

# On expanding foliations

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**Abstract.** Certain families of manifolds which support Anosov flows do not support foliations which both are expanding under a dynamical system and have quasi-isometrically embedded leaves.

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**Mathematical subject classification:** 57R30, 58F15, 37D40.

## 1 Introduction

This paper demonstrates that certain manifolds do not admit foliations which are both expanding and whose leaves satisfy a form of quasi-isometry. That is, if  $M$  belongs to one of several families of manifolds listed in the theorems below, it is impossible to find a diffeomorphism  $f : M \rightarrow M$  and a foliation  $W$  such that

- $W$  is *invariant*:  $f(W) = W$ ,
- $W$  is *expanding*: there is  $\lambda > 1$  such that  $\|Tf v\| \geq \lambda \|v\|$  for all  $v \in TW$ ,
- $W$  is *quasi-isometric*: letting  $\tilde{W}$  denote the lift of  $W$  to the universal cover  $\tilde{M}$ , there is a global constant  $Q > 1$  such that  $d_{\tilde{W}}(x, y) < Q d_{\tilde{M}}(x, y) + Q$  for all  $x$  and  $y$  on the same leaf of  $\tilde{W}$ .

A major motivation for investigating expanding, quasi-isometric foliations is the study of partially hyperbolic systems, diffeomorphisms of the form  $f : M \rightarrow M$  with an invariant splitting  $TM = E^u \oplus E^c \oplus E^s$  such that the *unstable*  $E^u$  subbundle is expanding under  $Tf$ , the *stable*  $E^s$  is contracting, and the *center*  $E^c$  neither expands as much as  $E^u$  nor contracts as much as  $E^s$ . In general, partially hyperbolic systems are difficult to analyze and classify. In the case where the foliations  $W^u$  and  $W^s$  tangent to  $E^u$  and  $E^s$  are quasi-isometric, the

situation is much improved. Under such an assumption, the center subbundle  $E^c$  is uniquely integrable [3], which is not true in general [5]. Moreover, the system enjoys a form of structural stability [8]. Any partially hyperbolic system on the 3-torus must have quasi-isometric invariant foliations [4], and this has been used to give a classification for these systems [10]. Both the establishment of quasi-isometry and the resulting classification can be extended to 3-manifolds with nilpotent fundamental group [17, 11]. Further results hold in higher dimensions [9, 12].

In light of the results cited above, a natural approach to analyze partially hyperbolic systems on a given manifold is to first establish quasi-isometry of the invariant foliations and then use this to prove further properties of the system. This paper shows that for many manifolds supporting partially hyperbolic diffeomorphisms this approach is impossible.

**Theorem 1.1.** *A closed manifold does not support an expanding quasi-isometric foliation if it is:*

- (1) *a  $d$ -dimensional Riemannian manifold of constant negative curvature where  $d \geq 3$ ,*
- (2) *the unit tangent bundle of a  $d$ -dimensional Riemannian manifold of constant negative curvature where  $d \geq 3$ , or*
- (3) *the suspension of a hyperbolic toral automorphism.*

Many examples of partially hyperbolic systems come from the time-one maps of Anosov flows, and a classic example of an Anosov flow is the geodesic flow on a negatively curved manifold  $M$ . This flow is defined on the unit tangent bundle  $T_1M$  as in case (2) above. Another example of an Anosov flow is the suspension of an Anosov diffeomorphism. If the diffeomorphism is defined on a torus  $\mathbb{T}^d$ , it corresponds to case (3). It is conjectured that every codimension one Anosov flow in dimension  $d \geq 4$  is of this form [7]. Note that Theorem 1.1 is not specific to the case of foliations coming from Anosov flows. In fact, it is easy to show that no Anosov flow (on any manifold) can have a quasi-isometric strong stable or unstable foliation.

In his original paper on the subject, Fenley showed that certain manifolds do not permit quasi-isometric codimension one foliations [6]. This paper considers foliations of any codimension with the additional condition of expanding dynamics. This extra condition is needed as in cases (2) and (3) above the orbits of the Anosov flows mentioned above give one-dimensional quasi-isometric foliations.

The proof of Theorem 1.1 relies on analyzing the fundamental group of the manifold, and the following generalization holds.

**Theorem 1.2.** *A closed manifold does not support an expanding quasi-isometric foliation if its fundamental group is isomorphic to the fundamental group of a manifold listed in Theorem 1.1.*

The proof involves Mostow Rigidity and the techniques could be easily applied to other locally symmetric spaces and certain manifolds with variable negative curvature [15, 2]. For the benefit of those dynamicists not well-versed in geometric group theory, this paper only treats the specific case of hyperbolic manifolds.

