A structure theorem of Dirac-harmonic maps between spheres

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Abstract For an arbitrary Dirac-harmonic map (ϕ, ψ) between compact oriented Riemannian surfaces, we shall study the zeros of $|\psi|$. With the aid of Bochner-type formulas, we explore the relationship between the order of the zeros of $|\psi|$ and the genus of M and N . On the basis, we could clarify all of non-trivial Dirac-harmonic maps from S^2 to S^2 .

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1 Introduction

Let (M, h) be an m -dimensional Riemannian spin manifold; $SpinM$ denotes the Spin-bundle on M , and $\eta : SpinM \rightarrow SOM$ is the bundle map, where SOM denotes the tangent orthonormal frame bundle on M . Denote by ΣM the spinor bundle associated to $SpinM$, i.e. $\Sigma M = SpinM \times_{\rho} \Sigma_m$, where $\rho : Spin_m \rightarrow \Sigma_m$ is the standard representation. On ΣM we can choose an Hermitian product $\langle \cdot, \cdot \rangle$, such that

$$\langle X \cdot \psi, \xi \rangle = -\langle \psi, X \cdot \xi \rangle \quad X \in \Gamma(TM), \psi, \xi \in \Gamma(\Sigma M). \quad (1.1)$$

Here

$$\mathbf{m} : X \otimes \psi \mapsto X \cdot \psi \quad (1.2)$$

is the Clifford multiplication. There is a connection on ΣM induced by the Levi-Civita connection of SOM ; denote it by ∇ ; and it is well known that ∇ is compatible with $\langle \cdot, \cdot \rangle$. Let ϕ be a smooth map from M to another Riemannian manifold (N, g) of dimension $n \geq 2$. Denote by $\phi^{-1}TN$ the pull-back bundle of TN and by $\Sigma M \otimes \phi^{-1}TN$ the twisted bundle. On it there is a metric induced from those on ΣM and $\phi^{-1}TN$. Similarly we have a natural connection $\tilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$. Based on it, we

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can define the Dirac operator along the map ϕ by

$$\not{D}\psi = \mathfrak{m} \circ \tilde{\nabla}\psi. \quad (1.3)$$

Here ψ is a smooth section of $\Sigma M \otimes \phi^{-1}TN$.

In Chen et al. [1], introduced a functional that couples the non-linear sigma model with a spinor field:

$$L(\phi, \psi) = \int_M [|d\phi|^2 + \langle \psi, \not{D}\psi \rangle] * 1 \quad (1.4)$$

The critical points of the functional are called Dirac-harmonic maps. In the paper, some geometric and analytic aspects of such maps were studied, especially a removable singularity theorem was established. Later in [2, 3, 9, 10], another geometric and analytic properties of Dirac-harmonic maps were studied.

Obviously there are two type of trivial Dirac-harmonic maps. One is $(\phi, 0)$, where ϕ is a harmonic map, and another is (y, ψ) , where y is a point in N viewed as a constant map, ψ is a harmonic spinor. In [1], the authors constructed non-trivial Dirac-harmonic maps (ϕ, ψ) from S^2 to S^2 , where ϕ is a (possible branched) conformal map, ψ could be written in the form

$$\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha), \quad (1.5)$$

$\{e_\alpha : \alpha = 1, 2\}$ is a local orthonormal frame field on S^2 , and Ψ is a twistor spinor. It is natural to ask whether there exists another form of Dirac-harmonic maps from S^2 to S^2 . And furthermore, is there a Dirac-harmonic map (ϕ, ψ) such that ϕ is not a harmonic map?

In the theory of harmonic maps between two compact Riemannian surfaces, Bochner formulas of $\log |\partial u|$ and $\log |\bar{\partial} u|$ play an important role (u denotes a harmonic map). From it several interesting formulas easily follow, which tell us the relationship between the order of the zeros of $\log |\partial u|$ and $\log |\bar{\partial} u|$ and the genus of M and N ; and moreover we can obtain some uniqueness theorems and non-existence theorems (see [7, Chapter I]). This phenomenon motives us to study the zeros of $|\psi|$.

Now we give a brief outline of the paper. In Sect. 2, the subjects we study are general Dirac-harmonic maps. In the viewpoint that $\Sigma M \otimes \phi^{-1}TN = \Sigma M \otimes (\phi^{-1}TN)^\mathbb{C}$, ψ could be written as $\psi = \psi^j \otimes W_j$, where $\{W_1, \dots, W_n\}$ is a local complex tangent frame field on N ; and we derive the Euler–Lagrange equation of L by using the above denotation. In Sect. 3, we assume M and N to be oriented Riemannian surfaces; the equations of harmonic spinor ψ along ϕ in the local complex coordinates are derived, which imply that the zeros of $|\pi_1^+(\psi)|$ are isolated, unless $|\pi_1^+(\psi)|$ is identically zero, so are $|\pi_0^+(\psi)|$, $|\pi_1^-(\psi)|$ and $|\pi_0^-(\psi)|$. (Here the definition of π_1^+ , π_0^+ , π_1^- , π_0^- is introduced in Sect. 3.) In Sects. 4 and 5, under the further assumption that M and N are both compact, we derive several Bochner formulas of $\log |\pi_1^+(\psi)|$, $\log |\pi_0^+(\psi)|$, $\log |\pi_1^-(\psi)|$, $\log |\pi_0^-(\psi)|$ on the basis of Weitzenböck-type formulas of ψ and furthermore give the proof of main theorems as follows, including a structure theorem of Dirac-harmonic maps from S^2 to S^2 . (In the process, it is necessary to use the results in Sects. 2 and 3).

