A classification of generic families of control-affine systems and their bifurcations

Marek W. Rupniewski · Witold Respondek

Received: 17 November 2008 / Accepted: 15 May 2010 / Published online: 5 June 2010
© The Author(s) 2010. This article is published with open access at Springerlink.com

Abstract We study control-affine systems with $n - 1$ inputs evolving on an n -dimensional manifold for which the distribution spanned by the control vector fields is involutive and of constant rank (equivalently, they may be considered as 1-dimensional systems with $n - 1$ inputs entering nonlinearly). We provide a complete classification of such generic systems and their one-parameter families. We show that a generic family for $n > 2$ is equivalent (with respect to feedback or orbital feedback transformations) to one of nine canonical forms which differ from those for $n = 2$ by quadratic terms only. We also describe all generic bifurcations of 1-parameter families of systems of the above form.

Keywords Feedback equivalence · Bifurcation · Control system · 1-Parameter family · Involutive distributions

This work was partially supported by Ministry of Research and Higher Education grant N201 039 32/2703 and by the European Community Marie Curie Fellowship through the CTS contract HMPT-CT-2001-00278.

M. W. Rupniewski
Institute of Electronic Systems, Warsaw University of Technology,
ul. Nowowiejska 15/19, 00-665 Warsaw, Poland
e-mail: m.rupniewski@ise.pw.edu.pl

W. Respondek (✉)
Laboratoire de Mathématique EA3226, INSA-Rouen, Pl. Emile Blondel,
76801 Saint-Etienne-du-Rouvray, France
e-mail: Witold.Respondek@insa-rouen.fr

1 Introduction

In this paper we deal with nonlinear control systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \sum_{i=1}^m u_i g_i(\mathbf{x}), \quad (\Sigma)$$

on an n -dimensional manifold M , and with 1-parameter families of such systems

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu) + \sum_{i=1}^m u_i g_i(\mathbf{x}, \mu) = f_\mu(\mathbf{x}) + \sum_{i=1}^m u_i g_{i,\mu}(\mathbf{x}), \quad (\Sigma_\mu)$$

in which case we replace the vector fields $f, g_1, \dots, g_m$ by 1-parameter families of vector fields $f_\mu, g_{1,\mu}, \dots, g_{m,\mu}$, respectively. Moreover, we assume throughout the paper that $m = n - 1$ and that the distribution $\text{span}\{g_1, \dots, g_{n-1}\}$ (respectively, $\text{span}\{g_{1,\mu}, \dots, g_{n-1,\mu}\}$) is involutive. We denote such class of systems by $\mathcal{A}_{\text{inv}}^{n,n-1}$ (for 1-parameter families we write $\mathcal{FA}_{\text{inv}}^{n,n-1}$). In the paper we provide a complete local classification of generic systems and 1-parametric families belonging to $\mathcal{A}_{\text{inv}}^{n,n-1}$ and $\mathcal{FA}_{\text{inv}}^{n,n-1}$, respectively. The classification is given with respect to feedback and orbital feedback equivalence. This generalises earlier results of Jakubczyk and Respondek (systems for $n = 2, m = 1$ in [24, 25] and families for $n = 2, m = 1$ in [28]) and of Jakubczyk (systems for $n = 2, m = 1$ and $m = n - 1$ in [19]).

We prove that a generic system (Σ) is locally feedback (or orbital feedback) equivalent to one of five canonical forms (in the case of feedback equivalence, one of them exhibiting a real continuous invariant and another a functional invariant). A generic family is equivalent either to one of those five canonical forms (meaning that we can get rid of the parameter via feedback) or to one of four canonical forms depending explicitly on the parameter. The classification Theorems 1 and 2, respectively for systems and for families, form our first main result of the paper. They are natural generalisations of the corresponding results for planar systems [24] and families [28] and differ from them just by adding quadratic terms.

Feedback classification and orbital feedback classification of control-affine systems on the plane is almost complete: geometry and invariants are well understood and normal forms (for generic and for analytic systems) are known, see [19, 24, 25, 27, 28, 36, 40], see also [14] and [11]. A study of feedback equivalence in the general case (arbitrary dimension n and arbitrary number of controls m) is less systematic, although many problems have been studied and solved: equivalence to a linear system [17, 23], orbital equivalence to a linear system [37, 42], classification of quadratic systems [9], classification of generic systems for $m = 2, n = 3$ in [41], classification of all simple germs (possible only if $m = n - 1$) in [47], equivalence to triangular forms [12, 38], formal normal and canonical forms for $m = 1$ [29, 30, 32, 46] and the survey [39], and for an arbitrary m in [45], equivalence and invariants studied via hamiltonian lifting [4, 8, 20, 21], invariants obtained via Cartan equivalence method [15, 16], and others.

The classification problem we study is equivalent to the local classification of 1-dimensional nonlinear systems (or 1-parameter families) with $n - 1$ controls entering nonlinearly as well as to the local classification of 1-parameter function deformations with respect to appropriate group of morphisms (see Sect. 3 for details).

The classification of families allows us to study bifurcations. Roughly speaking, a parameterised family of dynamical systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$$

bifurcates if the trajectories of the system change, topologically, their behaviour when we pass through a nominal value of the parameter. The theory of bifurcations of dynamical systems contains a huge variety of methods and results (including classifications), see, e.g., [1, 18, 34]. In the case of parameterised control systems of the general form

$$\dot{\mathbf{x}} = f(\mathbf{x}, u, \mu),$$

or of the control-affine form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu) + \sum_{i=1}^m u_i g_i(\mathbf{x}, \mu), \quad (\Sigma_\mu)$$

an analogous definition does not seem to be appropriate since the set of all trajectories (corresponding to all controls u) is too rich. Because of this, the following notion was introduced in [28] (see also [26, 27]). We attach to a control system three basic invariant objects corresponding to special trajectories: the equilibrium set E corresponding to stationary trajectories, the critical set C corresponding to critical trajectories and the canonical foliation $\mathcal{G}$ corresponding to fast trajectories (see Sect. 4 for precise definitions and interpretations). Those three basic invariants are crucial in feedback classification (compare Theorems 1 and 2) and we will also use them to define bifurcations. We say that a family of control systems bifurcates if the triple of basic invariants changes topologically when we pass through a nominal parameter value (see Sect. 6 for a precise statement).

Our second main result, Theorem 3 (see also its simplified formulation as Result 3 below), describes all generic bifurcations of 1-parameter families.

The study of bifurcations of control systems was initiated, in a different setting, by Abed and Fu [2, 3] for systems of the form $\dot{\mathbf{x}} = f(\mathbf{x}, u, \mu)$. They assumed that the uncontrolled system, defined by taking $u = 0$, undergoes a bifurcation at $\mu = \mu_0$ and they studied stabilizability of the system by quadratic and cubic feedbacks. Our approach is close in spirit to that of Kang [31] who studied bifurcations of the set of equilibria and of the linear approximation of the system at an equilibrium. A control system does not need a parameter to bifurcate, the control can play the same role. This point of view is presented by Krener et al. [33]. They consider systems $\dot{\mathbf{x}} = f(\mathbf{x}, u)$, for which the set of equilibria is conveniently parameterised by the control u . According to their definition, a control bifurcation takes place at an equilibrium if the linear approximation of the system loses stabilizability.

Clearly, the set of equilibria, the set of (time) critical trajectories and the foliation of the distribution Δ_μ spanned by the control vector fields play an important role in many control problems, not only in studying bifurcation and feedback classification. In the planar case, which was a starting point for our study, both the equilibria set and the critical set play a crucial role in constructing the time-optimal synthesis on $\mathbf{R}^2$ for the system $\dot{\mathbf{x}} = f(\mathbf{x}) + ug(\mathbf{x})$ with constraints $|u| \leq 1$ (see [5, 6, 10, 11, 43, 44]). In studying generic controllability problems and singularities of the boundary of the reachable set for such systems [13, 14], the equilibrium set is important as well as two foliations of oriented orbits of the vector fields, X_+ and X_- , $X_\pm = f \pm g$, called limiting directions [14]. We send the reader to monographs [14] and [11] summarising deep studies of planar systems from, respectively, controllability and optimal control points of view.

Main results of our paper can be summarized as follows.

Result 1 (cf. Theorem 1) A generic control system $\Sigma \in \mathcal{A}_{\text{inv}}^{n,n-1}$ on a manifold M is locally feedback equivalent around any fixed point $q \in M$ to the system

$$\dot{\mathbf{x}} = F(\mathbf{x}) \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + \sum_{i=2}^{n-1} u_i \frac{\partial}{\partial z_i},$$

where $\mathbf{x} = (x, y, z_2, \dots, z_{n-1})$ are local coordinates on M and the function F is equal to exactly one of the following:

1. $1 + y$,
2. y ,
3. $1 \pm y^2 + \sum_{i=2}^{n-1} \pm z_i^2$,
4. $\lambda x + y^2 + \sum_{i=2}^{n-1} \pm z_i^2$, $\lambda \neq 0$,
5. $y^3 + xy + a(x) + \sum_{i=2}^{n-1} \pm z_i^2$, $a(0) \neq 0$,

If the orbital feedback equivalence is considered, then we can replace both λ and $a(x)$ by ± 1 .

Result 2 (cf. Theorem 2) A generic family $\Sigma_\mu \in \mathcal{FA}_{\text{inv}}^{n,n-1}$ on a manifold M , $\mu \in \mathbf{R}$, is locally orbital feedback equivalent at any $(q, \mu) \in M \times \mathbf{R}$ to the family

$$\dot{\mathbf{x}} = F_\mu(\mathbf{x}) \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + \sum_{i=2}^{n-1} u_i \frac{\partial}{\partial z_i},$$

where $\mathbf{x} = (x, y, z_2, \dots, z_{n-1})$ are local coordinates on M and the family of functions F_μ is equal either to one of functions F given by 1.–5. of Result 1 (with λ and $a(x)$ replaced by ± 1) or to exactly one of the following families:

6. $x^2 - \mu \pm y^2 + \sum_{i=2}^{n-1} \pm z_i^2$,
7. $\pm y^3 + y(x^2 - \mu) + 1 + \sum_{i=2}^{n-1} \pm z_i^2$,
8. $y^3 + (x - \mu)y \pm x + \sum_{i=2}^{n-1} \pm z_i^2$,
9. $y^4 + (\pm x - \mu)y^2 + xy + a(x, \mu) + \sum_{i=2}^{n-1} \pm z_i^2$, $a(0, 0) \neq 0$,

Result 3 (cf. Theorem 3) A generic family $\Sigma_\mu \in \mathcal{FA}_{\text{inv}}^{n,n-1}$ bifurcates at (q, μ) if and only if it is locally equivalent at (q, μ) to one of the models 6.–9. of Result 2. Moreover, the models 6., 7., 8. and 9. correspond, respectively, to bifurcations of the equilibrium set E_μ (n non-equivalent bifurcations), of the critical set C_μ (two non-equivalent bifurcations), of the pair $(E_\mu, \mathcal{G}_\mu)$ (in which case also the pair (E_μ, C_μ) bifurcates), and of the pair $(C_\mu, \mathcal{G}_\mu)$.

