacting Through Mass Mixing

In the IF approach, one adds interaction terms to the free Lagrangian density. In the case of mass mixing terms, starting from

$$\mathcal{L} = \bar{\psi}(i\partial - \mu)\psi + \bar{\phi}(i\partial - \nu)\phi - m(\bar{\psi}\phi + \bar{\phi}\psi), \quad (14)$$

and following the same construction steps for quantum field operators as above, one obtains the corresponding canonical quantum Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{H}_I$, in which

$$\hat{H}_I = m \int d^3x : \left(\hat{\psi}^\dagger \gamma^0 \hat{\phi} + \hat{\phi}^\dagger \gamma^0 \hat{\psi} \right) :. \quad (15)$$

Having defined the quantum field operators as in the free theory, one arrives at

$$\begin{aligned} \hat{H}_I = m \sum_{\mu ps}^\dagger \sum_{s'} \frac{1}{2E_{\nu p}} & \left[\bar{u}_{\mu ps} u_{\nu ps'} b_{\mu ps}^\dagger b_{\nu ps'} + \bar{u}_{\mu ps} v_{\nu -ps'} b_{\mu ps}^\dagger d_{\nu -ps'}^\dagger \right. \\ & \left. + \bar{v}_{\mu ps} u_{\nu -ps'} d_{\mu ps} b_{\nu -ps'} - \bar{v}_{\mu ps} v_{\nu ps'} d_{\mu ps}^\dagger d_{\nu ps'} \right] + (\mu \leftrightarrow \nu). \end{aligned} \quad (16)$$

Physically, this result is not acceptable [3]. The interaction creates a badly divergent vacuum problem. To see the problem, consider the second term in $\hat{H}_I$ in Eq. (16),

$$\hat{h} = m \sum_{\mu ps}^\dagger \sum_{s'} \frac{1}{2E_{\nu p}} \bar{u}_{\mu ps} v_{\nu -ps'} b_{\mu ps}^\dagger d_{\nu -ps'}^\dagger, \quad (17)$$

where the spinor matrix element is

$$\bar{u}_{\mu ps} v_{\nu -ps'} = \left(\sqrt{\frac{E_{\nu p} + \nu}{E_{\mu p} + \mu}} + \sqrt{\frac{E_{\mu p} + \mu}{E_{\nu p} + \nu}} \right) \chi_s^\dagger \boldsymbol{\sigma} \mathbf{p} i \boldsymbol{\sigma}^2 \chi_{s'}, \quad (18)$$

and ask what the norm of state $|h\rangle = \hat{h}|0\rangle$ is. Direct evaluation yields

$$\langle h|h\rangle = \langle 0|\hat{h}^\dagger \hat{h}|0\rangle = V m^2 \sum_{\mu ps} \sum_{s'} \frac{1}{2E_{vp}} |\bar{u}_{\mu ps} v_{v-ps'}|^2 \sim V 4m^2 \int \frac{d^3 p}{(2\pi)^3 2E_{\mu p}} \frac{\mathbf{p}^2}{E_{vp}}, \quad (19)$$

where V denotes the volume of space at $t = 0$ and the approximation sign indicates that the large bracket in Eq. (18) is approximated by 2, with increasing accuracy for increasing $|\mathbf{p}|$. The result is infinite.

The consequence is that one does not know how to calculate the ground state of the interacting theory, or vacuum, in terms of the ground state of a free theory. All excited states contain similar divergences as the vacuum state.

To avoid the infinities and attempt a search for the spectrum of a regulated quantum Hamiltonian, one might impose a cutoff on $|\mathbf{p}|$. The price would be a violation of Lorentz symmetry, since boosts can change $|\mathbf{p}|$ by an arbitrary amount.

Similar interaction terms appear in QED. For example, there are terms in the QED IF Hamiltonian that are capable of creating the electron, positron, and photon out of bare vacuum and thus produce divergences in a similar pattern. These interaction terms and the resulting effects in states containing electrons and photons, are known how to handle in perturbative calculations [13]. But beyond the perturbative rules, one may question the existence of QED in the IF Schrödinger picture [3]. In the case of QCD, in the domain where perturbation theory does not apply, the role of interaction terms that alter the bare vacuum in the IF approach, is not understood yet.