Theorem 1.1 *M and N are both compact oriented Riemannian surfaces, and (ϕ, ψ) is a Dirac-harmonic map from M to N . If $g_M = 0$ or $|g_M - 1| < |\deg(\phi)| |2g_N - 2|$, then ϕ has to be a harmonic map.*

Theorem 1.2 *If $M = N = S^2$ equipped with arbitrary metric, (ϕ, ψ) is a non-trivial Dirac-harmonic map from M to N , then ϕ has to be holomorphic or anti-holomorphic, ψ could be*

written in the form

$$\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha), \quad (1.6)$$

where Ψ is a twistor spinor (possibly with isolated singularities).

Please note that here and in the sequel we use the summation convention and agree the range of indices:

$$1 \leq i, j, k \leq n; \quad 1 \leq \alpha, \beta \leq m.$$

We refer to [4–6] for more background material on spin structures and Dirac operators and to [7, 8] for more background material on harmonic maps.

2 Euler–Lagrange equations of Dirac-harmonic maps

Denote the complexification of $\phi^{-1}TN$ by $(\phi^{-1}TN)^\mathbb{C}$. Obviously $\Sigma M \otimes \phi^{-1}TN \subset \Sigma M \otimes (\phi^{-1}TN)^\mathbb{C}$. On the other hand, for any $\psi \in \Sigma M$, $X + \sqrt{-1}Y \in (\phi^{-1}TN)^\mathbb{C}$ (here $X, Y \in \phi^{-1}TN$),

$$\psi \otimes (X + \sqrt{-1}Y) = \psi \otimes X + \sqrt{-1}\psi \otimes Y \in \Sigma M \otimes \phi^{-1}TN;$$

which implies $\Sigma M \otimes (\phi^{-1}TN)^\mathbb{C} \subset \Sigma M \otimes \phi^{-1}TN$. Hence $\Sigma M \otimes (\phi^{-1}TN)^\mathbb{C} = \Sigma M \otimes \phi^{-1}TN$. The pull-back metric $\phi^{-1}g$ on $\phi^{-1}TN$ could be naturally extended to a Hermitian product on $(\phi^{-1}TN)^\mathbb{C}$; and there is a natural Hermitian product on $\Sigma M \otimes (\phi^{-1}TN)^\mathbb{C}$ induced from those on ΣM and $(\phi^{-1}TN)^\mathbb{C}$, which is also denoted by $\langle \cdot, \cdot \rangle$.

For each point $x \in M$, we can choose $\{W_i \in \Gamma((TU)^\mathbb{C}) : 1 \leq i \leq n\}$, where U is a neighborhood of $\phi(x)$, such that

$$(T_{\phi(y)}N)^\mathbb{C} = \bigoplus_{i=1}^n \mathbb{C}W_i(\phi(y)) \quad y \in \phi^{-1}(U), \quad (2.1)$$

then on $\phi^{-1}(U)$, ψ could be expressed by

$$\psi(y) = \psi_j(y) \otimes W_i(\phi(y)), \quad (2.2)$$

where $\psi_1, \dots, \psi_n \in \Gamma(\Sigma(\phi^{-1}(U)))$. We shall derive the Euler–Lagrange equations for L by using the above denotation.

Proposition 2.1 *Let $\{e_\alpha : 1 \leq \alpha \leq m\}$ be a local tangent orthonormal frame field, Then the Euler–Lagrange equations for L are*

$$\mathcal{D}\psi = 0 \quad (2.3)$$

$$\tau(\phi) = -\left\langle \psi^j, e_\alpha \cdot \psi^k \right\rangle R_{\bar{W}_j, W_k}^N \phi_*(e_\alpha). \quad (2.4)$$

Proof At first, we consider a family of ψ_t with $\frac{d\psi_t}{dt} = \eta$ at $t = 0$ and fix ϕ . Since $\mathcal{D}$ is formally self-adjoint (see [1]), we have

$$\begin{aligned} \left. \frac{dL}{dt} \right|_{t=0} &= \int_M \langle \eta, \mathcal{D}\psi \rangle + \langle \psi, \mathcal{D}\eta \rangle = \int_M \langle \eta, \mathcal{D}\psi \rangle + \langle \mathcal{D}\psi, \eta \rangle \\ &= 2 \int_M \operatorname{Re} \langle \eta, \mathcal{D}\psi \rangle. \end{aligned} \quad (2.5)$$

Since η could be chosen arbitrarily, (2.3) is easily followed.