The paper is organised as follows. In the next section, we describe systems and families we are dealing with and define feedback and orbital feedback equivalence. In Sect. 3 we give another formulations of the problems we consider. In Sect. 4, we introduce feedback invariant conditions which describe singularity classes of the classifications. We give our main results, classification of systems and families, in Sect. 5. In the same section we present also a diagram that illustrates consecutive degenerations that may occur for a system or a family. In Sect. 6, we define bifurcations, give all generic bifurcations, and analyse them in the case $n = 3$. Our main results are proved in Sect. 7. Finally, in Sect. 8 we study relations between generic systems and their restrictions to surfaces. In particular, we give necessary and sufficient conditions under which the original system is of the same type (in the sense of Result 1) as its restriction.

2 Notations

We consider a nonlinear control system, denoted by $f + (g_1, \dots, g_{n-1})$, of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \sum_{i=1}^{n-1} u_i g_i(\mathbf{x}), \quad (\Sigma)$$

where $u = (u_1, \dots, u_{n-1})$ is the control which takes values in $\mathbf{R}^{n-1}$, $\mathbf{x} = (x_1, \dots, x_n)$ are local coordinates on a smooth n -dimensional manifold M , f and $g_1, \dots, g_{n-1}$ are C^∞ -smooth vector fields on M . We denote the class of control systems of that type by $\mathcal{A}^{n,n-1}(M)$. If we deal with $M = \mathbf{R}^n$, which we can assume when studying local problems, we will write $\mathcal{A}^{n,n-1}$ for simplicity.

Denote by Δ the distribution spanned by the vector fields $g_1, \dots, g_{n-1}$. If the choice of the generators $g_1, \dots, g_{n-1}$ of Δ is irrelevant, we will identify the system (Σ) with the affine distribution $f + \Delta$.

Consider two systems $\Sigma^i = f^i + \Delta^i$ defined on M^i , $i = 1, 2$. They are said to be *feedback equivalent* if there exists a diffeomorphism $\phi: M^1 \rightarrow M^2$ such that

$$\phi_*(f^1 + \Delta^1) = f^2 + \Delta^2.$$

If the distributions Δ^1 and Δ^2 are involutive (see below), the systems are said to be *orbital feedback equivalent* if there exists a scalar function on M^1 , $h: M^1 \rightarrow \mathbf{R}$, constant on the leaves of the foliation defined by Δ^1 and such that $h\Sigma^1 = hf^1 + \Delta^1$ is feedback equivalent to Σ^2 . Two systems Σ^1 and Σ^2 are called *locally feedback equivalent* (respectively, locally orbital equivalent) at $q^1 \in M^1$, $q^2 \in M^2$, respectively, if

there exist neighbourhoods Ω^1 of q^1 and Ω^2 of q^2 such that Σ^1 restricted to Ω^1 and Σ^2 restricted to Ω^2 are feedback equivalent (respectively, orbital feedback equivalent).

In other words, local feedback equivalence of Σ^1 and Σ^2 means that there exist smooth functions α_i and β_{ij} , $1 \leq i, j \leq m$, on the neighbourhood Ω^1 , such that the diffeomorphism ϕ satisfies

$$\phi_* \left(f^1 + \sum_{i=1}^m \alpha_i g_i^1 \right) = f^2 \text{ and } \phi_* \left(\sum_{i=1}^m \beta_{ij} g_i^1 \right) = g_j^2.$$

In the case of local orbital feedback equivalence, the first equality is replaced by

$$\phi_* \left(hf^1 + \sum_{i=1}^m \alpha_i g_i^1 \right) = f^2$$

while the second equality remains unchanged.

In the paper we assume that Δ is of constant rank $(n-1)$ and involutive. In terms of vector fields it means that $g_1, \dots, g_{n-1}$ are linearly independent and $[g_i, g_j] \in \Delta$ for all $i, j = 1, \dots, n-1$. We denote the class of systems of that type by $\mathcal{A}_{\text{inv}}^{n,n-1}(M)$ or simply $\mathcal{A}_{\text{inv}}^{n,n-1}$. If $f + \Delta \in \mathcal{A}_{\text{inv}}^{n,n-1}(M)$ then, by the Frobenius theorem, Δ defines an $(n-1)$ -dimensional foliation on M . Around each point of M , there exist local coordinates $\mathbf{x} = (x_1, \dots, x_n)$, such that Δ is spanned by $\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$. Therefore, whenever it is convenient, we may bring locally, via an appropriate feedback transformation, any system from the class $\mathcal{A}_{\text{inv}}^{n,n-1}$ into the pre-normal form

$$\dot{\mathbf{x}} = F(\mathbf{x}) \frac{\partial}{\partial x_1} + \sum_{i=2}^n u_{i-1} \frac{\partial}{\partial x_i}, \quad (\Sigma_{\text{pnf}})$$

where F is a smooth function on M .

For the purpose of stating properties of a control system in a coordinate independent way we will use also the language of differential forms. If Δ is a distribution of constant corank 1 on M , then it can be given locally by the Pfaffian equation $\omega = 0$, where ω is a non-vanishing differential 1-form satisfying $\omega(g_i) = 0$, $i = 1, \dots, n-1$. Such a form is unique up to multiplication by a non-vanishing function. The involutiveness of Δ means, that

$$\omega \wedge d\omega \equiv 0.$$

For a control system $\Sigma = f + (g_1, \dots, g_{n-1})$ we will also use the notation (f, ω) , where ω is defined as above. Let us remark that, in this setting, a function h is constant on the leaves of the foliation Δ (cf. orbital feedback equivalence definition) if and only if

$$dh \wedge \omega \equiv 0.$$

In the paper we consider also 1-parameter families of control systems from the class $\mathcal{A}_{\text{inv}}^{n,n-1}$, i.e., families of the form

$$\dot{\mathbf{x}} = f_\mu(\mathbf{x}) + \sum_{i=1}^{n-1} u_i g_{i,\mu}(\mathbf{x}), \quad (\Sigma_\mu)$$

where $f_\mu, g_{1,\mu}, \dots, g_{n-1,\mu}$ are smooth families of vector fields, parameterised by $\mu \in \mathbf{R}$. The class of such families will be denoted by $\mathcal{FA}_{\text{inv}}^{n,n-1}$. We say that two families Σ_μ^1 and Σ_μ^2 are feedback (respectively, orbital feedback) equivalent if there exists a diffeomorphism

$$\Phi(\mathbf{x}, \mu) = (\phi(\mathbf{x}, \mu), \theta(\mu)): M^1 \times \mathbf{R} \rightarrow M^2 \times \mathbf{R}$$

(respectively, a diffeomorphism $\Phi(\mathbf{x}, \mu)$ and a family of smooth functions $h(\mathbf{x}, \mu)$, constant on the leaves of the foliation of Δ_μ^1) such that, for each value of μ , feedback equivalence of the two systems Σ_μ^1 and $\Sigma_{\theta(\mu)}^2$ is established by the diffeomorphism $\phi_\mu = \phi(\cdot, \mu)$ (resp. by the diffeomorphism ϕ_μ and the function $h_\mu = h(\cdot, \mu)$).

3 Another formulation of the problem

Consider a nonlinear 1-dimensional control system with $n - 1$ controls entering nonlinearly:

$$\frac{dx}{dt} = \dot{x} = f(x, u), \quad x \in \mathbf{R}, u \in \mathbf{R}^{n-1}. \quad (\Sigma_N)$$

A general feedback transformation of the above system is a diffeomorphism

$$\Phi: \mathbf{R} \times \mathbf{R}^{n-1} \rightarrow \mathbf{R} \times \mathbf{R}^{n-1}$$

of the form

$$(y, v) = \Phi(x, u) = (\phi(x), \psi(x, u)).$$

The map Φ transforms (Σ_N) into

$$\dot{y} = \tilde{f}(y, v), \quad (1)$$

where

$$\frac{\partial \phi}{\partial x}(x) f(x, u) = \tilde{f}(\phi(x), \psi(x, u)). \quad (2)$$

and the two systems are called general feedback equivalent (compare with the control-affine case where $u = \alpha(x) + \beta(x)v$). A general orbital feedback transformation

consists additionally of a positive function $h: \mathbf{R} \rightarrow \mathbf{R}$ that changes the time scale according to $\frac{dt}{d\tau} = h(y(\tau))$ and therefore transform system (Σ_N) into:

$$\frac{dy}{d\tau} = h(y) \tilde{f}(y, v), \quad (3)$$

where f and $\tilde{f}$ are related via (2). Notice that now the time rescaling is defined by an arbitrary positive function h while in the control-affine case the function is assumed to be constant on the leaves of the foliation of the distribution Δ .

We may extend the system (Σ_N) to a system Σ_{ext} from $\mathcal{A}_{inv}^{n,n-1}$ by introducing new state variables $x_2 = u_1, \dots, x_n = u_{n-1}$, and new controls $w_1 = \dot{u}_1, \dots, w_{n-1} = \dot{u}_{n-1}$:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \dots, x_n), \\ \dot{x}_2 &= w_1, \\ &\dots \\ \dot{x}_n &= w_{n-1}, \end{aligned} \quad (\Sigma_{ext})$$

where we write x_1 instead of x . Vice versa, by the inverse transformation (that identifies $x_2, \dots, x_n$ in (Σ_{pnf}) as the controls) we may bring any system that is in the pre-normal form (Σ_{pnf}) to the form (Σ_N) . In [19] it is shown that the problems of feedback and orbital feedback classification of systems from $\mathcal{A}_{inv}^{n,n-1}$ and systems of the form (Σ_N) are essentially the same. Indeed, two control systems Σ_N^1 and Σ_N^2 are (locally) general feedback equivalent if and only if their respective extensions Σ_{ext}^1 and Σ_{ext}^2 are (locally) feedback equivalent. Analogous statement holds for (local) orbital feedback.

The same remains true for 1-parameter families of control systems, where a 1-parameter family of the form:

$$\dot{x} = f(x, u, \mu), \quad x \in \mathbf{R}, u \in \mathbf{R}^{n-1}, \mu \in \mathbf{R} \quad (\mathcal{F}\Sigma_N)$$

is mapped by the feedback transformation

$$\begin{aligned} \Psi: \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} &\rightarrow \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R}, \\ (y, v, \varepsilon) &= \Psi(x, u, \mu) = (\phi(x, \mu), \psi(x, u, \mu), \theta(\mu)) \end{aligned}$$

into

$$\dot{y} = \tilde{f}(y, v, \varepsilon),$$

where

$$\frac{\partial \phi}{\partial x}(x, \mu) f(x, u, \mu) = \tilde{f}(\phi(x, \mu), \psi(x, u, \mu), \theta(\mu)). \quad (4)$$

For orbital feedback transformations we consider additionally a positive function $h: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, which appears as the factor $h(\varepsilon, y)$ on the right hand side of the above formula.

Eventually, we may treat the classification problem we consider as a problem concerning 1-parameter function deformations. To this end, consider the vector field f appearing in (Σ_N) as a 1-parameter deformation of the function $f(0, u)$. Then it is natural to interpret the diffeomorphism Φ in the case of feedback transformation (the pair (Φ, h) in the orbital case, respectively) as a morphism whose action on f is determined by (2) (by (2) with $\tilde{f}$ multiplied additionally by h , respectively). Analogously, the problem for 1-parameter families $(\mathcal{F}\Sigma_N)$ may be replaced by the problem of classification of 1-parameter μ -deformations of 1-parameter x -deformations of a function with respect to the action of morphisms Ψ , transforming a given deformation $f(x, u, \mu)$ according to (4) (multiplied additionally by $h(\varepsilon, y)$ in the case of the orbital feedback). We send the reader to [27] for details on the deformation based approach in the case $m = 1$ and to [22] for the general case.