In summary, the mass mixing model provides an elementary example of how the vacuum issue arises in relativistic theories of physical interest, although in the realistic theories the relevant interaction terms are much more complicated than in the model. As a consequence, the general vacuum issue remains unsolved. Concerning the physics of the vacuum in the context of comparison between the IF and FF approaches, especially in the context of QCD, see Refs. [11, 12] and the rich literature quoted there.

5 IF Re-Quantization Approach to the Mass Mixing Interaction and Vacuum Issue

In the IF approach, when one faces the vacuum problem due to the mass mixing interaction terms, one can step back to the Lagrangian density and re-write it using new field variables. Namely, one can write the density using a 2×2 -matrix notation, in which the fields ψ and ϕ form a doublet and a mass-matrix, say M , is introduced;

$$\mathcal{L} = \bar{\Psi}(i\partial - M)\Psi, \quad \Psi = \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \quad M = \begin{bmatrix} \mu & m \\ m & \nu \end{bmatrix}. \quad (20)$$

The eigenvalues and corresponding eigenvectors of M are

$$m_{1,2} = [\mu + \nu \pm (\mu - \nu)\epsilon]/2, \quad (21)$$

$$v_1 = \begin{bmatrix} \cos \varphi \\ -\sin \varphi \end{bmatrix}, \quad v_2 = \begin{bmatrix} \sin \varphi \\ \cos \varphi \end{bmatrix}, \quad (22)$$

where $\epsilon = \sqrt{1 + [2m/(\mu - \nu)]^2}$ and $\varphi = -\arctan \sqrt{(\epsilon - 1)/(\epsilon + 1)}$. One can write the doublet field Ψ in terms of the eigenvectors, $\Psi = \psi_1 v_1 + \psi_2 v_2$, and use the fields

$$\psi_1 = \cos \varphi \psi - \sin \varphi \phi \quad \text{and} \quad \psi_2 = \sin \varphi \psi + \cos \varphi \phi \quad (23)$$

as new degrees of freedom in a classical theory. In terms of the new fields, the Lagrangian density reads

$$\mathcal{L} = \bar{\psi}_1(i\partial - m_1)\psi_1 + \bar{\psi}_2(i\partial - m_2)\psi_2, \quad (24)$$

which is a theory of free fields corresponding to masses m_1 and m_2 .

Section 3 explains that the terms of type $\hat{h}$ in Eq. (17) do not appear in the corresponding IF quantum Hamiltonian once the right energies are used in the construction of quantum field operators, including boosted spinors and corresponding creation and annihilation operators.

The result is that one can avoid the vacuum issue in quantum theory when one constructs it by quantizing the field degrees of freedom that appear in Eq. (24) rather than those that appear in Eq. (14). Since one steps back

from a divergent quantum theory with interaction to the initial classical one and then quantizes the classical theory again using new degrees of freedom, for which the troublesome divergence turns out to cancel out, the IF cure for the vacuum problem can be called re-quantization.

In the re-quantization, one uses $E_{m_1 p}$ and $E_{m_2 p}$ instead of $E_{\mu p}$ and $E_{\nu p}$ in constructing quantum field operators $\hat{\psi}_1$ and $\hat{\psi}_2$, with new creation and annihilation operators. The resulting IF Hamiltonian,

$$\hat{H} = \sum_{m_1 p s} E_{m_1 p} \left(b_{m_1 p s}^\dagger b_{m_1 p s} + d_{m_1 p s}^\dagger d_{m_1 p s} \right) + \sum_{m_2 p s} E_{m_2 p} \left(b_{m_2 p s}^\dagger b_{m_2 p s} + d_{m_2 p s}^\dagger d_{m_2 p s} \right), \quad (25)$$

describes free fermions of masses m_1 and m_2 as desired in a relativistic theory.

The key question one faces in the IF of Hamiltonian dynamics is how to deal with the vacuum issue in the presence of complex interactions, i.e., when one does not know what masses to use. This becomes a serious roadblock when one does not even know if such masses exist. In case of theories such as QCD, which are expected to describe confined fermions, the very existence of “right” masses for IF quantization is doubtful.