Now we consider a variation ϕ_t ($t \in (-\varepsilon, \varepsilon)$) of ϕ such that $\phi_t = \phi$ outside a compact set $K \subset \subset \phi^{-1}(U)$ and $\phi_t(K) \subset U$; denote $\psi(y) = \psi^j(y) \otimes W_j(\phi(y))$ for each $y \in U$, then we define $\psi_t(y) = \psi^j(y) \otimes W_j(\phi_t(y))$. Denote $\xi = \frac{d\phi_t}{dt}|_{t=0}$. Obviously

$$\frac{dL}{dt}\Big|_{t=0} = \int_M \frac{d}{dt}\Big|_{t=0} |d\phi_t|^2 + \int_M \frac{d}{dt}\Big|_{t=0} \langle \psi, \mathcal{D}\psi \rangle = I + II, \quad (2.6)$$

and

$$I = -2 \int_M \langle \xi, \tau(\phi) \rangle. \quad (2.7)$$

Here $\tau(\phi)$ denotes the tension field of ϕ . Since $\psi = \psi^j \otimes W_j$, we have

$$\mathcal{D}\psi = \beta \psi^k \otimes W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} W_k, \quad (2.8)$$

where β denotes the usual Dirac operator. Then

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{D}\psi &= \beta \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} \nabla_{e_\alpha} W_k \\ &= e_\alpha \cdot \nabla_{e_\alpha} \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes R_{e_\alpha, \frac{\partial}{\partial t}} W_k \\ &= e_\alpha \cdot \nabla_{e_\alpha} (\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes R_{e_\alpha, \frac{\partial}{\partial t}} W_k \\ &= \mathcal{D}(\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes R_{\phi_*(e_\alpha), \xi}^N W_k. \end{aligned}$$

Please note that here and in the following text $R_{XY} = -[\nabla_X, \nabla_Y] + \nabla_{[X, Y]}$. In conjunction with (2.3), we have

$$\begin{aligned} II &= \int_M \left\langle \frac{d}{dt}\Big|_{t=0} \psi, \mathcal{D}\psi \right\rangle + \left\langle \psi, \frac{d}{dt}\Big|_{t=0} \mathcal{D}\psi \right\rangle \\ &= \int_M \left\langle \psi, \mathcal{D}(\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes R_{\phi_*(e_\alpha), \xi}^N W_k \right\rangle \\ &= \int_M \left\langle \mathcal{D}\psi, \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k \right\rangle + \int_M \left\langle \psi^j, e_\alpha \cdot \psi^k \right\rangle \left\langle W_j, R_{\phi_*(e_\alpha), \xi}^N W_k \right\rangle \\ &= \int_M \left\langle \psi^j, e_\alpha \cdot \psi^k \right\rangle \overline{\langle R_{\phi_*(e_\alpha), \xi}^N W_k, W_j \rangle} \\ &= - \int_M \left\langle \psi^j, e_\alpha \cdot \psi^k \right\rangle \left\langle R_{W_j, W_k}^N \phi_*(e_\alpha), \xi \right\rangle \end{aligned} \quad (2.9)$$

Substituting (2.7) and (2.9) into (2.6) yields

$$\frac{dL}{dt} = - \int_M \left\langle 2\tau(\phi) + \langle \psi^j, e_\alpha \cdot \psi^k \rangle R_{W_j, W_k}^N \phi_*(e_\alpha), \xi \right\rangle. \quad (2.10)$$

Thereby (2.4) follows. $\square$

Remark 2.1 The Euler–Lagrange equations of L was firstly derived in [1]. But our denotation is different.

3 Zeros of harmonic spinor fields

In this section, M and N are both oriented Riemannian surfaces. Then $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where

$$\Sigma^\pm M = \left\{ \xi \in \Sigma M : \sqrt{-1}e_1 \cdot e_2 \cdot \xi = \pm \xi \right\}. \quad (3.1)$$

(Here $\{e_1, e_2\}$ is an orthonormal basis of $T_{\pi(\xi)}M$, and π denotes the bundle projection of ΣM onto M .) In conjunction with $(\phi^{-1}TN)^\mathbb{C} = \phi^{-1}(T^{(1,0)}N) \oplus \phi^{-1}(T^{(0,1)}N)$, we have

$$\begin{aligned} \Sigma M \otimes (\phi^{-1}TN)^\mathbb{C} &= \left(\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)}N) \right) \oplus \left(\Sigma^+ M \otimes \phi^{-1}(T^{(0,1)}N) \right) \\ &\quad \oplus \left(\Sigma^- M \otimes \phi^{-1}(T^{(1,0)}N) \right) \oplus \left(\Sigma^- M \otimes \phi^{-1}(T^{(0,1)}N) \right). \end{aligned} \quad (3.2)$$

Denote by $\pi_1^+, \pi_0^+, \pi_1^-, \pi_0^-$ the projections of $\Sigma M \otimes (\phi^{-1}TN)^\mathbb{C}$ onto the subbundles, respectively. Let X be a tangent vector field on M , then ∇_X keeps $\Gamma(\Sigma^\pm M)$, $\Gamma(\phi^{-1}(T^{(1,0)}N))$ and $\Gamma(\phi^{-1}(T^{(0,1)}N))$ invariant, and $X \cdot \Sigma^\pm M \subset \Sigma^\mp M$; therefore

$$\begin{aligned} \mathcal{D}(\Gamma(\Sigma^\pm M \otimes \phi^{-1}(T^{(1,0)}N))) &\subset \Gamma(\Sigma^\mp M \otimes \phi^{-1}(T^{(1,0)}N)), \\ \mathcal{D}(\Gamma(\Sigma^\pm M \otimes \phi^{-1}(T^{(0,1)}N))) &\subset \Gamma(\Sigma^\mp M \otimes \phi^{-1}(T^{(0,1)}N)). \end{aligned} \quad (3.3)$$

Hence $\mathcal{D}\psi = 0$ yields that $\pi_1^+(\psi), \pi_0^+(\psi), \pi_1^-(\psi), \pi_0^-(\psi)$ are all harmonic spinor fields along ϕ .