4 Non-degeneracy conditions

One of the most natural feedback invariants of a control-affine system (or a family of such systems) is the *equilibrium set* E . For a system $f + \Delta$, the set E consists of points at which $f \in \Delta$. Since we deal with a local problem around a fixed point q , we have two complementary subclasses of systems depending on whether q belongs to E or not. If we represent the system by (f, ω) , then we can equivalently write

$$E = \{q \in M \mid f \lrcorner \omega(q) = 0\},$$

where $X \lrcorner \vartheta(\cdot) = \vartheta(X, \cdot)$ denotes the interior product of a vector field X with a differential form ϑ . Following the notation of [24, 28], we introduce the function

$$e: M \rightarrow \mathbf{R}, \quad e = f \lrcorner \omega.$$

Henceforth we will write $e(q) = 0$ to state that the point q belongs to E and $e(q) \neq 0$ in the opposite case.

Another important invariant set defined for control systems is the *critical set* C

$$C = \{q \in M \mid \eta(q) = 0\} = \{q \in M \mid [g, f] \lrcorner \omega(q) = 0, \forall g \in \Delta\}, \quad (5)$$

where

$$\eta = \omega^f \wedge \omega \text{ and } \omega^f = L_f \omega,$$

the latter being the Lie derivative of ω in the direction of f . The points of C are called critical points of the control system.

The two just defined invariant sets have very clear control theoretic interpretation. The set E consists of the potential equilibria of the system, that is, around any point

$q \in E$ we can find smooth functions α_i such that $f + \sum_{i=1}^m \alpha_i g_i$ has an equilibrium at q . The set E is defined by a single equation $f \lrcorner \omega = 0$ so typically it is a hypersurface in M . It has a particularly simple description for the pre-normal form Σ_{pnf} , namely, $E = \{F(\mathbf{x}) = 0\}$.

The critical set C consists of points, where the motion transversal to the leaves of the foliation $\mathcal{G}$ defined by Δ admits its extremal (critical) values. In particular, all time-minimal and time-maximal trajectories are entirely contained in the critical set C . This set is defined by $n - 1$ equations $[g_i, f] \lrcorner \omega(q) = 0$, $1 \leq i \leq n - 1$, so typically it is a curve and has a particularly simple description for the pre-normal form Σ_{pnf} , namely, $C = \{\frac{\partial F}{\partial x_i}(\mathbf{x}) = 0, 2 \leq i \leq n\}$. Time optimal trajectories correspond to the hessian of F being definite.

The curves tangent to the integrable distribution Δ , i.e., contained in the leaves of the foliation $\mathcal{G}$ defined by Δ , can be approximated by trajectories of the system Σ if we take sufficiently large values of the control u in order to neglect the effect of the drift f . Hence the name “fast trajectories”.

4.1 Degeneracies at critical points

Let us denote a fixed point belonging to C by q (thus we have $\eta(q) = 0$). At that point the following mapping $\theta: T_q M \rightarrow \Lambda^2(T_q^* M)$ is defined

$$\theta(v) = L_g \eta(q),$$

where g is an arbitrary vector field extending $v \in T_q M$ (the definition does not depend on the choice of g since $\eta(q) = 0$).

Introduce the following subset G of vector fields,

$$G = \{g \in \Delta \mid g(q) \in \ker \theta, g(q) \neq 0\}.$$

Vector fields belonging to G will be depicted by an extra bar over them, e.g., $\bar{g}$. In order to present another way of defining this set, consider the following bilinear symmetric form τ defined on $\Delta(q) \times \Delta(q)$

$$\tau(v_1, v_2) = L_{g_2} \left(g_1 \lrcorner \omega^f \right) (q) = [g_2, [g_1, f]] \lrcorner \omega(q),$$

where $g_i \in \Delta$ are arbitrary vector fields extending $v_i \in \Delta(q)$.

For a system in the pre-normal form Σ_{pnf} , we have $\tau(v_1, v_2) = v_1^T H v_2$, where $H = \frac{\partial^2 F}{\partial x_i \partial x_j}(q)$, $2 \leq i, j \leq n$ (recall that H being definite corresponds to time optimality of the critical trajectories contained in C).

Let us check that the above two variants of the definition of τ are equivalent and do not depend on the extensions of v_1, v_2 . We have

$$g_1 \lrcorner L_f \omega = L_f(g_1 \lrcorner \omega) - (L_f g_1) \lrcorner \omega.$$

Hence, by the fact that $g_1 \lrcorner \omega \equiv 0$, we get $g_1 \lrcorner L_f \omega = (L_{g_1} f) \lrcorner \omega$ and thus the equivalence of the definitions. The second variant of the definition is symmetric with respect to g_1, g_2 . Indeed, we have $[g_2, [g_1, f]] = [g_1, [g_2, f]] + [[g_2, g_1], f]$ and $[[g_2, g_1], f] \lrcorner \omega(q) = 0$ since $[g_2, g_1] \in \Delta$ and $q \in C$. The first variant of the definition does not depend on the choice of g_1 . Due to the equivalence of the variants, the irrelevance of the choice of the extensions of v_1, v_2 follows.

We have the following equivalent definition of the set G .

Proposition 1 *Let $\text{Ann } \tau$ denote the annihilator of τ . Then*

$$G = \{g \in \Delta \mid g(q) \in \text{Ann } \tau, g(q) \neq 0\}.$$

Proof Take $g \in \Delta$ and note that $\eta(q) = 0$ means that $\omega^f(q)$ is colinear with $\omega(q)$. Let us inspect both definitions of G .

The condition $g(q) \in \text{Ann } \tau$ means that for every $g_1 \in \Delta$

$$0 = \tau(g_1(q), g(q)) = (L_{g_1} g)(q) \lrcorner \omega^f(q) + g_1(q) \lrcorner L_g \omega^f(q).$$

The first summand above vanishes since $L_{g_1} g \in \Delta$ and $\eta(q) = 0$. Thus $g(q) \in \text{Ann } \tau$ means exactly that $\Delta(q)$ is a subset of the kernel of $L_g \omega^f(q)$ or, equivalently, $L_g \omega^f(q)$ is colinear with $\omega(q)$, i.e., $L_g \omega^f \wedge \omega(q) = 0$.

On the other hand, $g(q) \in \ker \theta(q)$ means that $L_g \omega^f \wedge \omega(q) + \omega^f \wedge L_g \omega(q) = 0$. The second term vanishes because $\eta(q) = 0$ and $L_g \omega(q)$ is colinear with $\omega(q)$. Eventually, we get

$$g(q) \in \text{Ann } \tau \Leftrightarrow (L_g \omega^f \wedge \omega)(q) = 0 \Leftrightarrow g(q) \in \ker \theta.$$

□

Further, we consider the simplest degeneracy of τ , i.e., the case where $\text{Ann } \tau$ is one-dimensional or, in other words, using the symbol crk to denote the corank,

$$\text{crk } \tau = 1.$$

Choose a vector field $\bar{g} \in G$ and introduce the following sequence of functions

$$\bar{e}_1(\bar{g}) = \bar{g} \lrcorner \omega^f, \quad \bar{e}_k(\bar{g}) = L_{\bar{g}} \bar{e}_{k-1}.$$

We have

$$\bar{e}_1(\bar{g}) = \bar{g} \lrcorner L_f \omega = L_f \underbrace{(\bar{g} \lrcorner \omega)}_0 - L_f \bar{g} \lrcorner \omega = [\bar{g}, f] \lrcorner \omega = \text{ad}_{\bar{g}} f \lrcorner \omega.$$

For the convenience of further consideration we adapt here the notation:

$$\text{ad}_g f = \text{ad}_g^1 f = [g, f], \quad \text{and} \quad \text{ad}_g^k f = \text{ad}_g \text{ad}_g^{k-1} f, \quad \text{for } k > 1.$$

Since $\omega \wedge d\omega \equiv 0$, the inner product $\bar{g} \lrcorner d\omega$ is colinear with ω and thus

$$L_{\bar{g}}\omega = d \underbrace{(\bar{g} \lrcorner \omega)}_0 + \bar{g} \lrcorner d\omega = \alpha\omega,$$

for some smooth function α . Now we may write

$$\bar{e}_2(\bar{g}) = L_{\bar{g}}(\text{ad}_{\bar{g}}^2 f \lrcorner \omega) = \text{ad}_{\bar{g}}^2 f \lrcorner \omega + \text{ad}_{\bar{g}} f \lrcorner \underbrace{L_{\bar{g}}\omega}_{\alpha\omega} = \text{ad}_{\bar{g}}^2 f \lrcorner \omega + \alpha \bar{e}_1(\bar{g}).$$

Consequently, for any $k > 1$ we get

$$\bar{e}_k(\bar{g}) = \text{ad}_{\bar{g}}^k \omega + \sum_{i=1}^k \alpha_{k,i} \bar{e}_i(\bar{g}),$$

for some smooth functions $\alpha_{k,i}$. Hence, whenever

$$\bar{e}_1(\bar{g})(q) = \cdots = \bar{e}_{k-1}(\bar{g})(q) = 0,$$

we have

$$\bar{e}_k(\bar{g})(q) = (\text{ad}_{\bar{g}}^k f \lrcorner \omega)(q).$$

Notice that, since $q \in C$, we have automatically $\bar{e}_1(\bar{g})(q) = 0$, and since $\bar{g} \in G$ we have $\bar{e}_2(\bar{g})(q) = 0$ as well.

In the cases where $e(q) \neq 0$ it is natural to consider the form

$$\omega_1 = \frac{\omega}{e}$$

and the sequence

$$\bar{c}_1(\bar{g}) = \bar{g} \lrcorner \omega_1^f, \quad \bar{c}_k(\bar{g}) = L_{\bar{g}} \bar{c}_{k-1},$$

where $\omega_1^f = L_f \omega_1$. Note that ω_1 is normalised in the sense that

$$f \lrcorner \omega_1 \equiv 1.$$

The same arguments, as those given after defining the functions $\bar{e}_k$, yield analogous conclusions for the functions $\bar{c}_k$. In particular,

$$\bar{c}_1(\bar{g})(q) = \bar{c}_2(\bar{g})(q) = 0$$

and whenever

$$\bar{c}_1(\bar{g})(q) = \cdots = \bar{c}_{k-1}(\bar{g})(q) = 0,$$

we have

$$\bar{c}_k(\bar{g})(q) = \left(\text{ad}_g^k f \lrcorner \omega_1 \right)(q).$$

Denote by $V(M)$ the space of C^∞ -vector fields on M and by $J_q^1 V(M)$ their first jets at the point q . Let $\theta_1: J_q^1 V(M) \rightarrow \Lambda^2(T_q^* M)$ be defined as

$$\theta_1(v) = L_g L_g \eta_1(q), \quad \eta_1 = L_f \omega_1 \wedge \omega_1,$$

where g is an arbitrary vector field with the 1-jet at q equal v . If the point q lies outside the equilibrium set, we define the following set of vector fields

$$G^1 = \left\{ \bar{g} \in G \mid j_q^1(g) \in \ker \theta_1, \bar{c}_5(\bar{g})(q) = 0 \right\}.$$

Vector fields belonging to G^1 will be depicted by an extra double bar over them, e.g., $\bar{\bar{g}}$. Whenever we consider the mapping $\bar{c}_k$ constructed with the help of a vector field from the set G^1 , we will rather write $\bar{\bar{c}}_k$ instead of $\bar{c}_k(\bar{\bar{g}})$. Please note that not only have we

$$\bar{\bar{c}}_1(q) = \bar{\bar{c}}_2(q) = 0$$

(which is already the case of the functions $\bar{c}_1$ and $\bar{c}_2$) but also

$$\bar{\bar{c}}_3(q) = 0,$$

which is a consequence of the definition of the set G^1 .