6 FF Canonical Quantum Theory for Fermions with Mass Mixing Interaction

Previous consideration suggests that one needs some alternative approach to the IF re-quantization in order to handle theories with complex interactions. The elementary fermion model with mass mixing interaction is used here to explain how a candidate for such alternative method, the RGPEP, avoids the vacuum problem and produces the same solution that the IF re-quantization allows one to find. The FF approach to the model does not require any re-quantization. With the preliminaries laid out above, the description of FF approach given below is considerably abbreviated.

The space-time hyperplane, which in the FF of Hamiltonian dynamics is used instead of the hyperplane $t = x^0 = 0$, is defined by the condition $x^+ = 0$. The notation is $x^\pm = x^0 \pm x^3$ and $x^\perp = (x^1, x^2)$, the same for all tensor indices. Other conventions used here are explained in Ref. [1].

The Euler–Lagrange equation for the doublet fermion field in the Lagrangian density of Eq. (20), namely $(i\not{\partial} - M)\Psi = 0$, can be written in terms of two orthogonal components $\Psi_\pm = \Lambda_\pm \Psi$, where $\Lambda_\pm = \gamma^0 \gamma^\pm / 2$ are projection matrices, as an equation of motion and a constraint, respectively,

$$i\partial^- \Psi_+ = (i\alpha^\perp \partial^\perp + \beta M) \Psi_- \quad \text{and} \quad i\partial^+ \Psi_- = (i\alpha^\perp \partial^\perp + \beta M) \Psi_+. \quad (26)$$

Following Refs. [14, 15], one obtains the FF Hamiltonian,

$$P^- = \frac{1}{2} \int dx^- d^2 x^\perp \mathcal{T}^{+-}, \quad (27)$$

where the relevant component of the energy-momentum density tensor is

$$\frac{1}{2} \mathcal{T}^{+-} = \Psi_+^\dagger i\partial^- \Psi_+ = \Psi_+^\dagger (i\alpha^\perp \partial^\perp + \beta M) \frac{1}{i\partial^+} (i\alpha^\perp \partial^\perp + \beta M) \Psi_+. \quad (28)$$

The resulting FF quantum Hamiltonian is obtained by replacing Ψ by the quantum field operator $\hat{\Psi}$ and evaluating

$$\hat{P}^- = \int dx^- d^2 x^\perp : \hat{\Psi}_+^\dagger \frac{-\partial^{\perp 2} + M^2}{i\partial^+} \hat{\Psi}_+ :. \quad (29)$$

In the FF representation of γ -matrices [1], the operator $\hat{\Psi}_+$ has the form

$$\Psi_+(x) = \begin{bmatrix} \hat{\zeta}(x) \\ 0 \\ \hat{\omega}(x) \\ 0 \end{bmatrix}, \quad (30)$$

where $\hat{\zeta}$ and $\hat{\omega}$ are two-component fields (the field Ψ has eight components). The anti-commutation relations at $x^+ = x'^+ = 0$ are set in the form

$$\{\hat{\zeta}(x), \hat{\zeta}^\dagger(x')\} = \{\hat{\omega}(x), \hat{\omega}^\dagger(x')\} = \delta^3(x - x'). \quad (31)$$

The FF Fourier expansions of operators $\hat{\zeta}$ and $\hat{\omega}$ are independent of mass parameters. Namely,

$$\hat{\zeta}(x) = \sum_{ps} \sqrt{p^+} \left[b_{\zeta ps} e^{-ipx} - d_{\zeta ps}^\dagger e^{ipx} \sigma^1 \right] \chi_s, \quad (32)$$

$$\hat{\omega}(x) = \sum_{ps} \sqrt{p^+} \left[b_{\omega ps} e^{-ipx} - d_{\omega ps}^\dagger e^{ipx} \sigma^1 \right] \chi_s, \quad (33)$$

where

$$\sum_{ps} = \sum_{s=\pm 1} \int_{-\infty}^{+\infty} \frac{d^2 p^\perp}{(2\pi)^2} \int_0^{+\infty} \frac{dp^+}{2(2\pi)p^+}, \quad (34)$$

and χ_s is the two-component Pauli spinor normalized to 1 (the Pauli matrix σ^1 takes care of the required spin flip and the negative sign reflects the requirement of Fermi–Dirac statistics). The creation and annihilation operators satisfy mass-independent anti-commutation relations

$$\{b_{\zeta ps}, b_{\zeta p's'}^\dagger\} = \{d_{\zeta ps}, d_{\zeta p's'}^\dagger\} = \{b_{\omega ps}, b_{\omega p's'}^\dagger\} = \{d_{\omega ps}, d_{\omega p's'}^\dagger\} = 2p^+(2\pi)^3 \delta^3(p - p') \delta_{ss'}. \quad (35)$$