As suggested by Ali Tahzibi, one could also consider *non-uniformly* expanding foliations and similar results hold under additional assumptions. For the benefit of those geometers not well-versed in non-uniform hyperbolicity, this discussion is left to the appendix.

## 2 Preliminaries

**Notation.** A lift of a continuous function  $f : M \rightarrow N$  is a choice of continuous function  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$  such that  $P_N \tilde{f} = f P_M$  where  $P_M : \tilde{M} \rightarrow M$  and  $P_N : \tilde{N} \rightarrow N$  are the universal coverings. Viewing the fundamental group as the set of deck transformations on  $\tilde{M}$ ,  $\tilde{f}$  uniquely determines a group homomorphism  $f_* : \pi_1(M) \rightarrow \pi_1(N)$  which satisfies  $f_*(\alpha) \tilde{f}(x) = \tilde{f}(\alpha(x))$  for  $x \in \tilde{M}$  and  $\alpha \in \pi_1(M)$ .

For a foliation to be expanding as defined above, we require that the function  $f : M \rightarrow M$  is  $C^1$  and that each leaf of the foliation is  $C^1$  as a submanifold. The foliation itself need only be continuous, as is commonly the case for foliations encountered when studying dynamical systems. Also, since the proofs of Theorems 1.1 and 1.2 do not use the fact that a foliation covers the entire manifold, the results also hold for laminations in place of foliations.

The following is an immediate consequence of the definitions of expanding and quasi-isometric.

**Lemma 2.1.** *If the foliation  $W$  is quasi-isometric and expanding under  $f : M \rightarrow M$  then for a lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$  and distinct points  $x$  and  $y$  on the same leaf of the lifted foliation  $\tilde{W}$ , the sequence  $\{d_{\tilde{M}}(\tilde{f}^n(x), \tilde{f}^n(y))\}$  grows exponentially.*

If we can establish that for any homeomorphism  $f : M \rightarrow M$  with lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$  and any  $x, y \in \tilde{M}$ , the sequence  $\{d(\tilde{f}^n(x), \tilde{f}^n(y))\}$  grows subexponentially, then there can be no expanding quasi-isometric foliation on  $M$ . This is the technique used to prove Theorem 1.1.

**Lemma 2.2.** *Let  $M$  and  $N$  be manifolds,  $M$  be closed, and  $f, g : M \rightarrow N$  be continuous functions with lifts  $\tilde{f}, \tilde{g} : \tilde{M} \rightarrow \tilde{N}$  such that the induced homomorphisms  $f_*, g_* : \pi_1(M) \rightarrow \pi_1(N)$  are equal. Then, there is  $C > 0$  such that  $d_{\tilde{N}}(\tilde{f}(x), \tilde{g}(x)) < C$  for all  $x \in \tilde{M}$ .*

**Proof.** The function  $\tilde{M} \rightarrow \mathbb{R}, x \mapsto d_{\tilde{N}}(\tilde{f}(x), \tilde{g}(x))$  is invariant under deck transformations. It descends to a function  $M \rightarrow \mathbb{R}$  and is therefore bounded.  $\square$

**Corollary 2.3.** *If a foliation  $W$  is quasi-isometric and expanding under  $f : M \rightarrow M$  then the induced homomorphism  $f_*$  is not equal to the identity.*

**Corollary 2.4.** *No time-one map of an Anosov flow or perturbation thereof has a quasi-isometric strong stable or unstable foliation.*

Theorem 1.1 follows from Theorem 1.2. However, since cases (1) and (2) of Theorem 1.1 have short, direct proofs, we give them first for illustrative purposes.

**Proposition 2.5.** *Let  $M$  be a closed manifold of constant negative curvature,  $\dim M \geq 3$ , and let  $f : M \rightarrow M$  be a homeomorphism with lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ . Then, there is  $C > 0$  such that  $d(\tilde{f}^n(x), \tilde{f}^n(y)) < d(x, y) + Cn$  for  $x, y \in \tilde{M}$ .*