4.2 Degeneracies at critical points in local coordinates

The advantage of the definitions and formulations presented in the previous section is their invariance (under coordinates and the choice of 1-form ω), which significantly simplifies the proofs of our main results and allows us to use the above definitions in any coordinate system and for any choice of vector fields g_i that span the considered distribution Δ . Non-degeneracy conditions listed in Theorems 1 and 2 below, that are constructed with a use of notions introduced in the previous section, can be expressed in local coordinates for the pre-normal form (Σ_{pnf}). The correspondence between coordinate-free and coordinate versions of these conditions, for $n = 2$, can be recovered by comparing conditions (S1)–(S5) of Theorem 1 to (J1)–(J5) of Theorem 5 and conditions (F1)–(F9) of Theorem 2 to (R1)–(R9) of Theorem 6. This correspondence, as well as the correspondence valid for arbitrary $n > 2$, can be easily obtained by the fact that, in a fixed coordinate system, we may define a form ω by declaring its value on an arbitrary vector field h as

$$h \lrcorner \omega = \omega(h) = \det(h, g_1, \dots, g_{n-1}),$$

where by $\det(h, g_1, \dots, g_{n-1})$ we mean the determinant of the matrix whose columns are formed by the components of the vector fields $h, g_1, \dots, g_{n-1}$, respectively.

Clearly, $e(q) = \det(f, g_1, \dots, g_{n-1})$. We have

$$E = \{q \in M \mid \det(f, g_1, \dots, g_{n-1})(q) = 0\} = \{q \in M \mid e(q) = 0\},$$

$$C = \{q \in M \mid \det(\operatorname{ad}_f g_i, g_1, \dots, g_{n-1})(q) = 0, i = 1, \dots, n-1\}$$

and, for $q \in C$,

$$G = \{g \in \Delta \mid g(q) \neq 0 \text{ and } \det(\operatorname{ad}_g \operatorname{ad}_f g_i, g_1, \dots, g_{n-1})(q) = 0, i = 1, \dots, n-1\}.$$

The set $L = \{g(q) \mid g \in G\} \cup \{0\}$ is a linear space. Its dimension is equal to the corank of τ , i.e.,

$$\operatorname{crk} \tau = \dim L.$$

For $\bar{g} \in G$, we have

$$\bar{e}_1(\bar{g}) = \det(\operatorname{ad}_{\bar{g}} f, g_1, \dots, g_{n-1}), \quad \bar{e}_k(\bar{g}) = L_{\bar{g}} \bar{e}_{k-1}(\bar{g}).$$

The functions $\bar{e}_1(\bar{g})$ and $\bar{e}_2(\bar{g})$ vanish at q which is due to $q \in C$ and $\bar{g} \in G$ (compare Sect. 4.1). Recall also that if $\bar{e}_1(\bar{g})(q) = \dots = \bar{e}_{k-1}(\bar{g})(q) = 0$, then

$$\bar{e}_k(\bar{g})(q) = \det\left(\operatorname{ad}_{\bar{g}}^k f, g_1, \dots, g_{n-1}\right)(q).$$

For $q \in C \setminus E$, we have the form ω_1 defined as

$$h \lrcorner \omega_1 = \omega(h) = \frac{\det(h, g_1, \dots, g_{n-1})}{\det(f, g_1, \dots, g_{n-1})} = \frac{1}{e} \det(h, g_1, \dots, g_{n-1})$$

and the functions $\bar{c}_k$ as

$$\bar{c}_1(\bar{g}) = \frac{1}{e} \det(\operatorname{ad}_{\bar{g}} f, g_1, \dots, g_{n-1}), \quad \bar{c}_k(\bar{g}) = L_{\bar{g}} \bar{c}_{k-1}(\bar{g}).$$

Recall that (we refer the reader to the previous subsection once again)

$$\bar{c}_1(\bar{g})(q) = \bar{c}_2(\bar{g})(q) = 0$$

and whenever $\bar{c}_1(\bar{g})(q) = \dots = \bar{c}_{k-1}(\bar{g})(q) = 0$, we get

$$\bar{c}_k(\bar{g})(q) = \frac{1}{e(q)} \det\left(\operatorname{ad}_{\bar{g}}^k f, g_1, \dots, g_{n-1}\right)(q).$$

Eventually,

$$G^1 = \left\{ \bar{g} \in G \mid \bar{c}_5(\bar{g})(q) = 0 \text{ and } \det \left(\text{ad}_{\bar{g}}^2 \text{ad}_f g_i(q), g_1, \dots, g_{n-1} \right)(q) = 0, \right. \\ \left. i = 1, \dots, n-1 \right\}.$$

In particular, as we already stated in Sect. 4.1, for $\bar{g} \in G^1$ we have

$$\bar{c}_1(\bar{g})(q) = \dots = \bar{c}_3(\bar{g})(q) = 0, \quad \text{i.e.,} \quad \bar{c}_1(q) = \bar{c}_2(q) = \bar{c}_3(q) = 0.$$

5 Classification results

For $\dim M = n > 2$, we denote local coordinates by $x, y, z_2, \dots, z_{n-1}$. Then an analogy with planar systems (compare Theorems 1 and 2 with Theorems 5 and 6, respectively) is more evident. In the statement of various conditions, d will denote the exterior derivative on M or on $M \times \{0\}$ in the case of families. For the exterior derivative on $M \times \mathbf{R}$ (for the families) we will use the symbol D .

Notice that any control system $\Sigma \in \mathcal{A}_{\text{inv}}^{n,n-1}$ on a manifold M can be equivalently interpreted as the vector field f on the manifold M equipped with the $(n-1)$ foliation $\mathcal{G}$. By a generic system on M , we will mean a generic vector field f on the foliated manifold $(M, \mathcal{G})$, where $\mathcal{G}$ is a fixed $(n-1)$ -dimensional foliation. Similarly, by a generic family $\Sigma_\mu \in \mathcal{F}\mathcal{A}_{\text{inv}}^{n,n-1}$ on M , we will mean a generic family of vector field f_μ on the foliated manifold $(M, \mathcal{G}_\mu)$, where $\mathcal{G}_\mu$ is a fixed family of $(n-1)$ -dimensional foliations on M .

The proofs of the following theorems will be given in the next section.

Theorem 1 Consider a control system $\Sigma \in \mathcal{A}_{\text{inv}}^{n,n-1}$ around a fixed point $q \in M$. Assume that one of the following conditions holds at q :

- (S1) $e \neq 0, \eta \neq 0$,
- (S2) $e = 0, \eta \neq 0$,
- (S3) $e \neq 0, \eta = 0, \text{crk } \tau = 0$,
- (S4) $e = 0, \eta = 0, \text{crk } \tau = 0, de \neq 0$,
- (S5) $e \neq 0, \eta = 0, \text{crk } \tau = 1, \bar{c}_3 \neq 0, d\bar{c}_1 \neq 0$.

Then Σ is, locally at q , feedback equivalent to the system $(F \frac{\partial}{\partial x}, dx) \in \mathcal{A}_{\text{inv}}^{n,n-1}$, at $0 \in \mathbf{R}^n$, where F is a function equal to exactly one of the following (function of the i -th form corresponds to condition (Si)):

- 1. $1 + y, \quad (O^n)$
- 2. $y, \quad (E^n)$
- 3. $1 + \epsilon_1 y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2, \quad (C^n)$
- 4. $\lambda x + y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2, \lambda \neq 0, \quad (EC^n)$
- 5. $y^3 + xy + a(x) + \sum_{i=2}^{n-1} \epsilon_i z_i^2, a(0) \neq 0, \quad (C\mathcal{G}^n)$

where, in each case, (ϵ_i) denotes a nondecreasing ± 1 sequence. If the orbital feedback equivalence is considered, then we can replace both λ and $a(x)$ by ± 1 .

Moreover, systems from $\mathcal{A}_{inv}^{n,n-1}$ for which, at each point, one of the conditions (S1)–(S5) holds are generic.

The above classification under a different, but equivalent, formulation of conditions (S1)–(S5) was obtained by [19].

The right column in the above list is given for consistency with [28]. We also follow this convention in the next theorem concerning 1-parameter families.

Theorem 2 Consider a 1-parameter family $\Sigma_\mu \in \mathcal{FA}_{inv}^{n,n-1}$ around a fixed point $(q, \mu) \in M \times \mathbf{R}$. Assume that one of the following conditions holds at (q, μ) :

- (F1) $e \neq 0, \eta \neq 0$,
- (F2) $e = 0, \eta \neq 0$,
- (F3) $e \neq 0, \eta = 0, \text{crk } \tau = 0$,
- (F4) $e = 0, \eta = 0, \text{crk } \tau = 0, de \neq 0$,
- (F5) $e \neq 0, \eta = 0, \text{crk } \tau = 1, \bar{c}_3 \neq 0, d\bar{c}_1 \neq 0$,
- (F6) $e = 0, \eta = 0, \text{crk } \tau = 0, de = 0, De \neq 0$,
- (F7) $e \neq 0, \eta = 0, \text{crk } \tau = 1, \bar{c}_3 \neq 0, d\bar{c}_1 = 0, D\bar{c}_1 \neq 0, \det \text{hess}|_V \bar{c}_1 \neq 0$,
where V is the kernel of the mapping assigning $L_{v\eta}$ to each vector $v \in T_q M$,
- (F8) $e = 0, \eta = 0, \text{crk } \tau = 1, de \neq 0, d\bar{e}_1 \neq 0, \bar{e}_3 \neq 0, D\bar{e}_1 \wedge De \neq 0$,
- (F9) $e \neq 0, \eta = 0, \text{crk } \tau = 1, \bar{c}_3 = 0, \bar{\bar{c}}_4 \neq 0, d\bar{c}_1 \neq 0, d\bar{\bar{c}}_2 \neq 0, D\bar{c}_1 \wedge D\bar{\bar{c}}_2 \neq 0$.