For comparison with the IF theory, it is important to stress that the FF Fourier expansion of fields does not require any knowledge of mass or “energy” p^- of the field quanta, in stark contrast to the IF construction where one assumes a value for the mass of field quanta and change of mass requires re-quantization. No re-quantization is needed in the FF version of the interacting theory.

Evaluation of the FF Hamiltonian yields

$$\begin{aligned} \hat{P}^- = & \sum_{ps} \left[\left(p_\mu^- + \frac{m^2}{p^+} \right) (b_{\zeta ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\zeta ps}) + \left(p_v^- + \frac{m^2}{p^+} \right) (b_{\omega ps}^\dagger b_{\omega ps} + d_{\omega ps}^\dagger d_{\omega ps}) \right. \\ & \left. + \frac{m(\mu + v)}{p^+} (b_{\zeta ps}^\dagger b_{\omega ps} + d_{\omega ps}^\dagger d_{\zeta ps} + b_{\omega ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\omega ps}) \right], \end{aligned} \quad (36)$$

where $p_\mu^- = (p^{\perp 2} + \mu^2)/p^+$ and $p_v^- = (p^{\perp 2} + v^2)/p^+$. The free fermion theory, the one before the mass mixing terms are added, corresponds to the Hamiltonian denoted by $\hat{P}_f^-$, which is obtained from $\hat{P}^-$ by setting $m = 0$.

Note that in the FF Hamiltonian $\hat{P}^-$ there are no terms of the type $\hat{h}$ that cause trouble in the IF version of the theory. The result is that $\hat{P}^-|0\rangle = 0$. Therefore, one may assume that the bare vacuum $|0\rangle$, which by definition is annihilated by all the annihilation operators, can represent the physical vacuum. This is precisely what happens in the model. In more complex theories, one has to find out if the interactions allow for this interpretation of $|0\rangle$. For example, if very strong interactions produced eigenstates of $\hat{P}^-$ with negative eigenvalues, one could not interpret $|0\rangle$ with eigenvalue zero as a ground state of a physical theory.

In FF Hamiltonians, the absence of interaction terms that can alter the bare vacuum state $|0\rangle$ is a generic feature of all theories of physical interest. This feature is a consequence of translation invariance, which implies conservation of $+$ -component of total momentum in the dynamics. Since all quanta included in the theory have positive p^+ , as indicated in the Fourier spectrum of momenta in Eq. (34), the quanta cannot carry zero momentum, which characterizes the state $|0\rangle$. Thus, the translation invariant Hamiltonian cannot create quanta from or annihilate them into the bare vacuum.

The only exception to stability of $|0\rangle$ is its interaction with the Fourier modes with $p^+ = 0$. But for quanta with non-zero masses such modes correspond to infinite eigenvalues of $\hat{P}_f^-$ and by definition these modes lie outside the cutoffs used to regulate a theory. If there is a need to include any effects due to such modes, the possibility which is not excluded in singular gauge theories, these effects should be identifiable through the cutoff dependence that appears when the needed modes are absent. However, this means that their effects can be describable using counter terms to the cutoff dependence. This is explained Ref. [11]. The cutoff dependence and counter terms can be studied using the RGPEP. The FF of dynamics thus appears to provide new ways for handling the vacuum problem that is otherwise prohibitively divergent in the IF of quantum dynamics. In the

fermion model with mass mixing, no cutoff dependence arises that would require questioning $|0\rangle$ as a good vacuum state.

In the model example, with terms of type $\hat{h}$ being absent for reasons found in all physically interesting theories with regularization, the FF Hamiltonian still contains interaction terms that mix fermions of masses μ and ν with arbitrary strength. The question is how to find the spectrum. This is done below using the RGPEP within one and the same quantum theory, no re-quantization being involved.