**Proof.** By Mostow rigidity, there is an isometry  $g : M \rightarrow M$  and lift  $\tilde{g} : \tilde{M} \rightarrow \tilde{M}$  such that  $f_* = g_*$  as automorphisms of  $\pi_1(M)$ . By Lemma 2.2, for  $x, y \in \tilde{M}$ ,

$$\begin{aligned} d(\tilde{f}(x), \tilde{f}(y)) &\leq d(\tilde{f}(x), \tilde{g}(x)) + d(\tilde{g}(x), \tilde{g}(y)) + d(\tilde{g}(y), \tilde{f}(y)) \\ &\leq C + d(x, y) + C \end{aligned}$$

and the claim follows by induction.  $\square$

**Proposition 2.6.** *Let  $M$  be a closed manifold of constant negative curvature,  $\dim M \geq 3$ , and let  $T_1M$  be the unit tangent bundle. If  $f : T_1M \rightarrow T_1M$  is a homeomorphism with lift  $\tilde{f} : \tilde{T}_1M \rightarrow \tilde{T}_1M$ , there is  $C > 0$  such that  $d(\tilde{f}^n(x), \tilde{f}^n(y)) < d(x, y) + Cn$  for  $x, y \in \tilde{T}_1M$ .*

**Proof.** The unit tangent bundle  $T_1M$  fibers over  $M$  with fiber  $\mathbb{S}^k, k > 2$ . The long exact sequence of homotopy groups for a fibration

$$\dots \rightarrow \pi_1(\mathbb{S}^k) \rightarrow \pi_1(T_1M) \rightarrow \pi_1(M) \rightarrow \pi_0(\mathbb{S}^k) \rightarrow \dots$$

shows that the projection  $p : T_1M \rightarrow M$  induces an isomorphism  $p_*$  on the fundamental groups. By Mostow rigidity, there is an isometry  $g : M \rightarrow M$

such that  $p_* f_* p_*^{-1} = g_*$ . After lifting,  $d_{\tilde{M}}(\tilde{p}(\tilde{f}(x)), \tilde{g}(\tilde{p}(x)))$  is bounded for  $x \in \tilde{T}_1 \tilde{M}$ . Arguing as in the last proof, for any  $x$  and  $y$

$$d(\tilde{p}(\tilde{f}(x)), \tilde{p}(\tilde{f}(y))) < d(\tilde{p}(x), \tilde{p}(y)) + C$$

so that

$$d(\tilde{p}(\tilde{f}^n(x)), \tilde{p}(\tilde{f}^n(y))) < d(\tilde{p}(x), \tilde{p}(y)) + nC,$$

and as  $p_*$  is an isomorphism, one can show that there is a global constant  $R > 1$  such that  $d_{\tilde{M}}(\tilde{p}(x), \tilde{p}(y)) < R d_{\tilde{T}_1 \tilde{M}}(x, y) + R$ . From these inequalities the proof follows. □

### 3 The general proof

To prove Theorem 1.2 and case (3) of Theorem 1.1, we reason more abstractly. Suppose  $M$  is a compact manifold with universal covering  $\tilde{M}$  and  $f : M \rightarrow M$  is a diffeomorphism with lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ . This lift induces an automorphism  $f_* : \pi_1(M) \rightarrow \pi_1(M)$ .

Fix a fundamental domain  $K \subset \tilde{M}$  and for a subset  $A \subset \pi_1(M)$  define  $AK = \{\alpha x : \alpha \in A, x \in K\}$ . Observe that  $\tilde{f}(AK) = f_*(A)\tilde{f}(K)$  and if  $A'$  is another subset of  $\pi_1(M)$ , then  $AA'K = (AA')K = A(A'K)$  is well-defined.

Fix a finite set of generators for  $\pi_1(M)$  and define a metric on the group by word distance. There is a constant  $C > 0$  such that  $d_{\tilde{M}}(\alpha_i x, x) < C$  for every generator  $\alpha_i$  of  $\pi_1(M)$  and all  $x \in \tilde{M}$ . Consequently, for a subset  $A \subset \pi_1(M)$ ,

$$\text{diam}(AK) \leq C \text{diam}(A) + \text{diam}(K)$$

where the diameters of  $AK$  and  $K$  are measured on  $\tilde{M}$  and  $\text{diam}(A)$  is with respect to the word metric.