Then Σ_μ is locally at (q, μ) orbital feedback equivalent to $(F_\mu \frac{\partial}{\partial x}, dx) \in \mathcal{FA}_{inv}^{n,n-1}$ at $(0, 0) \in \mathbf{R}^n \times \mathbf{R}$, where F_μ is a family of functions equal to exactly one of the following (function of the i -th form corresponds to condition (Fi)):

- 1. $1 + y$, (O^n)
- 2. y , (E^n)
- 3. $1 + \epsilon_1 y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (C^n)
- 4. $\pm x + y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (EC^n)
- 5. $y^3 + xy \pm 1 + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (CG^n)
- 6. $x^2 - \mu + \epsilon_1 y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (E_{bif}^n)
- 7. $\pm y^3 + y(x^2 - \mu) + 1 + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (C_{bif}^n)
- 8. $y^3 + (x - \mu)y \pm x + \sum_{i=2}^{n-1} \epsilon_i z_i^2$, (EG_{bif}^n)
- 9. $y^4 + (\pm x - \mu)y^2 + xy + a(x, \mu) + \sum_{i=2}^{n-1} \epsilon_i z_i^2, a(0, 0) \neq 0$, (CG_{bif}^n)

where, in each case, (ϵ_i) denotes a nondecreasing ± 1 sequence, and $a(x, \mu)$ is a smooth function.

Moreover, 1-parameter families for which at each point one of the conditions (F1)–(F9) holds are generic among 1-parameter families of systems from $\mathcal{FA}_{inv}^{n,n-1}$.

5.1 Diagram of consecutive degenerations

On Fig. 1 we illustrate degeneracies of the families satisfying any of the conditions (F1)–(F9) of Theorem 2, in particular, any of the conditions (S1)–(S5) of Theorem 1. Further in this section, we concentrate on the case of families.

Fig. 1 Diagram of degenerations of a 1-parameter family

A collection of subconditions along an oriented path starting from $\emptyset$ forms a condition (see Fig. 1). If we take a path ending at a box we get one of the conditions (F1)–(F9). The numbers in the top row of the diagram denote the corank of the degenerations of 1-parameter families satisfying all subconditions along a path ended at a given column. By the corank of degeneration we mean the codimension of the algebraic set formed by all families satisfying the conditions. Because we are interested in generic objects only, and due to Thom transversality theorem (see the second step of the proof of Theorem 1 in Sect. 7), we may ignore degeneracies of corank greater than $n+1$ in the case of 1-parameter families and of corank greater than n in the case of systems. Thus we skip all subconditions that would bring us out of our domain of interest, e.g., there is no arrow indicating the condition $\text{crk } \tau = 2$ since, together with $\eta = 0$, it would correspond to a degeneration of corank at least $n+2$. So the diagram describes completely all generic families.

Let us stress the fact that the corank increases by $n-1$ when the subcondition $\eta = 0$ appears along a path. This clarifies why the shape of conditions (F1)–(F9) is independent of the dimension n .

6 Bifurcations

In this section we consider bifurcations of 1-parameter families $\Sigma_\mu = f_\mu + \Delta_\mu \in \mathcal{FA}_{\text{inv}}^{n,n-1}(M)$. Based on Theorem 2 we give a complete list of all generic bifurcations. For illustration purposes we provide pictures in the case $n = 3$.

We say that a family Σ_μ bifurcates, if the triple $(E_\mu, C_\mu, \mathcal{G}_\mu)$ changes topologically around a nominal value μ_0 of the parameter (see the next paragraph for a precise definition). Otherwise we say that Σ_μ does not bifurcate. For the class $\mathcal{FA}_{\text{inv}}^{n,n-1}(M)$ only finitely many generic bifurcations may occur and they are represented by the families listed in Theorem 2 and denoted by symbols with the subscript “bif”. All of them have clear geometric interpretations in terms of qualitative changes of the equilibrium set, the critical set, and the foliation $\mathcal{G}_\mu$ defined by the distribution Δ_μ .

More precisely, a family $\Sigma_\mu \in \mathcal{FA}_{\text{inv}}^{n,n-1}(M)$ does not bifurcate at (q_0, μ_0) if there exists an open neighbourhood $\Omega \subset M \times \mathbf{R}$ of (q_0, μ_0) and a family of homeomorphisms $\chi_\mu: \Omega_\mu \rightarrow \Omega_{\mu_0}$, continuous with respect to (q, μ) , such that for all μ close enough to μ_0 :

$$\chi_\mu(E_\mu \cap \Omega_\mu) = E_{\mu_0}, \quad \chi_\mu(C_\mu \cap \Omega_\mu) = C_{\mu_0}, \quad \text{and} \quad \chi_\mu(\mathcal{G}_\mu \cap \Omega_\mu) = \mathcal{G}_{\mu_0},$$

where $\Omega_\mu = \{q \in X \mid (q, \mu) \in \Omega\}$. Otherwise we say that the family Σ_μ bifurcates. Following [28], we distinguish a few particular types of bifurcations: *E*-bifurcations, if E_μ is the only element of the triple $(E_\mu, C_\mu, \mathcal{G}_\mu)$ that bifurcates; *C*-bifurcations, if C_μ is the only element of the triple $(E_\mu, C_\mu, \mathcal{G}_\mu)$ that bifurcates; *EG*-bifurcations, if none of the elements of the triple $(E_\mu, C_\mu, \mathcal{G}_\mu)$ bifurcates but the pair $(E_\mu, \mathcal{G}_\mu)$ does; and *CG*-bifurcations, if none of the elements of the triple $(E_\mu, C_\mu, \mathcal{G}_\mu)$ bifurcates but the pair $(C_\mu, \mathcal{G}_\mu)$ does.

Theorem 2 implies the following result.

Theorem 3 *Let $\Sigma_\mu \in \mathcal{FA}_{\text{inv}}^{n,n-1}$ be a generic (in the sense described at the beginning of Sect. 5) 1-parameter family. If Σ_μ bifurcates locally at (q_0, μ_0) , then the bifurcation is of exactly one of the following types:*

- (1) *an E-bifurcation, which can be of n non-equivalent types, observed for families equivalent to the form (E_{bif}^n) ;*
- (2) *a C-bifurcation, which can be of two non-equivalent types, observed for families equivalent to the form (C_{bif}^n) ;*
- (3) *an EG-bifurcation observed for families equivalent to the form $(E\mathcal{G}_{\text{bif}}^n)$;*
- (4) *a CG-bifurcation, observed for families equivalent to the form $(C\mathcal{G}_{\text{bif}}^n)$;*

We study all the above types, for the case $n = 3$, in the following sections.

Remark 1 If the family undergoes an *EG*-bifurcation, the pair (E_μ, C_μ) bifurcates as well, so it is simultaneously an *EC*-bifurcation (see Sect. 6.4).

Fig. 2 Lack of bifurcations

6.1 Lack of bifurcations

Let $\Sigma_\mu \in \mathcal{FA}^{3,2}$. If one of the conditions (F1)–(F5) is satisfied at the origin, we may assume (see Theorem 2) that $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where F_μ is equal to, respectively,

1. $1 + y$, (O^3)
2. y , (E^3)
3. $1 + \epsilon_1 y^2 + \epsilon_2 z^2$, (C^3)
4. $\epsilon_0 x + y^2 + \epsilon_2 z^2$, (EC^3)
5. $y^3 + xy + \epsilon_0 + \epsilon_2 z^2$, (CG^3)

where $\epsilon_0, \epsilon_1, \epsilon_2 = \pm 1$ and $\epsilon_1 \leq \epsilon_2$. In all these cases we observe no bifurcations. The triple $(E, C, \mathcal{G})$ is constant with respect to the parameter μ and looks like one of those presented in the Fig. 2 (for clarity we draw only one leaf $L \ni q$ of the foliation $\mathcal{G}$).

Fig. 3 E bifurcation, $\epsilon_1, \epsilon_2 > 0$

Fig. 4 E bifurcation, $\epsilon_1, \epsilon_2 < 0$

In the first case, the origin belongs neither to E_0 nor to C_0 . In the second one, the origin is a point of $E_0 \setminus C_0$.

In the (C^3) -case and $(C\mathcal{G}^3)$ -case the origin belongs to $C_0 \setminus E_0$ and C_0 is transversal in the former case and tangent in the latter to the leaf of the foliation passing through the origin.

In the (EC^3) case, the origin lies in $E_0 \cap C_0$ and it is a point of tangency of E_0 to the foliation. If we consider the set E together with the foliation $\mathcal{G}$ we can distinguish two non-homeomorphic cases depending on the sign of ϵ_2 (see Fig. 2).

6.2 E bifurcations

If $\Sigma_\mu \in \mathcal{FA}^{3,2}$ and condition (F6) is satisfied, we observe a bifurcation of the equilibrium set E . Indeed, according to Theorem 2, we may bring (by an orbital feedback transformation) the family to the form $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where

$$F_\mu(x, y, z) = x^2 - \mu + \epsilon_1 y^2 + \epsilon_2 z^2.$$

Thus the sets E_μ and C_μ are determined by the following equations

$$E_\mu: x^2 + \epsilon_1 y^2 + \epsilon_2 z^2 = \mu, \quad C_\mu: y = z = 0.$$

For $\epsilon_1, \epsilon_2 > 0$, the set E_μ is empty for $\mu < 0$, is a point for $\mu = 0$, and becomes a sphere for $\mu > 0$, see Fig. 3.

For other values of ϵ_1, ϵ_2 , the set E_μ remains a hyperboloid but the number of its sheets changes when the parameter varies, see Figs. 4 and 5. If we consider the set E together with the foliation $\mathcal{G}$ we can distinguish two non-homeomorphic cases. For $\epsilon_1, \epsilon_2 < 0$, the intersection of E and a leaf of the foliation may be of three types: an

Fig. 5 E bifurcation, $\epsilon_1 \epsilon_2 < 0$

Fig. 6 C bifurcations

empty set, a point, or a circle, see Fig. 4. For $\epsilon_1 \epsilon_2 < 0$ this intersection may consist of the two branches of a hyperbola or two lines, see Fig. 5.

6.3 C bifurcations

If $\Sigma_\mu \in \mathcal{FA}^{3,2}$ and condition (F7) is satisfied, we observe a bifurcation of the critical set C . According to Theorem 2, we may bring (by an orbital feedback transformation) the family to the form $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where

$$F_\mu(x, y, z) = \epsilon_0 y^3 + y(x^2 - \mu) + 1 + \epsilon_2 z^2, \quad \epsilon_0 = \pm 1.$$

Then the critical set C_μ is defined by the following set of equations

$$3\epsilon_0 y^2 + x^2 = \mu, \quad z = 0.$$

The bifurcation may be of two types depending on the sign of ϵ_0 . Both of them are presented in Fig. 6.

Fig. 7 $(E, \mathcal{G})$ bifurcation

Fig. 8 $(C, \mathcal{G})$ bifurcation

6.4 $(E, \mathcal{G})$ bifurcation

If $\Sigma_\mu \in \mathcal{FA}^{3,2}$ and condition (F8) is satisfied, then we observe a bifurcation of the pair $(E, \mathcal{G})$, which is presented in the Fig. 7. According to Theorem 2, we may bring the family to the form $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where

$$F_\mu(x, y, z) = y^3 + (x - \mu)y + \epsilon_0 x + \epsilon_2 z^2, \quad \epsilon_0 = \pm 1.$$

The leaves of the foliation $\mathcal{G}_\mu$ are $\{x = \text{const}\}$ and the sets E_μ and C_μ are determined by the following equations

$$E_\mu: y^3 + (x - \mu)y + \epsilon_0 x + \epsilon_2 z^2 = 0, \quad C_\mu: 3y^2 + x = \mu, \quad z = 0.$$

It is then clear that the both pairs $(E_\mu, \mathcal{G}_\mu)$ and (E_μ, C_μ) bifurcate (see Remark following Theorem 3).