7 The RGPEP in Quantum Field Theory

General non-perturbative definition of the RGPEP in quantum field theory can be found in Ref. [16]. In its essence, the procedure amounts to transforming quantum fields from the bare ones to effective ones by a unitary rotation, $\psi_t = \mathcal{U}_t \psi_0 \mathcal{U}_t^\dagger$. This is done by expressing the bare, point-like quanta of a local theory in the canonical Hamiltonian with counter terms by the quanta of size $s = t^{1/4}$. As a result, one obtains a non-local theory of quanta that are called effective particles. So, for fermions, $b_{tp} = \mathcal{U}_t b_{0p} \mathcal{U}_t^\dagger$ and $d_{tp} = \mathcal{U}_t d_{0p} \mathcal{U}_t^\dagger$. The Hamiltonian remains the same, $\mathcal{H}_t(b_t, d_t) = \mathcal{H}_0(b_0, d_0)$, but it is written using different degrees of freedom. Below, where necessary for brevity, the annihilation operators b and d are commonly denoted by q .

The canonical theory with counter terms has a Hamiltonian of the form

$$\mathcal{H}_0(q_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_0(i_1, \dots, i_n) q_{0i_1}^\dagger \dots q_{0i_n}. \quad (37)$$

The effective theory Hamiltonian differs by a change of operators q_0 to q_t and coefficients c_0 to c_t ,

$$\mathcal{H}_t(q_t) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_t(i_1, \dots, i_n) q_{ti_1}^\dagger \dots q_{ti_n}. \quad (38)$$

The goal of RGPEP is to find the coefficients c_t . This is done by defining $\mathcal{H}_t(q_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(q_0) \mathcal{U}_t$ and solving the equation

$$\mathcal{H}'_t(q_0) = [\mathcal{G}_t(q_0), \mathcal{H}_t(q_0)], \quad (39)$$

where prime denotes the derivative with respect to t and $\mathcal{G}_t = -\mathcal{U}_t^\dagger \mathcal{U}'_t$. Knowing $\mathcal{G}_t$, one can find

$$\mathcal{U}_t = T \exp \left(- \int_0^t d\tau \mathcal{G}_\tau \right), \quad (40)$$

where T denotes ordering operators according to the size of effective particles. In its simplest version, the RGPEP defines $\mathcal{G}_t$ by the formula

$$\mathcal{G}_t = [\mathcal{H}_f, \mathcal{H}_{Pt}], \quad (41)$$

where $\mathcal{H}_f$ is the free part of the initial Hamiltonian and

$$\mathcal{H}_{Pt}(q_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_t(i_1, \dots, i_n) \left(\frac{1}{2} \sum_{k=1}^n p_{ik}^+ \right)^2 q_{0i_1}^\dagger \dots q_{0i_n}, \quad (42)$$

which means that $\mathcal{H}_{Pt}$ differs from $\mathcal{H}_t$ only by multiplication of the coefficients c_t by a square of the total $+$ -momentum carried by quanta involved in a term.

In summary, the simplest version of RGPEP is carried out by solving the equation

$$\mathcal{H}'_t = [[\mathcal{H}_f, \mathcal{H}_{Pt}], \mathcal{H}_t] \quad (43)$$

with initial condition $\mathcal{H}_0 = \mathcal{H}_{canonical} + \mathcal{H}_{CT}$, where $\mathcal{H}_{CT}$ denotes the counter terms. The counter terms are also found using the RGPEP; one adjusts the initial conditions so that the effective theories match physical data by their solutions.

8 Application of the RGPEP to the Fermion Mass-Mixing Model

The starting point is the Hamiltonian of Eq. (36), which is treated as an initial condition for a solution to

$$\mathcal{P}_t^{-'} = \left[\left[\mathcal{P}_f^-, \mathcal{P}_{Pt}^- \right], \mathcal{P}_t^- \right]. \quad (44)$$

A great simplification occurs in the fermion model: the initial condition does not require counter terms that diverge as functions of regularization parameters when regularization is being removed. The model does not require regularization of divergences; only finite adjustments of initial masses μ and ν and mass mixing parameter m are required in order to obtain desired values of fermion masses in the resulting spectrum.