As  $\tilde{f}(K)$  is compact, there is an integer  $N$  such that  $\tilde{f}(K) \subset B_N K$  where  $B_N = \{\alpha \in \pi_1(M) : |\alpha| \leq N\}$ . The word metric is defined such that the  $N$ -neighbourhood  $U_N(A)$  of a set  $A \subset \pi_1(M)$  is given by  $AB_N$ , and therefore.

$$\tilde{f}(AK) = f_*(A)\tilde{f}(K) \subset f_*(A)B_N K = U_N(f_*(A))K. \tag{1}$$

See Figure 1. Starting with a subset  $A_0 \subset \pi_1(M)$ , define a sequence  $\{A_k\}$  by  $A_{k+1} = U_N(f_*(A_k))$ . One can prove by induction that  $\tilde{f}^k(A_0 K) \subset A_k K$  for all  $k \geq 1$ . If the diameter of  $A_k$  grows at most polynomially, then the diameter of  $\tilde{f}^n(A_0 K)$  does as well. The above reasoning is summed up in the following proposition.

**Proposition 3.1.** *Suppose  $G$  is a finitely generated group with the following property:*

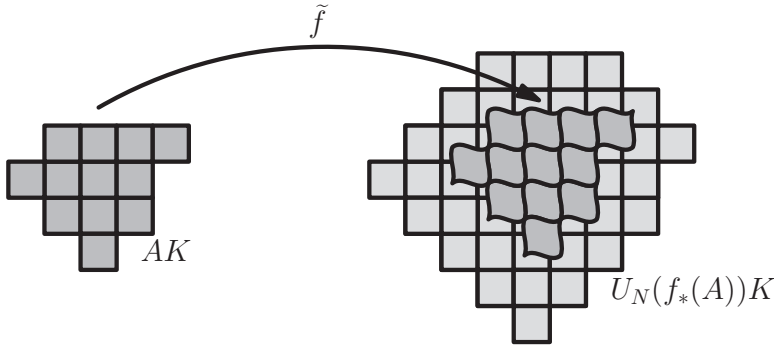


Figure 1: A visual depiction of the set inclusion given in (1).

For every automorphism  $\phi : G \rightarrow G$ , integer  $N > 0$ , and starting set  $A_0 \subset G$ , the sequence  $\{A_k\}$  defined by  $A_{k+1} = U_N(\phi(A_k))$  grows at most polynomially in diameter.

Then, for any manifold  $M$  with  $\pi_1(M) = G$ , diffeomorphism  $f : M \rightarrow M$  with lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$  and bounded subset  $K \subset \tilde{M}$ , the diameter of  $\tilde{f}^n(K)$  grows at most polynomially as  $n \rightarrow \infty$ .

**Notation.** For lack of a better word, call any group  $G$  satisfying the hypothesis of Proposition 3.1 *unstretchable*.

**Corollary 3.2.** *There is no expanding quasi-isometric foliation on a manifold with unstretchable fundamental group.*

We consider the fundamental groups of hyperbolic manifolds at the end of this section. For now, consider the fundamental group arising from a manifold included in case (3) of Theorem 1.1.

**Proposition 3.3.** *The fundamental group of a suspension of a hyperbolic toral automorphism is unstretchable.*

To prove this proposition, consider  $\pi_1(M)$  as an abstract group  $G$ . It fits into an exact sequence

$$0 \rightarrow \mathbb{Z}^d \rightarrow G \rightarrow \mathbb{Z} \rightarrow 0.$$

Let  $H \triangleleft G$  be the image of  $\mathbb{Z}^d$  in this sequence and fix an element  $z \in G$  such that its image under the projection  $G \rightarrow \mathbb{Z}$  generates  $\mathbb{Z}$ . Every element of  $G$  may then be written uniquely as  $x \cdot z^k$  where  $x \in H$  and  $k \in \mathbb{Z}$ . Further, there is an

automorphism  $A : H \rightarrow H$ , coming from the hyperbolic toral automorphism, such that  $z \cdot x = (Ax) \cdot z$  for all  $x \in H$ .

**Lemma 3.4.** *The automorphisms of  $G$  are exactly those of the form  $\phi(x) = Bx$  for  $x \in H$  and  $\phi(z) = v \cdot z^e$  where  $B \in \text{Aut}(H) \approx GL(d, \mathbb{Z})$ ,  $v \in H$ ,  $e = \pm 1$ , and  $A^e B = BA$ .*

This result is well known, at least in the case  $d = 2$ . For completeness, we give a short proof for general  $d$ , starting with the following claim.

**Lemma 3.5.**  *$H$  is a characteristic subgroup: if  $\phi$  is an automorphism of  $G$ , then  $\phi(H) = H$ .*

**Proof.** We will show that  $H = \text{rad}([G, G])$ ; that is,  $v \in H$  if and only if there is  $k \in \mathbb{Z}$  such that  $v^k$  is in  $[G, G]$ . As this is a purely group-theoretic characterization, it is preserved under isomorphism. Note that the image of a commutator  $uvu^{-1}v^{-1}$  under a map  $G \rightarrow \mathbb{Z}$  must be zero. By this property, the above short exact sequence implies that  $[G, G] < H$ . Since  $\mathbb{Z}$  is torsion-free, it further implies that  $\text{rad}([G, G]) < H$ .