6.5 $(E, \mathcal{G})$ bifurcation

If $\Sigma_\mu \in \mathcal{FA}^{3,2}$ and condition (F9) is satisfied, we observe a bifurcation of the pair $(C, \mathcal{G})$, which is presented in the Fig. 8. The canonical representation of such a family is, see Theorem 2, $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where

$$F_\mu(x, y, z) = y^4 + (\epsilon_0 x - \mu)y^2 + xy + a(x, \mu) + \epsilon_2 z^2, \quad a(0) \neq 0, \quad \epsilon_0 = \pm 1.$$

The leaves of the foliation $\mathcal{G}_\mu$ are $\{x = \text{const}\}$ and the set C_μ is described by the following set of equations

$$4y^3 + 2(\epsilon_0 x - \mu)y + x = 0, \quad z = 0,$$

so, clearly, the pair $(C_\mu, \mathcal{G}_\mu)$ bifurcates.

7 Proofs of the classification theorems

Introductory steps of the proofs of Theorems 1 and 2 are based on the following version of the Mather theorem on universal unfoldings (cf., e.g., Theorem 14.8 in [7] and Chapters IV and IX in [35]).

Theorem 4 *Let $\phi = \phi(s, w)$ be a C^∞ -function from a neighbourhood of $(0, 0) \in \mathbf{R} \times \mathbf{R}^p$ into $\mathbf{R}$. Assume that for an integer $k \geq 2$*

$$\frac{\partial^i \phi}{\partial s^i}(0, 0) = 0, \quad \text{for } 1 \leq i \leq k-1, \quad \frac{\partial^k \phi}{\partial s^k}(0, 0) \neq 0.$$

Then there exists a local transformation $s = \psi(\hat{s}, w)$, invertible with respect to $\hat{s}$, such that

$$\phi(\psi(\hat{s}, w), w) = \sigma \hat{s}^k + \sum_{i=0}^{k-2} a_i(w) \hat{s}^i,$$

where $a_i(0) = 0$ for $0 \leq i \leq k-2$ and $\sigma = 1$ for odd k and ± 1 when k is an even number.

We will use the following result of Jakubczyk and Respondek [24].

Theorem 5 *Consider a system $\Sigma = (f(x, y) \frac{\partial}{\partial x}, dx) \in \mathcal{A}^{2,1}$ around the origin. Sufficient and necessary conditions for Σ to be locally feedback equivalent to the system $(F \frac{\partial}{\partial x}, dx)$ are, respectively,*

$$(J1) \quad f \neq 0, \frac{\partial f}{\partial y} \neq 0, \text{ for } F(x, y) = 1 + y,$$

$$(J2) \quad f = 0, \frac{\partial f}{\partial y} \neq 0, \text{ for } F(x, y) = y,$$

$$(J3) \quad f \neq 0, \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2} \neq 0, \text{ for } F(x, y) = 1 \pm y^2$$

$$(J4) \quad f = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2} \neq 0, \frac{\partial f}{\partial x} \neq 0, \text{ for } F = \lambda x + y^2, \lambda \neq 0,$$

$$(J5) \quad f \neq 0, \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^3 f}{\partial y^3} \neq 0, \frac{\partial^2 f}{\partial x \partial y} \neq 0, \text{ for } F = y^3 + xy + a(x), a(0) \neq 0.$$

If we consider the orbital feedback equivalence, then we can replace λ and $a(x)$ by ± 1 .

In [24] the authors state the conditions (J1)–(J5) in a coordinate-free way. For the purposes of this paper it is more convenient to write them as above.

Remark 2 As proved in [28], the systems from $\mathcal{A}^{2,1}$ which, at each point, are feedback equivalent to one of the models listed in Theorem 5, constitute a countable union of open dense subsets of $\mathcal{A}^{2,1}$, i.e., they are generic.

Proposition 2 *The set G (see Sect. 4.1) is orbital feedback invariant. The conditions (S1)–(S5) of Theorem 1, formulated with the help of a fixed $\bar{g} \in G$, are (orbital) feedback invariant and do not depend on the choice of ω (i.e., multiplying ω by a non-vanishing factor does not change them).*

Remark 3 Actually, a stronger result is true. Namely, the conditions (S1)–(S5) do not depend on the choice of $\bar{g} \in G$, which will be shown in the proof of Theorem 1.

Proof (Proposition 2) Let $\Sigma = f + \Delta = (f, \omega)$. We must check whether the conditions are invariant with respect to the transformation

$$f \mapsto hf + g = \hat{f}, \quad k\omega = \hat{\omega}$$

where g is a vector field from Δ , i.e., $g \lrcorner \omega \equiv 0$, k is a non-vanishing function and h is a positive function constant on the leaves of the foliation defined by Δ , i.e., $dh \wedge \omega \equiv 0$. We denote objects of the system $\hat{\Sigma} = (\hat{f}, \hat{\omega})$ by an extra hat over them. It is enough to show that, at the considered point, they differ from their counterparts without hats by a non-vanishing factor only.

$$\begin{aligned} \hat{e} &= (hf + g) \lrcorner \hat{\omega} = hkf \lrcorner \omega + hg \lrcorner \omega = hke, \\ L_{\hat{f}}\hat{\omega} &= L_{hf}\hat{\omega} + L_g\hat{\omega} = hkL_f\omega + kedh + kg \lrcorner d\omega + (L_{\hat{f}}k)\omega. \end{aligned}$$

Using the equalities $\omega \wedge d\omega \equiv 0$, $dh \wedge \omega \equiv 0$, we thus get

$$\hat{\eta} = L_{\hat{f}}\hat{\omega} \wedge \hat{\omega} = hk^2 L_f\omega \wedge \omega = hk^2 \eta.$$

If $e \neq 0$ we have also

$$\hat{c}_1 = \bar{g} \lrcorner L_{\hat{f}}\hat{\omega}_1 = \bar{g} \lrcorner \left(\left(L_{\hat{f}} \frac{k}{\hat{e}} \right) \omega + \frac{k}{\hat{e}} L_{\hat{f}}\omega \right) = \frac{k}{hke} h\bar{g} \lrcorner L_f\omega = \bar{c}_1$$

and hence $\hat{c}_k = \bar{c}_k$ and $d\hat{c}_k = d\bar{c}_k$ for arbitrary k . Similarly, we check that $d\hat{e} = hkd e$, provided $e = 0$ and $\hat{\tau} = hk\tau$, provided $\eta = 0$. In particular, $\text{Ann } \hat{\tau} = \text{Ann } \tau$ and thus $\hat{G} = G$. $\square$

Proof (Theorem 1) Proposition 2 guarantees that validity of each of the conditions (S1)–(S5) is independent of the feedback transformation applied to the system. Thus we may assume Σ is of the form (Σ_{pnf}) . Therefore, we may choose local coordinates $\mathbf{x} = (x, y, z_2, \dots, z_{n-1})$ which map the point q into $0 \in \mathbf{R}^n$ and such that

$$\Sigma = \left(F(\mathbf{x}) \frac{\partial}{\partial x}, dx \right),$$

where F is a smooth function. We split the proof into two steps. First we fine-tune the coordinates to highlight the structure of the system. Next we restrict Σ to a surface and, using Theorem 5, deduce the existence of an appropriate feedback transformation. Step 1.

- (S1) Since $e \neq 0$ and $\eta \neq 0$, we have $F(0) \neq 0$ and $(\frac{\partial F}{\partial y}, \frac{\partial F}{\partial z_2}, \dots, \frac{\partial F}{\partial z_{n-1}})(0) \neq 0$. Without loss of generality we can assume $\frac{\partial F}{\partial y}(0) \neq 0$. Replacing y by $\hat{y} = F(\mathbf{x}) - F(0)$, we bring Σ into the form

$$\left((F(0) + \hat{y}) \frac{\partial}{\partial x} + F_r(x, \hat{y}, z_2, \dots, z_{n-1}) \frac{\partial}{\partial \hat{y}}, dx \right).$$

Using an appropriate feedback we get rid of the second component of the vector field above. Thus, up to a feedback transformation and a change of coordinates, we have

$$F(x, \hat{y}, z_2, \dots, z_{n-1}) = \xi + \hat{y}, \quad \xi \neq 0.$$

- (S2) As above, we can bring the system into the form with

$$F(x, \hat{y}, z_2, \dots, z_{n-1}) = \xi + \hat{y}, \quad \xi = 0.$$

- (S3) In this case we have $e = 0$, $\eta = 0$ and $\text{crk } \tau = 0$. Hence the Hessian of F with respect to $(y, z_2, \dots, z_{n-1})$, i.e.,

$$\frac{\partial^2 F}{\partial z_i \partial z_j}, \quad 1 \leq i, j \leq n-1, \text{ where } z_1 = y,$$

is non-degenerated. By an appropriate change of the $(z_1, \dots, z_{n-1})$ -coordinates we can assume it is diagonal. Applying the Mather theorem (for $s = \hat{y}$ and $w = (x, z_2, \dots, z_{n-1})$) followed by an appropriate feedback transformation, we obtain

$$F(x, \hat{y}, z_2, \dots, z_{n-1}) = \hat{\varepsilon}_1 \hat{y}^2 + F_1(x, z_2, \dots, z_{n-1}),$$

where $\hat{\varepsilon}_1 = \pm 1$. We can repeat the arguments with respect to F_1 to get

$$F_1 = \hat{\varepsilon}_1 \hat{y}^2 + \hat{\varepsilon}_2 \hat{z}_2^2 + F_2(x, z_3, \dots, z_{n-2}),$$

where $\hat{\varepsilon}_1, \hat{\varepsilon}_2 = \pm 1$. Continuing the procedure we obtain

$$F(x, \hat{y}, \hat{z}_2, \dots, \hat{z}_{n-1}) = \xi(x) + \hat{\varepsilon}_1 \hat{y}^2 + \sum_{i=2}^{n-1} \hat{\varepsilon}_i \hat{z}_i^2,$$

where $\hat{\varepsilon}_i = \pm 1$ and $\xi(0) \neq 0$.

(S4) As above, we can bring the system into the form

$$F = \xi(x) + \hat{\varepsilon}_1 \hat{y}^2 + \sum_{i=2}^{n-1} \hat{\varepsilon}_i \hat{z}_i^2,$$

where $\hat{\varepsilon}_i = \pm 1$, $\xi(0) = 0$, $\xi'(0) \neq 0$. The last inequality is implied by $de \neq 0$.
(S5) Since the corank of Hessian of F is one, we can repeat the arguments of the three former cases to obtain

$$F = \xi(x, y) + \sum_{i=2}^{n-1} \hat{\varepsilon}_i \hat{z}_i^2,$$

where $\hat{\varepsilon}_i = \pm 1$, $\xi(0) = 0$, $\frac{\partial \xi}{\partial y}(0) = 0$, $\frac{\partial^2 \xi}{\partial y^2}(0) = 0$. In the $(x, y, \hat{z})$ -coordinates,

$$G = \left\{ a \frac{\partial}{\partial y} + g \mid a \neq 0, g \lrcorner dx \equiv 0, g(0) = 0 \right\}$$

and $\bar{c}_3 \neq 0$ means precisely $\frac{\partial^3 F}{\partial y^3} \neq 0$ independently of the choice of $\bar{g}$. Thus applying the Mather theorem to F , with $s = \hat{y}$ and $w = (x, \hat{z}_2, \dots, \hat{z}_{n-1})$, we can assume

$$F = \hat{y}^3 + \hat{y} \xi_1(x) + \xi_0(x) + \sum_{i=2}^{n-1} \hat{\varepsilon}_i \hat{z}_i^2,$$

where $\hat{\varepsilon}_i = \pm 1$, $\xi_0(0) \neq 0$, $\xi_1(0) = 0 \neq \xi'_1(0)$. The last inequality comes from $d\bar{c}_1 \neq 0$, which means exactly $\frac{\partial^2 F}{\partial y \partial x}(0) \neq 0$ independently of the choice of $\bar{g}$. We showed that both conditions $\bar{c}_3 \neq 0$ and $d\bar{c}_1 \neq 0$ do not depend on the choice of $\bar{g} \in G$, in fixed coordinates. So far, we have applied a feedback transformation only and both G and the condition (S5) with a fixed $\bar{g}$ are invariant with respect to such transformations. Thus we proved also the irrelevance of the choice of $\bar{g}$ in the general situation.