By examining the RGPEP equations in the model, one finds that the solution for $\mathcal{P}_t^-$ has the form

$$\begin{aligned} \mathcal{P}_t^- = \sum_{ps}^f & \left[A_{tp} \left(b_{\zeta ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\zeta ps} \right) + B_{tp} \left(b_{\omega ps}^\dagger b_{\omega ps} + d_{\omega ps}^\dagger d_{\omega ps} \right) \right. \\ & \left. + C_{tp} \left(b_{\zeta ps}^\dagger b_{\omega ps} + b_{\omega ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\omega ps} + d_{\omega ps}^\dagger d_{\zeta ps} \right) \right], \end{aligned} \quad (45)$$

where the coefficients can be written in the forms

$$A_{tp} = \frac{p^{\perp 2} + \mu_t^2}{p^+}, \quad B_{tp} = \frac{p^{\perp 2} + \nu_t^2}{p^+}, \quad C_{tp} = \frac{m_t^2}{p^+}, \quad (46)$$

and the initial conditions for the mass parameters are

$$\mu_0^2 = \mu^2 + m^2, \quad \nu_0^2 = \nu^2 + m^2, \quad m_0^2 = m(\mu + \nu). \quad (47)$$

The RGPEP generator in Eq. (44) involves

$$\mathcal{P}_f^- = \sum_{ps}^f \left[p_\mu^- \left(b_{\zeta ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\zeta ps} \right) + p_\nu^- \left(b_{\omega ps}^\dagger b_{\omega ps} + d_{\omega ps}^\dagger d_{\omega ps} \right) \right], \quad (48)$$

$$\begin{aligned} \mathcal{P}_{Pt}^- = \sum_{ps}^f p^{+2} & \left[A_{tp} \left(b_{\zeta ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\zeta ps} \right) + B_{tp} \left(b_{\omega ps}^\dagger b_{\omega ps} + d_{\omega ps}^\dagger d_{\omega ps} \right) \right. \\ & \left. + C_{tp} \left(b_{\zeta ps}^\dagger b_{\omega ps} + b_{\omega ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\omega ps} + d_{\omega ps}^\dagger d_{\zeta ps} \right) \right], \end{aligned} \quad (49)$$

and hence has the form

$$[\mathcal{P}_f^-, \mathcal{P}_{Pt}^-] = \sum_{ps}^f C_{tp} p^{+2} (p_\mu^- - p_\nu^-) \left(b_{\zeta ps}^\dagger b_{\omega ps} - b_{\omega ps}^\dagger b_{\zeta ps} + d_{\zeta ps}^\dagger d_{\omega ps} - d_{\omega ps}^\dagger d_{\zeta ps} \right). \quad (50)$$

By equating coefficients in front of the same products of creation and annihilation operators on both sides of Eq. (44), one obtains three spin-independent equations for every value of momentum p ,

$$A'_{tp} = 2p^{+2} (p_\mu^- - p_\nu^-) C_{tp}^2, \quad (51)$$

$$B'_{tp} = -2p^{+2} (p_\mu^- - p_\nu^-) C_{tp}^2, \quad (52)$$

$$C'_{tp} = -p^{+2} (p_\mu^- - p_\nu^-) (A_{tp} - B_{tp}) C_{tp}. \quad (53)$$

But when one inserts in this set the coefficients A_t , B_t and C_t in their forms shown in Eq. (46), the momenta $p^\perp$ and p^+ drop out and all these RGPEP equations reduce to just three for the mass parameters,

$$(\mu_t^2)' = 2 (\mu^2 - \nu^2) (m_t^2)^2, \quad (54)$$

$$(\nu_t^2)' = -2 (\mu^2 - \nu^2) (m_t^2)^2, \quad (55)$$

$$(m_t^2)' = -(\mu^2 - \nu^2) (\mu_t^2 - \nu_t^2) m_t^2. \quad (56)$$

This simplification has to happen because the RGPEP Eq. (44) is designed to respect the seven kinematical Poincaré symmetries of the FF of Hamiltonian dynamics. The three equations for mass parameters can be written in the form of one matrix equation,