To show the other inclusion, note that for  $x \in H$ ,

$$[x, z] = x \cdot z \cdot (-x) \cdot z^{-1} = (x - Ax) \in H$$

and therefore  $(A - I)H \subset [G, G]$  where  $I : H \rightarrow H$  denotes the identity. Taking  $A - I$  to be an  $n \times n$  matrix, if  $(A - I)\mathbb{Z}^d$  did not have full rank, it would mean  $A - I$  has a nullspace (in both  $\mathbb{Z}^d$  and  $\mathbb{R}^d$ ), but  $A$  is hyperbolic, implying that  $A - I$  is invertible over  $\mathbb{R}^d$ . Therefore,  $(A - I)\mathbb{Z}^d$  has full rank and  $\text{rad}((A - I)\mathbb{Z}^d) = \mathbb{Z}^d$ . Consequently,  $H = \text{rad}((A - I)H) < \text{rad}([G, G])$ .  $\square$

**Proof of Lemma 3.4.** Let  $\phi : G \rightarrow G$  be an automorphism. From Lemma 3.5,  $\phi(H) = H$ , so define  $B := \phi|_H \in \text{Aut}(H)$ . Further,  $\phi$  induces an automorphism on the quotient  $G/H \approx \mathbb{Z}$  which must be of the form  $\pm \text{id} : \mathbb{Z} \rightarrow \mathbb{Z}$ . Therefore, the coset  $zH$  maps to the coset  $z^{\pm 1}H$  which is the case exactly when  $\phi(z) = v \cdot z^{\pm 1}$  for some  $v \in H$ . To be well-defined,  $\phi$  must satisfy  $\phi(z) \cdot \phi(x) = \phi(Ax) \cdot \phi(z)$  for all  $x \in H$ . This is equivalent to the condition  $A^e B = BA$ . The converse direction is straightforward to verify.  $\square$

Now fix  $\phi \in \text{Aut}(G)$  and define  $B, v$ , and  $e$  as in Lemma 3.4. For simplicity, assume  $e = 1$ . The case with  $e = -1$  is similar. Define a metric  $\| \cdot \|$  on

$H \approx \mathbb{Z}^d \subset \mathbb{R}^d$  using the standard metric on  $\mathbb{R}^d$ . Fix a very large positive constant  $\lambda$ , and define for  $\ell, h \in \mathbb{N}$  the set

$$B(\ell, h) = \{x \cdot z^k \in G : \|x\| \leq \lambda^\ell, |k| \leq h\}.$$

If  $x_1, \dots, x_m$  is a list of all elements of  $H$  with norm one, then  $\{x_1, \dots, x_m, z, z^{-1}\}$  is a generating set for  $G$ . This determines a word metric on  $G$ .

**Lemma 3.6.** *For all  $\ell, h \geq 1$ ,*

$$U_1(B(\ell, h)) \subset B(\ell + h, h + 1)$$

and for  $N \geq 1$ ,

$$U_N(B(\ell, h)) \subset B(\ell + N(h + N), h + N).$$

**Proof.** Suppose  $x \cdot z^k \in B(\ell, h)$ . Then if  $y \in H$  is a generator,  $\|y\| = 1$  and  $(x \cdot z^k) \cdot y = (x + A^k y) \cdot z^k$ . As  $A$  is fixed, we may assume  $\lambda$  was chosen large enough that  $\|A^{\pm 1} u\| < \lambda \|u\|$  for all  $u \in H$ . Then (assuming also  $\lambda > 2$ ),

$$\|x + A^k y\| < \|x\| + \lambda^k \|y\| \leq \lambda^\ell + \lambda^h \leq \lambda^{\ell+h}$$

proving  $(x \cdot z^k) \cdot y \in B(\ell + h, h + 1)$ . The case  $(x \cdot z^k) \cdot z^{\pm 1} = x \cdot z^{k \pm 1}$  is immediate. The second half of the lemma is proved by induction using  $U_{n+1}(A) = U_1(U_n(A))$ . □