Let $\psi \in \Gamma(\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)}N))$ be harmonic, we shall derive the equation of ψ in local complex coordinates.

Let $z = x + \sqrt{-1}y$, $w = u + \sqrt{-1}v$ be complex coordinates of M, N , respectively. Then the metric of M, N are of the forms $\lambda(z)|dz|^2, \rho(w)|dw|^2$, respectively. Denote

$$s = \{e_1, e_2\}, \quad \text{where } e_1 = \lambda^{-\frac{1}{2}} \frac{\partial}{\partial x}, \quad e_2 = \lambda^{-\frac{1}{2}} \frac{\partial}{\partial y}. \quad (3.4)$$

then s is a local tangent orthonormal frame field; i.e. s is a smooth section of $SOU, U \subset M$. Let $\tilde{s} \in \Gamma(\text{Spin}U)$ be a lift of s , i.e. $\eta \circ \tilde{s} = s$. Denote

$$\psi^+ = [\tilde{s}, \sigma], \quad \psi^- = e_1 \cdot \psi^+, \quad (3.5)$$

where σ is a unit vector in Σ_2^+ . (The definition of Σ_2^+ is introduced in [5, Proposition 2.16]). From (3.1),

$$\begin{aligned} e_1 \cdot \psi^+ &= \psi^-, \quad e_1 \cdot \psi^- = -\psi^+, \\ e_2 \cdot \psi^+ &= \sqrt{-1}\psi^-, \quad e_2 \cdot \psi^- = \sqrt{-1}\psi^+. \end{aligned} \quad (3.6)$$

And furthermore,

$$\begin{aligned} \frac{\partial}{\partial z} \cdot \psi^+ &= \lambda^{\frac{1}{2}} \psi^-, \quad \frac{\partial}{\partial z} \cdot \psi^- = 0, \\ \frac{\partial}{\partial \bar{z}} \cdot \psi^+ &= 0, \quad \frac{\partial}{\partial \bar{z}} \cdot \psi^- = -\lambda^{\frac{1}{2}} \psi^+. \end{aligned} \quad (3.7)$$

By the definition of the connection on ΣM (see [6, p. 110] and [5, Proposition 4.3]), we have

$$\begin{aligned}
 \partial \psi^+ &= e_\alpha \cdot \nabla_{e_\alpha} \psi^+ = \frac{1}{2} e_\alpha \cdot \langle \nabla_{e_\alpha} e_1, e_2 \rangle e_1 \cdot e_2 \cdot \psi^+ \\
 &= \frac{1}{2} \langle \nabla_{e_1} e_1, e_2 \rangle e_1 \cdot e_1 \cdot e_2 \cdot \psi^+ + \frac{1}{2} \langle \nabla_{e_2} e_1, e_2 \rangle e_2 \cdot e_1 \cdot e_2 \cdot \psi^+ \\
 &= \frac{1}{2} \frac{\partial \lambda^{-\frac{1}{2}}}{\partial y} (-\sqrt{-1}) \psi^- - \frac{1}{2} \frac{\partial \lambda^{-\frac{1}{2}}}{\partial x} \psi^- \\
 &= -\frac{\partial \lambda^{-\frac{1}{2}}}{\partial \bar{z}} \psi^-.
 \end{aligned} \tag{3.8}$$

Let f be a smooth function on U , such that $\psi = f \psi^+ \otimes \frac{\partial}{\partial w}$, then

$$\begin{aligned}
 0 &= \mathcal{D} \psi = \mathcal{D} \left(f \psi^+ \otimes \frac{\partial}{\partial w} \right) \\
 &= \partial \psi^+ \otimes f \frac{\partial}{\partial w} + \frac{2}{\lambda} \left(\frac{\partial}{\partial \bar{z}} \cdot \psi^+ \otimes \nabla_{\frac{\partial}{\partial \bar{z}}} \left(f \frac{\partial}{\partial w} \right) + \frac{\partial}{\partial \bar{z}} \cdot \psi^+ \otimes \nabla_{\frac{\partial}{\partial \bar{z}}} \left(f \frac{\partial}{\partial w} \right) \right) \\
 &= -\frac{\partial \lambda^{-\frac{1}{2}}}{\partial \bar{z}} \psi^- \otimes f \frac{\partial}{\partial w} + 2\lambda^{-\frac{1}{2}} \psi^- \otimes \left(\frac{\partial f}{\partial \bar{z}} \frac{\partial}{\partial w} + f \nabla_{\frac{\partial w}{\partial \bar{z}} \frac{\partial}{\partial w} + \frac{\partial \bar{w}}{\partial \bar{z}} \frac{\partial}{\partial w}} \frac{\partial}{\partial w} \right) \\
 &= 2\lambda^{-\frac{1}{2}} \left(\frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} f + \frac{\partial \log \rho}{\partial w} \frac{\partial w}{\partial \bar{z}} f + \frac{\partial f}{\partial \bar{z}} \right) \psi^- \otimes \frac{\partial}{\partial w}.
 \end{aligned} \tag{3.9}$$

Thereby we get the equation of f as follows:

$$\frac{\partial f}{\partial \bar{z}} + \left(\frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} + \frac{\partial \log \rho}{\partial w} \frac{\partial w}{\partial \bar{z}} \right) f = 0. \tag{3.10}$$

From it, we can prove the following proposition.