Step 2. Consider the restriction of our system to the surface $z_2 = 0, \dots, z_{n-2} = 0$ in the cases (S1), (S2) or $\hat{z}_2 = 0, \dots, \hat{z}_{n-2} = 0$ in the others. It is a matter of a direct computation to check that conditions (S1)–(S5) (precisely, their parts relevant for the surface) mean exactly that the conditions (J1)–(J5) for the restricted system are satisfied, respectively. Hence by Theorem 5, a feedback transformation brings the restricted system to the one of the canonical form listed in that theorem. The same transformation (precisely, its lift to the original space) brings the original system Σ to the form $(F \frac{\partial}{\partial x}, dx)$ with $F(x, y, z_2, \dots, z_{n-1})$ equal to, respectively,

1. $1 + y$,
2. y ,
3. $1 + \hat{\varepsilon}_1 y^2 + \psi(x, y) \sum_{i=2}^{n-1} \hat{\varepsilon}_i \hat{z}_i^2$,

4. $\lambda x + y^2 + \psi(x, y) \sum_{i=2}^{n-1} \hat{\epsilon}_i z_i^2, \quad \lambda \neq 0,$
5. $y^3 + xy + a(x) + \psi(x, y) \sum_{i=2}^{n-1} \hat{\epsilon}_i z_i^2, \quad a(0) \neq 0,$

where $\psi(0) \neq 0, \hat{\epsilon}_i = \pm 1, i = 2, \dots, n-1$. Now in the three latter cases we replace the coordinates z_i by $z_i |\psi(x, y)|^{1/2}$ and then reorder them to get

3. $1 + \epsilon_1 y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2,$
4. $\lambda x + y^2 + \sum_{i=2}^{n-1} \epsilon_i z_i^2, \quad \lambda \neq 0,$
5. $y^3 + xy + a(x) + \sum_{i=2}^{n-1} \epsilon_i z_i^2, \quad a(0) \neq 0,$

where (ϵ_i) denotes a nondecreasing ± 1 sequence.

To obtain the orbital feedback classification the same method is used.

Let us eventually give the proof of the last part of the theorem. Denote by $j^k Q$, the set of k -jets at the origin of systems belonging to $Q = \mathcal{A}_{\text{inv}}^{n, n-1}(\mathbf{R}^n)$ (the k -jet of the system $f + (g_1, \dots, g_{n-1})$ is, by definition, the k -jet of the n -tuple $(f, g_1, \dots, g_{n-1})$). Consider a singularity class $S \subset Q$, i.e., an orbital feedback invariant set of the form

$$S = \left\{ \Sigma \in Q \mid j^k \Sigma \in D \right\},$$

where $D \subset j^k Q$ is a stratified submanifold. For a fixed system $\Sigma \in \mathcal{A}_{\text{inv}}^{n, n-1}(M)$, $\dim M = n$, denote by S_Σ , the set of points around which Σ is locally orbital feedback equivalent to a germ at the origin of a system from S . Thom transversality theorem, applied on the manifold M equipped with the fixed foliation $\mathcal{G}$ (for a version for control systems, we refer the reader to [41]), says that if $\text{codim } S > n$ then there exists an everywhere dense intersection of a countable number of open sets of $\mathcal{A}_{\text{inv}}^{n, n-1}(M)$ such that for any Σ from this intersection the set S_Σ is empty.

Consider a subset $R \subset Q$ for which none of the conditions (S1)–(S5) is satisfied at the origin. By Proposition 2, R is orbital feedback invariant. It remains to prove that there exists a number k and a stratified submanifold $D \subset j^k Q$ such that $R = \{\Sigma \in Q \mid j^k \Sigma \in D\}$. Direct computations in a coordinate system show that one may take $k = 3$ and $D = j^3 R$. $\square$

In the proof of Theorem 2, we will use the following classification [28] of 1-parameter families of systems on the plane.

Theorem 6 Consider a 1-parameter family $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx) \in \mathcal{FA}^{2,1}$ and the following conditions computed at $(0, 0) \in \mathbf{R}^2 \times \mathbf{R}$ (below we skip the subscript μ in F_μ , for brevity):

- (R1) $F \neq 0, \frac{\partial F}{\partial y} \neq 0,$
- (R2) $F = 0, \frac{\partial F}{\partial y} \neq 0,$
- (R3) $F \neq 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} \neq 0,$
- (R4) $F = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} \neq 0, \frac{\partial F}{\partial x} \neq 0,$
- (R5) $F \neq 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} = 0, \frac{\partial^2 F}{\partial y \partial x} \neq 0, \frac{\partial^3 F}{\partial y^3} \neq 0,$
- (R6) $F = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} \neq 0, \frac{\partial F}{\partial x} = 0, \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \neq \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2, \frac{\partial F}{\partial x} \frac{\partial^2 F}{\partial y^2} \neq \frac{\partial F}{\partial y} \frac{\partial^2 F}{\partial x \partial y},$

$$\begin{aligned}
\text{(R7)} \quad & F \neq 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} = 0, \frac{\partial^3 F}{\partial y^3} \neq 0, \frac{\partial^2 F}{\partial y \partial x} = 0, \frac{\partial^3 F}{\partial y \partial x^2} \frac{\partial^3 F}{\partial y^3} \neq \left(\frac{\partial^3 F}{\partial y^2 \partial x} \right)^2, \\
\text{(R8)} \quad & F = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} = 0, \frac{\partial F}{\partial x} \neq 0, \frac{\partial^2 F}{\partial y \partial x} \neq 0, \frac{\partial^3 F}{\partial y^3} \neq 0, \frac{\partial F}{\partial x} \frac{\partial^2 F}{\partial y^2} \neq \frac{\partial F}{\partial y} \frac{\partial^2 F}{\partial x \partial y}, \\
\text{(R9)} \quad & F \neq 0, \frac{\partial F}{\partial y} = 0, \frac{\partial^2 F}{\partial y^2} = 0, \frac{\partial^3 F}{\partial y^3} = 0, \frac{\partial^4 F}{\partial y^4} \neq 0, \frac{\partial^2 F}{\partial y \partial x} \neq 0, \frac{\partial^2 F}{\partial y \partial x} \frac{\partial^2 F}{\partial y \partial \mu} \neq \\
& \frac{\partial^3 F}{\partial y^2 \partial x} \frac{\partial^3 F}{\partial y^2 \partial \mu}, 7 \frac{\partial^3 F}{\partial y^2 \partial x} \frac{\partial^4 F}{\partial y^4} \neq \frac{\partial^5 F}{\partial y^5} \frac{\partial^2 F}{\partial y \partial x}.
\end{aligned}$$

If one of the above conditions holds, then Σ_μ at the origin is locally orbital feedback equivalent to the family $(F_\mu \frac{\partial}{\partial x}, dx) \in \mathcal{FA}^{2,1}$, where F_μ is a family of functions equal to the exactly one of the following, respectively,

1. $1 + y$,
2. y ,
3. $1 \pm y^2$,
4. $\pm x + y^2$,
5. $\pm y^3 + xy + 1$,
6. $x^2 - \mu \pm y^2$,
7. $\pm y^3 + y(x^2 - \mu) + 1$
8. $y^3 + (x - \mu)y \pm x$
9. $y^4 \pm (x - \mu)y^2 + xy + a(x, \mu)$, $a(0) \neq 0$.

Remark 4 As proved in [28], the systems from $\mathcal{FA}^{2,1}$ which, at each point, are feedback equivalent to one of the models listed in Theorem 5, constitute a countable union of open dense subsets of $\mathcal{FA}^{2,1}$, i.e., they are generic.

Proposition 3 The sets G and G^1 as well as $\ker \theta$ (see Sect. 4.1) are orbital feedback invariant. The conditions (F1)–(F9), formulated for a fixed $\bar{g} \in G$, are orbital feedback invariant and do not depend on the choice of ω (i.e. multiplying ω by a non-vanishing factor does not change them).

Remark 5 Actually, a stronger result is true. Namely, the conditions (F1)–(F9) depend neither on the choice of $\bar{g} \in G$ nor on the choice of $\hat{\bar{g}} \in G^1$, which will be shown in the proof of Theorem 2.

Proof (Proposition 3) Conditions (F1)–(F5) are the same, geometrically speaking, as (S1)–(S5). (The only difference being that the involved objects may now depend on μ as well). Thus their invariance, as well as the invariance of G , comes from Proposition 2. We skip the proof of invariance of those subconditions of conditions (F6)–(F9), for which it is directly implied by Proposition 2.

We consider two feedback equivalent systems $\Sigma = (f, \omega)$ and $\hat{\Sigma} = (hf + g, k\omega)$, where $g \lrcorner \omega \equiv 0$ and $dh \wedge \omega \equiv 0$. Let us denote all the objects corresponding to the second system by an extra hat over them. In particular, $\hat{f} = hf + g$.

(F6) $\hat{e} \equiv hke$ and $e = 0$, thus we obtain (at the fixed point (q, μ)) $D\hat{e} = hkDe$.

(F7) We have (see the proof of Proposition 2) $\hat{\bar{c}}_1 \equiv \bar{c}_1$.

(F8) The invariance of the condition $D\bar{c}_1 \wedge De \neq 0$ comes from already obtained equalities $D\hat{e} = hkDe$ and $D\hat{\bar{c}}_1 = hD\bar{c}_1$. Moreover,

$$\hat{\bar{e}}_3 = (L_{\bar{g}})^3(hke) = h \left(eL_{\bar{g}}^3k + 3\bar{e}_1L_{\bar{g}}^2k + 3\bar{e}_2L_{\bar{g}}k + k\bar{e}_3 \right).$$

We have also $e = 0$, $\bar{e}_1 = 0$ since $\eta = 0$ and $\bar{e}_2 = 0$ since $\bar{g} \in \text{Ann } \tau$. Thus $\hat{e}_3 = hk\bar{e}_3$.

(F9) The invariance is implied by $\hat{c}_1 \equiv \bar{c}_1$.