**Lemma 3.7.** *For  $\ell, h \geq 2$ ,  $\phi(B(\ell, h)) \subset B(\ell + h, h)$ .*

**Proof.** Recall  $\phi$  is defined by  $\phi(x) = Bx$  for  $x \in H$  and  $\phi(z) = v \cdot z$ . Then for  $k > 0$ ,  $\phi(z^k) = (v \cdot z)^k = (\sum_{i=0}^{k-1} A^i v) \cdot z^k$  as can be proved by induction. As  $v$  is fixed, we may assume  $\lambda$  was chosen large enough that  $\|v\| + \|Av\| < 1 + \lambda$  and  $\|A^i v\| < \lambda^i$  for all  $i > 2$ . These conditions imply  $\|\sum_{i=0}^{k-1} A^i v\| < \sum_{i=0}^{k-1} \lambda^i < \lambda^k$ . Also, assume  $\|Bx\| < \lambda x$  for all  $x \in H$ . If  $x \cdot z^k \in B(\ell, h)$  with  $k \geq 0$ , then

$$\phi(x \cdot z^k) = (Bx + \sum_{i=0}^{k-1} A^i v) \cdot z^k$$

and

$$\|Bx + \sum_{i=0}^{k-1} A^i v\| \leq \lambda \|x\| + \lambda^k \leq \lambda^{\ell+1} + \lambda^h \leq \lambda^{\ell+h},$$

so  $\phi(x \cdot z^k) \in B(\ell + h, h)$ . The case of  $x \cdot z^k$  with  $k$  negative follows by the same reasoning with  $A^{-1}$  in place of  $A$ . □



**Remark.** We assumed  $h \geq 2$  above so that  $\lambda^{\ell+1} + \lambda^h \leq \lambda^{\ell+h}$  would hold.

Now, as in the hypothesis of Proposition 3.1, assume  $N$  is fixed, and  $A_0$  is a finite subset of  $G$  which defines a sequence  $\{A_k\}$  by  $A_{k+1} = U_N(\phi(A_k))$ . As  $A_0$  is finite, it is contained in some  $B(\ell, h)$  for large enough  $\ell$  and  $h$ . Then,

$$\begin{aligned} A_1 &\subset U_N(\phi(B(\ell, h))) \\ &\subset U_N(B(\ell + h, h)) \\ &\subset B(\ell + h + N(h + N), h + N) \\ &\subset B(\ell + 2(h + N)^2, h + N). \end{aligned}$$

By induction,  $A_k \subset B(\ell + p(k), h + Nk)$  where  $p(k) := \sum_{i=1}^k 2(h + Ni)^2$  grows at most polynomially in  $k$ . To show  $\text{diam}(A_k)$  is growing polynomially, it is enough to show that the diameter of  $B(\ell, h)$  is polynomial in  $\ell$  and  $h$ . In fact, the dependence is linear.

**Lemma 3.8.** *There is  $C > 0$  such that  $\text{diam}(B(\ell, h)) < C\ell + h$ .*

**Proof.** To prove this, we move our study from the group back to the manifold. The automorphism  $A$  on  $H \approx \mathbb{Z}^d$  may be thought of as a hyperbolic toral automorphism defining the manifold  $M = \mathbb{T}^d \times \mathbb{R} / \sim$  under the relation  $(x, t + 1) \sim (Ax, t)$ . Define a Riemannian metric on  $M$  such that the submanifold  $\mathbb{T}^d \times \{0\}$  is equipped with the usual flat metric on  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  and such that the flow on  $M$  defined by  $\varphi_t(x, s) = (x, s + t)$  flows at unit speed.

Lift the metric to the universal cover  $\tilde{M} = \mathbb{R}^d \times \mathbb{R}$ . The lifted flow  $\tilde{\varphi}_t(x, s) = (x, s + t)$  is Anosov and the strong stable manifold through the origin,  $W^s(0, 0)$ , is a linear subspace of  $\mathbb{R}^d \times \{0\}$  corresponding to the stable manifold of  $A$ . Suppose  $(x, 0)$  is a point on  $W^s(0, 0)$  with  $\|x\| > 1$ . Then, there is  $\sigma > 0$  such that

$$d_s(\tilde{\varphi}_t(x, 0), \tilde{\varphi}_t(0, 0)) \leq e^{-\sigma t} d_s((x, 0), (0, 0))$$

for all  $t > 0$  where  $d_s$  is distance measured along the strong stable leaf. Choosing  $t$  such that  $e^{-\sigma t} \|x\| = 1$ , one can show that  $d((x, 0), (0, 0)) \leq \frac{2}{\sigma} \log \|x\| + 1$ . A similar formula holds for points on the same unstable leaf. As the stable and unstable foliations are linear and transverse, it follows that there is a constant  $C > 0$  such that  $d((x, 0), (0, 0)) \leq C \log \|x\|$  for any  $x \in \mathbb{R}^d$  with  $\|x\| \geq 1$ .