Proposition 3.1 *Let ψ be a harmonic spinor field along ϕ , which is a smooth map between two oriented Riemannian surfaces. Then $|\pi_1^+(\psi)|$ has isolated zeroes and the order of the zeros are well defined, unless $|\pi_1^+(\psi)|$ is identically zero. So are $|\pi_0^+(\psi)|$, $|\pi_1^-(\psi)|$ and $|\pi_0^-(\psi)|$.*

Proof As we have seen, $\pi_1^+(\psi)$ is harmonic whenever ψ is harmonic. Denote $\pi_1^+(\psi) = f \psi^+ \otimes \frac{\partial}{\partial w}$, then f satisfies (3.10). Denote

$$h(z) = \frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} + \frac{\partial \log \rho}{\partial w} \frac{\partial w}{\partial \bar{z}},$$

then $\frac{\partial f}{\partial \bar{z}} + hf = 0$. Let ζ be a local solution of $\frac{\partial \zeta}{\partial \bar{z}} = h$, then

$$\frac{\partial (fe^\zeta)}{\partial \bar{z}} = -hfe^\zeta + hfe^\zeta = 0.$$

i.e. fe^ζ is holomorphic. Hence the conclusion follows from the well-known fact that the zeros of a holomorphic function are isolated and the order of zeros are well defined, unless it is identically zero. And the proof for $|\pi_0^+(\psi)|$, $|\pi_1^-(\psi)|$ and $|\pi_0^-(\psi)|$ is similar. $\square$

4 Weitzenböck-type formulas and Bochner-type formulas

For a spinor field ψ along a map $\phi : M \rightarrow N$, where (M^m, h) is a Riemannian spin manifold, we can proceed as [1, Proposition 3.4] to have the following Weitzenböck-type formula.

$$\mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{4} S \psi + \frac{1}{2} \sum_{\alpha \neq \beta} e_\beta \cdot e_\alpha \cdot \psi^j \otimes R_{\phi_*(e_\alpha), \phi_*(e_\beta)}^N W_j. \quad (4.1)$$

Here $\psi = \psi^j \otimes W_j$, $\{e_\alpha\}$ is a local tangent orthonormal frame field on M such that $\nabla_{e_\alpha} = 0$ at the considered point, and S is the scalar curvature. When $m = 2$, $S = 2K_M$, where K_M denote the Gauss curvature of M , then

$$\mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{2} K_M \psi + e_2 \cdot e_1 \cdot \psi_j \otimes R_{\phi_*(e_1), \phi_*(e_2)}^N W_j. \quad (4.2)$$

Now we assume M, N are both oriented Riemannian surfaces; $\psi \in \Gamma(\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)} N))$, $\psi \neq 0$ and $\mathcal{D} \psi = 0$. For arbitrary $x \in M$ satisfying $\psi(x) \neq 0$, let $r > 0$ such that $\exp_x : B(r) \rightarrow M$ is injective; denote $U = \exp_x(B(r))$, then we can define a local section of $\Sigma^+ M$ (denoted by ψ^+) and a local section of $\phi^{-1}(T^{(1,0)} N)$ (denoted by W) on U , such that for any geodesic γ starting from x ,

$$\nabla_{\dot{\gamma}} \psi^+ = \nabla_{\dot{\gamma}} W = 0,$$

and $\langle \psi^+, \psi^+ \rangle = 1$, $\langle W, W \rangle = 1$; hence at x ,

$$\nabla_{e_\alpha} \psi^+ = \nabla_{e_\alpha} \nabla_{e_\alpha} \psi^+ = \nabla_{e_\alpha} W = \nabla_{e_\alpha} \nabla_{e_\alpha} W = 0.$$

ψ could locally be expressed by $\psi = f \psi^+ \otimes W$, then at x ,

$$\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi = \Delta f \psi^+ \otimes W. \quad (4.3)$$

Denote $W = \frac{\sqrt{2}}{2}(V_1 - \sqrt{-1}V_2)$, then $V_2 = J^N V_1$, where J^N is the complex structure on N , and $g(V_i, V_j) = \delta_{ij}$. Denote

$$\phi_* e_i = \phi_{ij} V_j,$$

then

$$\begin{aligned} R_{\phi_* e_1, \phi_* e_2}^N W &= R_{\phi_{11} V_1 + \phi_{12} V_2, \phi_{21} V_1 + \phi_{22} V_2}^N \left(\frac{\sqrt{2}}{2} (V_1 - \sqrt{-1} V_2) \right) \\ &= \det(\phi_{ij}) R_{V_1, V_2}^N \left(\frac{\sqrt{2}}{2} (V_1 - \sqrt{-1} V_2) \right) \\ &= \sqrt{-1} J(\phi) K_N W. \end{aligned} \quad (4.4)$$