$$\begin{pmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & v_t^2 \end{pmatrix}' = \left[\left[\begin{pmatrix} \mu^2 & 0 \\ 0 & v^2 \end{pmatrix}, \begin{pmatrix} 0 & m_t^2 \\ m_t^2 & 0 \end{pmatrix} \right], \begin{pmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & v_t^2 \end{pmatrix} \right], \quad (57)$$

which would be Wegner's flow equation [17] for a 2×2 Hamiltonian matrix, if the first, diagonal matrix on the right-hand side contained μ_t^2 and v_t^2 instead of μ^2 and v^2 , respectively. Solutions are

$$\mu_t^2 = m^2 + \frac{1}{2}(\mu^2 + v^2) + \frac{1}{2}\delta\mu_t^2, \quad (58)$$

$$v_t^2 = m^2 + \frac{1}{2}(\mu^2 + v^2) - \frac{1}{2}\delta\mu_t^2, \quad (59)$$

$$\delta\mu_t^2 = (\mu^2 - v^2) \frac{\cosh x_t + \epsilon \sinh x_t}{\cosh x_t + \epsilon^{-1} \sinh x_t}, \quad (60)$$

$$m_t^2 = \frac{m(\mu + v)}{\cosh x_t + \epsilon^{-1} \sinh x_t}, \quad (61)$$

where $x_t = (\mu^2 - v^2)^2 \epsilon t$ and ϵ is introduced in Eq. (21).

It is visible that m_t^2 tends to zero when $t \rightarrow \infty$. As a result, one obtains in the limit a theory of two types of free fermions that do not mix with each other. Masses squared of these fermions are $\lim_{t \rightarrow \infty} \mu_t^2 = m_1^2$ and $\lim_{t \rightarrow \infty} v_t^2 = m_2^2$, where m_1 and m_2 are the same as the IF re-quantization masses in Eq. (21). This is how the RGPEP recovers the IF result in a quantum theory with empty vacuum and without re-quantization. The spectrum of solutions to the eigenvalue problem of FF Hamiltonian found using the RGPEP, consists of states of free fermions with masses m_1 and m_2 in empty vacuum. The solutions form a Poincaré invariant spectrum in as large a range of momenta as one wishes.

9 Conclusion

The model discussion can be concluded by observing that fermions cannot become tachyons. The issue arises because the free FF Hamiltonian depends on the squares of masses of fermions. A priori, one could expect that, in the presence of mass-mixing interactions, the matrix of masses squared could have negative eigenvalues, and thus describe tachyonic fermions. However, the RGPEP does not allow this to happen. One can write

$$\lim_{t \rightarrow \infty} \begin{pmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & v_t^2 \end{pmatrix} = \lim_{t \rightarrow \infty} \begin{pmatrix} m_{1t} & m_{It} \\ m_{It} & m_{2t} \end{pmatrix}^2 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \quad (62)$$

and see that the squares of masses cannot become negative in the fermion model since the eigenvalues m_1 and m_2 are real.

In the model, the RGPEP avoids the vacuum problem and provides information about what happens when the mass mixing interaction terms are so large, $|m| > \sqrt{\mu v}$, that the smaller one of the two IF mass eigenvalues becomes negative. Namely, a chiral rotation can restore the positive sign of the eigenmass, while the RGPEP mass squared never approaches zero when t changes from zero to infinity. Instead, μ_t^2 and v_t^2 go directly to the positive squares of the eigenmasses m_1 and m_2 irrespective of the signs of m_1 and m_2 . More detailed explanation of this result is available in [1].

The elementary model of fermions with mass mixing interaction illustrates the RGPEP as a new tool for studying quantum field theory. The method passes a test of providing solutions one expects to be valid on the basis of the IF re-quantization approach. It is important to observe that nowhere in the RGPEP any assumption was made to the effect that the interactions are small; no perturbative expansion was employed and the theory is solved using the RGPEP for arbitrary strength of the interaction terms. Still, in complex theories where non-perturbative calculations require considerable gain in understanding what actually happens and perturbative studies may provide helpful signposts, one may also apply the RGPEP using a perturbative expansion [16].

One may conclude by quoting Dirac [18], who said that the FF “offers new opportunities, while the familiar IF seems to be played out.”

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