The embedding  $i : G \rightarrow \tilde{M}, x \cdot z^k \mapsto (x, k)$  agrees with the standard method of embedding a fundamental group in the universal cover. In particular, it is a quasi-isometric function and there is  $Q > 1$  such that

$$d_G(x, 0) < Q d_{\tilde{M}}(i(x), i(0)) < CQ \log \|x\|$$

for all non-zero  $x \in H$ . From here it is straightforward to show that  $d_G(x \cdot z^k, 0) < CQ \log(\lambda)\ell + h$  for all  $x \cdot z^k \in B(\ell, h)$ , completing the proof.  $\square$

Lemma 3.8, with the discussion preceding it, concludes the proof of Proposition 3.3. We have shown that a manifold constructed as the suspension of a hyperbolic toral automorphism does not have an expanding quasi-isometric foliation. This completes case (3) of Theorem 1.1 and also part of Theorem 1.2, since the application of Proposition 3.1 depends only on the group  $\pi_1(M)$  and not the manifold itself. To finish the proof of Theorem 1.2, we consider groups coming from hyperbolic manifolds.

**Proposition 3.9.** *The fundamental group of a  $d$ -dimensional manifold of constant negative curvature ( $d \geq 3$ ) is unstretchable.*

**Proof.** Let  $G$  be such a group, and let  $\phi$ ,  $N$ , and the sequence  $\{A_k\}$  be as in Proposition 3.1. For a subset  $A \subset G$ , note that  $\phi(U_N(A)) \subset U_C(\phi(A))$  where  $C = \max\{|\phi(x)| : x \in G, |x| \leq N\}$ . Therefore, for any  $p > 0$ ,  $A_{k+p} \subset U_{N'}(\phi^p(A_k))$  for some integer  $N'$  depending on  $p$ ,  $N$ , and  $C$ .

As a consequence of Mostow rigidity, the group of outer automorphisms,  $Out(G)$ , is finite (see remark below). Hence, there is  $p$  such that  $\phi^p$  is an inner automorphism  $x \mapsto g^{-1}xg$ . For  $x, y \in G$ ,

$$d(\phi(x), \phi(y)) = |g^{-1}x^{-1}gg^{-1}yg| \leq |g| + |x^{-1}y| + |g| = d(x, y) + 2|g|,$$

Thus,  $A_{k+p} \subset U_{N'}(U_{2|g|}(A_k))$  from which it follows that  $\{A_k\}$  grows at most polynomially.  $\square$

**Remark.** The fact that  $Out(G)$  is finite is well known to those studying rigidity. However, I was unable to find a citable elementary proof. For readers not familiar with the result, I give an outline of the proof here.

As  $M$  is aspherical, an automorphism  $\phi$  of  $\pi_1(M, x_0)$  is induced by a homotopy equivalence  $h : (M, x_0) \rightarrow (M, x_0)$ . By Mostow rigidity,  $h$  is homotopic to an isometry  $g : (M, x_0) \rightarrow (M, x_0)$ . As this homotopy does not preserve the base point  $x_0$ , the automorphisms  $\phi = h_*$  and  $g_*$  are conjugate, but not necessarily identical. Now choose paths  $\alpha_i$  ( $i \in \{1, \dots, n\}$ ) which represent the generators of  $\pi_1(M, x_0)$ . For each  $i$ , the path  $g \circ \alpha_i$  is the same length as  $\alpha_i$  and so there are a finite number of possibilities for the element of  $\pi_1(M, x_0)$  which it represents. Hence, there are only a finite number of possibilities for  $g_*$ .

### A Non-uniform expansion

Suppose a diffeomorphism  $f : M \rightarrow M$  has an invariant, one-dimensional foliation  $W$ . By Oseledets theorem, there is a full probability set  $R \subset M$  such that for  $x \in R$ , the Lyapunov exponent

$$\lambda^W(x) := \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tf^n|_{TW(x)}\|$$

exists [16].