Here $J(\phi)$ denotes the Jacobian of ϕ and K_N denotes the Gauss curvature of N . Substituting (4.3) and (4.4) into (4.2) yields

$$\begin{aligned} 0 &= \mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{2} K_M \psi + e_2 \cdot e_1 \cdot \psi^+ \otimes R_{\phi_* e_1, \phi_* e_2}^N W \\ &= \left(-\Delta f + \frac{1}{2} K_M f - K_N J(\phi) f \right) \psi^+ \otimes W. \end{aligned} \quad (4.5)$$

i.e.

$$\Delta f = \frac{1}{2} K_M f - K_N J(\phi) f \quad \text{at } x. \quad (4.6)$$

Furthermore,

$$\begin{aligned}\Delta|\psi|^2 &= \Delta|f|^2 = \bar{f}\Delta f + f\Delta\bar{f} + 2|\nabla f|^2 \\ &= K_M|\psi|^2 - 2K_N J(\phi)|\psi|^2 + 2|\nabla\psi|^2.\end{aligned}\quad (4.7)$$

From

$$\begin{aligned}0 &= e_1 \cdot \not{D}\psi = e_1 \cdot e_1 \cdot \nabla_{e_1}\psi + e_1 \cdot e_2 \cdot \nabla_{e_2}\psi \\ &= -(\nabla_{e_1}f)\psi^+ \otimes W + (\nabla_{e_2}f)e_1 \cdot e_2 \cdot \psi^+ \otimes W \\ &= -(\nabla_{e_1}f + \sqrt{-1}\nabla_{e_2}f)\psi^+ \otimes W\end{aligned}\quad (4.8)$$

we have

$$\nabla_{\bar{Z}}f = 0. \quad (4.9)$$

Here $Z = \frac{\sqrt{2}}{2}(e_1 - \sqrt{-1}e_2)$ and $\bar{Z} = \frac{\sqrt{2}}{2}(e_1 + \sqrt{-1}e_2)$, which satisfy $h(Z, \bar{Z}) = 1$, $h(Z, Z) = h(\bar{Z}, \bar{Z}) = 0$. Then $\nabla f = (\nabla_Z f)\bar{Z}$ and

$$|\nabla\psi|^2 = |\nabla f|^2 = |\nabla_Z f|^2. \quad (4.10)$$

Furthermore, from

$$\begin{aligned}\nabla|\psi|^2 &= \nabla|f|^2 = f\nabla\bar{f} + \bar{f}\nabla f \\ &= f(\nabla_{\bar{Z}}\bar{f})Z + \bar{f}(\nabla_Z f)\bar{Z}\end{aligned}\quad (4.11)$$

we arrive at

$$|\nabla|\psi|^2|^2 = 2|f|^2|\nabla_Z f|^2 = 2|\psi|^2|\nabla\psi|^2. \quad (4.12)$$

Substituting (4.12) into (4.7) yields

$$\Delta|\psi|^2 = K_M|\psi|^2 - 2K_N J(\phi)|\psi|^2 + \frac{|\nabla|\psi|^2|^2}{|\psi|^2} \quad (4.13)$$

at $x \in M$ whenever $\psi(x) \neq 0$.

And at last we derive the following Bochner-type formula

$$\Delta \log |\psi| = \frac{1}{2}K_M - K_N J(\phi). \quad (4.14)$$

Similarly, when $\psi \in \Sigma^+M \otimes \phi^{-1}(T^{(0,1)}N)$, $\Sigma^-M \otimes \phi^{-1}(T^{(1,0)}N)$ or $\Sigma^-M \otimes \phi^{-1}(T^{(0,1)}N)$, the corresponding Bochner-type formulas could be derived. We write those results as the following theorem.

Theorem 4.1 *Let M and N are both oriented Riemannian surfaces. If ψ is a harmonic spinor field along $\phi : M \rightarrow N$, then $\log |\pi_1^+(\psi)|$, $\log |\pi_0^+(\psi)|$, $\log |\pi_1^-(\psi)|$ and $\log |\pi_0^-(\psi)|$ satisfy Bochner-type formulas as follows:*

$$\Delta \log |\pi_1^+(\psi)| = \frac{1}{2}K_M - K_N J(\phi), \quad (4.15)$$

$$\Delta \log |\pi_0^+(\psi)| = \frac{1}{2}K_M + K_N J(\phi), \quad (4.16)$$

$$\Delta \log |\pi_1^-(\psi)| = \frac{1}{2}K_M + K_N J(\phi), \quad (4.17)$$

$$\Delta \log |\pi_0^-(\psi)| = \frac{1}{2}K_M - K_N J(\phi). \quad (4.18)$$

When M and N are both compact, since the zeros of $|\pi_1^+(\psi)|$ are isolated, there exist a finite number of zeros $p_1, \dots, p_k$ in M . And similarly the zeros of $|\pi_0^+(\psi)|$, $|\pi_1^-(\psi)|$ and $|\pi_0^-(\psi)|$ are finite. Integrating both side of (4.15)–(4.18) on M , in conjunction with divergence theorem and Gauss–Bonnet formula, we can proceed as [7, pp. 11–12] to get the proposition:

Theorem 4.2 *Let M and N are both compact oriented Riemannian surfaces, ψ is a harmonic spinor field along $\phi : M \rightarrow N$. If $|\pi_1^+(\psi)|$ is not identically zero, then*

$$\sum_{p \in M, |\pi_1^+ \psi|(p)=0} n_p^+ = g_M - 1 - \deg(\phi)(2g_N - 2). \quad (4.19)$$

If $|\pi_0^+(\psi)|$ is not identically zero, then

$$\sum_{p \in M, |\pi_0^+ \psi|(p)=0} m_p^+ = g_M - 1 + \deg(\phi)(2g_N - 2). \quad (4.20)$$

If $|\pi_1^-(\psi)|$ is not identically zero, then

$$\sum_{p \in M, |\pi_1^- \psi|(p)=0} n_p^- = g_M - 1 + \deg(\phi)(2g_N - 2). \quad (4.21)$$

If $|\pi_0^-(\psi)|$ is not identically zero, then

$$\sum_{p \in M, |\pi_0^- \psi|(p)=0} m_p^- = g_M - 1 - \deg(\phi)(2g_N - 2). \quad (4.22)$$

Here n_p^+ , m_p^+ , n_p^- , m_p^- are respectively the order of $|\pi_1^+(\psi)|$, $|\pi_0^+(\psi)|$, $|\pi_1^-(\psi)|$, $|\pi_0^-(\psi)|$ at p ; $\deg(\phi)$ denotes the degree of mapping; g_M and g_N are genus of M and N , respectively.

5 Proof of main theorems

In conjunction with Proposition 2.1 and Theorem 4.2, it is not difficult to obtain:

Theorem 5.1 *M and N are both compact oriented Riemannian surfaces, and (ϕ, ψ) is a Dirac-harmonic map from M to N . If $g_M = 0$ or $|g_M - 1| < |\deg(\phi)||2g_N - 2|$, then ϕ has to be a harmonic map.*

Proof If $g_M = 0$ or $|g_M - 1| < |\deg(\phi)||2g_N - 2|$, then $g_M - 1 - \deg(\phi)(2g_N - 2) < 0$ or $g_M - 1 + \deg(\phi)(2g_N - 2) < 0$. Hence from Theorem 4.2, either $\pi_1^+(\psi) = \pi_0^-(\psi) = 0$ or $\pi_0^+(\psi) = \pi_1^-(\psi) = 0$ could be obtained. If $\pi_1^+(\psi) = \pi_0^-(\psi) = 0$, then there exist smooth functions f and g , such that

$$\psi = f\psi^+ \otimes \frac{\partial}{\partial \bar{w}} + g\psi^- \otimes \frac{\partial}{\partial w},$$

where the definition of ψ^+ , ψ^- , $\frac{\partial}{\partial \bar{w}}$, $\frac{\partial}{\partial w}$ is similar to Sect. 3. Hence by (2.4),

$$\begin{aligned} \tau(\phi) &= -\frac{1}{2} \langle f\psi^+, e_\alpha \cdot f\psi^+ \rangle R_{\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial \bar{w}}}^N \phi_*(e_\alpha) - \frac{1}{2} \langle g\psi^-, e_\alpha \cdot g\psi^- \rangle R_{\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial w}}^N \phi_*(e_\alpha) \\ &\quad - \frac{1}{2} \langle f\psi^+, e_\alpha \cdot g\psi^- \rangle R_{\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial w}}^N \phi_*(e_\alpha) - \frac{1}{2} \langle g\psi^-, e_\alpha \cdot f\psi^+ \rangle R_{\frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}}^N \phi_*(e_\alpha) \\ &= 0. \end{aligned} \quad (5.1)$$

So ϕ is a harmonic map. When $\pi_0^+(\psi) = \pi_1^-(\psi) = 0$, the proof is similar. $\square$

It is well known that S^2 is biholomorphically isomorphic to $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$; hence in the following text we identify S^2 and $\mathbb{C}^*$. Let $h = \lambda(z)|dz|^2$ be a metric on $\mathbb{C}^*$; denote $\tilde{z} = z^{-1}$, then

$$\lambda(z)|dz|^2 = \lambda(\tilde{z}(z))|\tilde{z}|^{-4}d\tilde{z}^2;$$

hence $\lambda(\tilde{z}(z))|\tilde{z}|^{-4}$ is regular at $\tilde{z} = 0$; then there exists a constant $c > 0$, such that

$$\lim_{z \rightarrow \infty} \lambda(z)|z|^4 = c. \quad (5.2)$$

And the definition of e_1, e_2, ψ^+, ψ^- is similar to Sect. 3.

Theorem 5.2 *If $M = S^2 = \mathbb{C}^*$ equipped with metric $h = \lambda(z)|dz|^2$ and $N = S^2$ equipped with arbitrary metric, (ϕ, ψ) is a non-trivial Dirac-harmonic map from M to N , then ϕ has to be holomorphic or anti-holomorphic, ψ could be written in the form*

$$\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha); \quad (5.3)$$

and there exist two meromorphic function u_1, u_2 on $\mathbb{C}^*$ such that

$$\Psi = \bar{u}_1 \lambda^{\frac{1}{4}} \psi^+ + u_2 \lambda^{\frac{1}{4}} \psi^-; \quad (5.4)$$

if u_i ($i = 1$ or 2) has a pole of order k at $z_0 \in \mathbb{C}$, then $|d\phi|(z_0) = 0$ and the order of $|d\phi|$ at z_0 is no less than k ; if ∞ is a pole of order $k \geq 2$, then $|d\phi|(\infty) = 0$ and the order of $|d\phi|$ at ∞ is no less than $k - 1$. And vice versa.