**Proposition A.1.** *Suppose  $f : M \rightarrow M$  is a diffeomorphism of a manifold with unstretchable fundamental group, and  $W$  is an invariant quasi-isometric foliation. Then,*

$$R' := \{x \in R : \lambda^W(x) \neq 0\}$$

*intersects each leaf of  $W$  in a set of (one-dimensional) Lebesgue measure zero.*

*Moreover, if  $W$  is absolutely continuous, then  $R'$  has Lebesgue measure zero as a subset of  $M$ .*

**Remark.** There are several possible ways to define absolute continuity (see, for example, §2.6 of [19]). Here, we take absolute continuity of a foliation to mean that any set  $X$  which intersects each leaf in a null set is itself a null set on  $M$ . Then, the second half of the proposition follows immediately from the first half.

**Proof.** The proof is an adaptation of an idea explained in [1, Proposition 0.5]. There, it is originally attributed to Mañé.

Assume the proposition is false for some  $f$  and  $W$ . By replacing  $f$  with  $f^{-1}$  if necessary, we may assume there is a constant  $c > 0$  and a precompact subset  $A$  of a leaf  $L$  of  $W$  such that  $A$  has positive Lebesgue measure and  $\lambda^W(x) > c$  for all  $x \in A$ .

For a positive integer  $k$ , let  $A_k$  denote the set of all points  $x \in A$  such that  $\frac{1}{n} \log \|Tf^n|_{TW(x)}\| > c$  for all  $n > k$ . As  $\bigcup A_k = A$ , there is  $k$  such that  $A_k$  has positive Lebesgue measure as a subset of  $L$ . Further, the Lebesgue measure of  $f^n(A_k)$  grows exponentially fast. By quasi-isometry, the diameter of  $A$  (as a subset of  $M$ ) grows exponentially fast, contradicting Proposition 3.1.  $\square$

In several cases, non-zero Lyapunov exponents have been used to show that the center foliations of partially hyperbolic systems are not absolutely continuous [18, 1]. We show that the same technique applies here.

Let  $m$  be a measure equivalent to Lebesgue on a compact manifold  $M$ . Let  $\text{Diff}_m^1(M)$  denote all  $C^1$  diffeomorphisms on  $M$  which preserve  $m$ , and let the subset  $\text{PH}_m^1(M)$  denote partially hyperbolic diffeomorphisms with one-dimensional center.  $\text{PH}_m^1(M)$  is open with respect to the  $C^1$  topology on  $\text{Diff}_m^1(M)$ .

If  $f \in \text{PH}_m^1(M)$  has a center foliation  $W_f^c$  which satisfies a technical condition known as *plaque expansiveness*, it follows that there is a neighbourhood  $U$  of  $f$  such that every  $g \in U$  also has a center foliation  $W_g^c$ . Moreover, the foliations are equivalent; there is a homeomorphism  $h$  (depending on  $g$ ) taking leaves of  $W_f^c$  to those of  $W_g^c$ . Plaque expansiveness can be established in many specific cases, and it is an open question if all center foliations are plaque expansive (see [8], [13], and [14] for more details).

**Proposition A.2.** *Suppose  $M$  has unstretchable fundamental group and  $f \in \text{PH}_m^1(M)$ . Further, suppose  $W_f^c$  exists and is plaque expansive and quasi-isometric. Then, for an open and dense set of  $g$  close to  $f$ ,  $W_g^c$  is not absolutely continuous. To be precise, there are open subsets  $U, V \subset \text{PH}_m^1(M)$  such that  $f \in U \subset \bar{V}$  and  $W_g^c$  is not absolutely continuous for any  $g \in V$ .*

**Proof.** Let  $U$  be the open neighbourhood of  $f$  given by plaque expansiveness [14]. In particular, there exists a foliation  $W_g^c$  tangent to the center direction  $E_g^c$  for every  $g \in U$ . As this foliation is equivalent to  $W_f^c$ , it is also quasi-isometric. For  $g \in U$ , define

$$\lambda^c(g) := \int_M \log \|Tg|_{E_g^c(x)}\| dm(x)$$

and  $V := \{g \in U : \lambda^c(g) \neq 0\}$ . As the function  $g \mapsto \lambda^c(g)$  is continuous,  $V$  is open. It follows from [1, Proposition 0.3] that  $V$  is dense in  $U$ . Suppose  $g \in V$ . By the Birkhoff ergodic theorem,

$$\lambda^c(x) := \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tg^n|_{E_g^c(x)}\|$$

is defined almost everywhere and  $\int_M \lambda^c(x) = \lambda^c(g)$ . Therefore,  $\lambda^c(x)$  is non-zero on a positive measure set and, by Proposition A.1,  $W_g^c$  is not absolutely continuous.  $\square$

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