Proof By Theorem 5.1, ϕ has to be a harmonic map. It is well known that when $\deg \phi = 0$, ϕ is a constant mapping; when $\deg \phi \geq 1$, ϕ is holomorphic; and when $\deg \phi \leq -1$, ϕ is anti-holomorphic (cf. [7, pp. 11–12]). From Theorem 4.2, when $\deg \phi = 0$, $|\pi_1^+(\psi)|, |\pi_0^+(\psi)|, |\pi_1^-(\psi)|, |\pi_0^-(\psi)|$ are all identically zero, hence $\psi = 0$; it is a trivial solution of (2.3) and (2.4). When $\deg \phi \geq 1$, we have $|\pi_0^+(\psi)| = |\pi_1^-(\psi)| = 0$; since ϕ is holomorphic, $\phi_*(e_1) - \sqrt{-1}\phi_*(e_2) \in T^{(1,0)}N$, $\phi_*(e_1) + \sqrt{-1}\phi_*(e_2) \in T^{(0,1)}N$, there exist two functions f, g (possibly with isolated singularities), such that

$$\psi = f\psi^+ \otimes \left(\phi_*(e_1) - \sqrt{-1}\phi_*(e_2) \right) + g\psi^- \otimes \left(\phi_*(e_1) + \sqrt{-1}\phi_*(e_2) \right). \quad (5.5)$$

From (3.6), it is easy to obtain

$$\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha); \quad (5.6)$$

where $\Psi = g\psi^+ - f\psi^-$. When $\deg \phi \leq -1$, similarly we can construct a spinor Ψ (possibly with isolated singularities) satisfying (5.6).

By [1, Proposition 2.2], Ψ is a twistor spinor, i.e.

$$\nabla_v \Psi + \frac{1}{2} v \cdot \not\partial \Psi = 0 \quad (5.7)$$

for any $v \in T_p S^2$, where p is an arbitrary regular point of Ψ . If $\Psi \in \Gamma(\Sigma^+ M)$, then from (3.1), (5.7) is equivalent to

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \Psi = 0. \quad (5.8)$$

Denote

$$\Phi_0^+ = \lambda^{\frac{1}{4}} \psi^+, \quad (5.9)$$

then from

$$\begin{aligned}
 \nabla_{\frac{\partial}{\partial \bar{z}}} \psi^+ &= \frac{1}{2} \langle \nabla_{\frac{\partial}{\partial \bar{z}}} e_1, e_2 \rangle e_1 \cdot e_2 \cdot \psi^+ \\
 &= -\frac{1}{4} \sqrt{-1} \lambda^{\frac{1}{2}} \langle \nabla_{e_1 - \sqrt{-1} e_2} e_1, e_2 \rangle \psi^+ \\
 &= -\frac{1}{4} \sqrt{-1} \lambda^{\frac{1}{2}} \left(\frac{\partial(\lambda^{-\frac{1}{2}})}{\partial y} + \sqrt{-1} \frac{\partial(\lambda^{-\frac{1}{2}})}{\partial x} \right) \psi^+ \\
 &= -\frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} \psi^+,
 \end{aligned} \tag{5.10}$$

we have

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \Phi_0^+ = \frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} \lambda^{\frac{1}{4}} \psi^+ - \frac{1}{4} \frac{\partial \log \lambda}{\partial \bar{z}} \lambda^{\frac{1}{4}} \psi^+ = 0. \tag{5.11}$$

Let u_1 be a function on C^* (possibly with isolated singularities) such that

$$\Psi = \bar{u}_1 \Phi_0^+,$$

then it is easy to obtain $\frac{\partial u_1}{\partial \bar{z}} = 0$. Similarly, if $\Psi \in \Gamma(\Sigma^- M)$ is a twistor spinor, then we could obtain

$$\Psi = u_2 \Phi_0^-,$$

where u_2 is a meromorphic function and $\Phi_0^- = \lambda^{\frac{1}{4}} \psi^-$. Thereby (5.4) follows. The last statement is followed from $\lambda > 0$ on $\mathbb{C}$ and (5.2).

On the other hand, let $\psi = e_\alpha \cdot \psi \otimes \phi_*(e_\alpha)$, where ϕ is holomorphic or anti-holomorphic, and Ψ is a twistor spinor (possibly with isolated singularities) satisfying (5.4). Denote by $z_1, \dots, z_l \in \mathbb{C}^*$ the singularities of Ψ , then it is easily to check that (ϕ, ψ) satisfies (2.3) and (2.4) on $\mathbb{C}^* - \{z_1, \dots, z_l\}$, and by the assumption on the poles of u_i and the zeros of $|d\phi|$, $|\psi|$ is bounded on $\mathbb{C}^* - \{z_1, \dots, z_l\}$. Hence the energy of (ϕ, ψ) on $\mathbb{C}^*$

$$E(\phi, \psi, \mathbb{C}^*) = \int_{\mathbb{C}^*} (|d\phi|^2 + |\psi|^4)$$

is finite. The removable singularity theorem (see [1]) yields that (ϕ, ψ) is a Dirac-harmonic map from M to N . $\square$

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