The invariance of $\ker \theta$ is a direct consequence of $\hat{\eta} = hk^2\eta$ (see the proof of Proposition 2) and the fact that when considering $\ker \theta$ we assume $\eta = 0$. Thus $\hat{\theta} = hk^2\theta$. For similar reasons we have $\hat{\theta}_1 = hk^2\theta_1$ and hence the invariance of G^1 follows. $\square$

Proof (Theorem 2) We proceed in the same way as in the proof of Theorem 1. First, we change coordinates (and apply a feedback) to get $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where F_μ is equal, respectively:

- 1) $y + \xi$,
- 2) y ,
- 3) $\hat{e}_1 y^2 + \xi(x) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 4) $\hat{e}_1 y^2 + \xi(x) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 5) $y^3 + y\xi_1(x) + \xi_0(x) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 6) $\hat{e}_1 y^2 + \xi(x, \mu) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 7) $y^3 + y\xi_1(x, \mu) + \xi_0(x, \mu) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 8) $y^3 + y\xi_1(x, \mu) + \xi_0(x, \mu) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$,
- 9) $y^4 + y^2\xi_2(x, \mu) + y\xi_1(x, \mu) + \xi_0(x, \mu) + \sum_{i=2}^{n-1} \hat{e}_i z_i^2$.

The rest of subconditions of (F1)–(F9) gives us precisely (where each function is computed at the origin):

- 1) $\xi \neq 0$,
- 2) no additional information,
- 3) $\xi \neq 0$,
- 4) $\xi = 0$, $\frac{\partial \xi}{\partial x} \neq 0$,
- 5) $\xi_1 \neq 0$, $\xi_0 \neq 0$,
- 6) $\xi = 0$, $\frac{\partial \xi}{\partial x} = 0$, $\frac{\partial^2 \xi}{\partial x^2} \neq 0$, $\frac{\partial \xi}{\partial \mu} \neq 0$,
- 7) $\xi_0 \neq 0$, $\xi_1 = 0$, $\frac{\partial \xi_1}{\partial x} = 0$, $\frac{\partial^2 \xi_1}{\partial x^2} \neq 0$, $\frac{\partial \xi_1}{\partial \mu} \neq 0$,
- 8) $\xi_0 = 0$, $\frac{\partial \xi_0}{\partial x} \neq 0$, $\xi_1 \neq 0$, $\frac{\partial \xi_1}{\partial x} \frac{\partial \xi_0}{\partial \mu} \neq \frac{\partial \xi_1}{\partial \mu} \frac{\partial \xi_0}{\partial x}$,
- 9) $\xi_0 \neq 0$, $\xi_1 = 0$, $\frac{\partial \xi_1}{\partial x} \neq 0$, $\xi_2 = 0$, $\frac{\partial \xi_2}{\partial x} \neq 0$ and $\left(\frac{\partial \xi_1}{\partial x} \frac{\partial \xi_2}{\partial \mu} - \frac{\partial \xi_1}{\partial \mu} \frac{\partial \xi_2}{\partial x} \right) \neq 0$.

In particular, direct computations show that the choice of $\bar{g}$ and $\bar{\bar{g}}$ is irrelevant.

After restricting to the plane $z_2 = \dots = z_{n-1} = 0$ we get the families from the class $\mathcal{FA}^{2,1}$, which satisfy conditions (R1)–(R9). Next we proceed as in the second step of the proof of Theorem 1 (referring to Theorem 6 instead of Theorem 5).

The proof of genericity is analogous to the last part of the proof of Theorem 1. The differences are that now we consider the manifold M with the fixed family of foliations $\mathcal{G}_\mu$ and jets of 1-parameter families instead of jets of systems. Moreover, when formulating Thom transversality theorem, we consider $\text{codim } S > n + 1$. Finally we take $D = j^5 R$, where R is the set of those 1-parameter families of $\mathcal{FA}_{\text{inv}}^{n,n-1}(\mathbf{R}^n)$ for which none of the conditions (F1)–(F9) is satisfied at the origin. For reader's convenience, in Fig. 1 we give a diagram of the consecutive degenerations which may occur for

a 1-parameter family at a fixed point (see Sect. 5.1 for a detailed description of the diagram). $\square$

8 Restrictions to surfaces

Given any system $\Sigma = f + \Delta \in \mathcal{A}^{n,m}(M)$ and a surface $S \subset M$, we can consider the restriction $\Sigma|_S$ of Σ to S by defining $\Sigma|_S$ pointwise as the (possibly singular) affine distribution $(f + \Delta)(q) \cap T_q S$ on S . If $\Sigma \in \mathcal{A}_{\text{inv}}^{n,n-1}(M)$ and the surface S is transversal to the distribution Δ , then the restriction $\Sigma|_S$ is a well defined smooth system on S belonging to the class $\mathcal{A}^{2,1}(S)$. A natural question is whether the equivalence class of a system is determined by that of its restriction. In particular, whether the restricted system is of the same type as the original one, where by the type of a system Σ we mean the class of all systems satisfying the same condition (Si) as the system Σ does. Therefore we distinguish five types of systems: (O) , (E) , (C) , (EC) , (CG) . We say that a system, satisfying a condition (Si), and its restriction to a given surface S are of the same type, if the restricted system satisfies the planar counterpart of (Si). In some special cases, the type, being usually preserved after restriction, may change. It may also happen that the restricted system fails to satisfy any of the conditions (S1)–(S5). The aim of this section is to describe type-preserving restrictions.

Example 1 Let $\Sigma = (y \frac{\partial}{\partial x}, dx) \in \mathcal{A}_{\text{inv}}^{3,2}$. This system is of type (E^3) (cf. Theorem 1). Consider two surfaces:

$$\begin{aligned} S_1 &= \{(x, y, 0) \mid (x, y) \in \mathbf{R}^2\}, \\ S_2 &= \{(x, \lambda x + z^2, z) \mid (x, z) \in \mathbf{R}^2\}. \end{aligned}$$

$\Sigma_1 = \Sigma|_{S_1}$ is feedback equivalent to the system

$$\Sigma_1 = \left(y \frac{\partial}{\partial x}, dx \right) \in \mathcal{A}^{2,1}.$$

Hence Σ_1 and Σ are of the same type (E^n) . It is not so, however, for $\Sigma_2 = \Sigma|_{S_2}$. We have, up to a feedback equivalence,

$$\Sigma_2 = \left((\lambda x + y^2) \frac{\partial}{\partial x}, dx \right) \in \mathcal{A}^{2,1}.$$

Hence, Σ_2 is of type (EC^2) .

Given a surface S , let us denote the original system by Σ and the restricted one by $\Sigma' = \Sigma|_S$. Similarly, let C and C' denote the critical sets of Σ and Σ' , respectively. A general rule is that these systems have the same type if $C = C'$. Obviously, for this to happen, the surface S must contain the set C . If so, then $C \subset C'$. For the converse inclusion to hold, S must satisfy some non-degeneracy conditions. Suppose that S satisfies them. As one may expect, if we restrict the system Σ to a surface S_1 close enough to S , then $\Sigma|_S$ and $\Sigma|_{S_1}$ are of the same type. The appropriate measure of the

Fig. 9 Restriction to the surface S_2

nearness is the order of tangency of C to the surfaces. Indeed, if the order of tangency of C to the surface S is high enough (two will be sufficient in our considerations) and some non-degeneracy conditions concerning S hold, then the restricted and the original systems are of the same type. Similarly, one may restate the above considerations in the framework of 1-parameter families. In the rest of this section, for a fixed family Σ_μ , we give conditions that guarantee that the family restricted to a fixed surface $S \ni q$ is of the same type (described by one of the conditions (F1)–(F9)) as the original one. This problem is a natural generalisation of that regarding systems because every system may be treated as a constant (with respect to the parameter) 1-parameter family. For brevity, we consider families around points of the form $(q, \mu) = (q, 0)$. Now, the critical set C_0 constructed for the family Σ_μ with $\mu = 0$ plays the same role as the set C for the system Σ in the above considerations.

Let us start with the simplest cases in which we do not impose conditions on the tangency of C_0 to the surface S .

Proposition 4 Consider a 1-parameter family $\Sigma_\mu = f_\mu + \Delta_\mu \in \mathcal{A}_{inv}^{n,n-1}$ around a fixed point $(q, 0) \in M \times \mathbf{R}$. Let $S \ni q$ be an arbitrary surface transversal to Δ_0 at q , i.e., $T_q S \not\subset \Delta_0(q)$. Assume that Σ_μ satisfies either one of the condition (F1)–(F4) or (F6) at q . Then $\Sigma_\mu|_S$ satisfies the same conditions (precisely, their planar counterparts) if and only if

$$\begin{array}{ll} \Sigma_\mu \text{ satisfies (F1) or (F2) and} & de(v) \neq 0, \\ \Sigma_\mu \text{ satisfies (F3), (F4), or (F6) and} & \tau(v, v) \neq 0, \end{array}$$

where v is any non-zero vector belonging to $T_q S \cap \Delta_0(q)$ and inequalities are meant to be satisfied at q .

Proof We may assume that $q = 0$ and $\Sigma_\mu = (F_\mu \frac{\partial}{\partial x}, dx)$, where F_μ has either one of the first four or the sixth normal form listed in Theorem 2. Transversality of S to $\Delta_0(q)$ at q means that S may be parametrised in the following way

$$p : (r, s) \mapsto (r, y(r, s), z_2(r, s), \dots, z_{n-1}(r, s)), \quad (6)$$

where $y, z_2, \dots, z_{n-1}$ are smooth functions and there exists at least one among them with non-vanishing partial derivative in the s -direction. Thus the restricted family $\Sigma_\mu|_S$ is represented, up to a feedback transformation, by

$$\begin{aligned}\dot{r} &= F_\mu(r, y(r, s), z_2(r, s), \dots, z_{n-1}(r, s)), \\ \dot{s} &= u.\end{aligned}$$

If Σ_μ is of one of the first two forms listed in Theorem 2, then $de = dy$ and $\dot{r} = \gamma + y(r, s)$ with γ being zero for the first form and one for the second. For the restricted system to be of the appropriate type, $\eta(q) = \eta(0)$ must be non-zero. In the coordinates (r, s) , it means exactly that $\frac{\partial y}{\partial s}(0) \neq 0$. This, in turn, is equivalent to the condition

$$\frac{\partial p}{\partial s}(0) \lrcorner dy(0) \neq 0.$$

Since $de = dy$ and $\frac{\partial p}{\partial s}(0)$ spans the space $T_q S \cap \Delta_0(q)$, the assertion for the first two cases is proved.

Consider the cases where Σ_μ satisfies one of the conditions (F3), (F4), or (F6). By the normal form of F_μ and direct computations we see that the restricted family is of the appropriate type if and only if the bilinear form τ constructed for that restricted family is non-degenerated (all the other subconditions of (Fi) are evidently satisfied). To omit ambiguity we denote this form by τ' keeping the symbol τ for the form constructed for the original family Σ_μ . Non-degeneracy of τ' means that $\tau'(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) \neq 0$ which is equivalent to $\tau(w, w) \neq 0$, where $w = \frac{\partial}{\partial s}p(0)$. Since $T_q S \cap \Delta_0(q) = \text{span}(w)$, the assertion follows. $\square$

Proposition 5 *Consider a 1-parameter family $\Sigma_\mu = f_\mu + \Delta_\mu \in \mathcal{A}_{inv}^{n,n-1}$ around a fixed point q . Assume that Σ_μ satisfies one of the conditions: either (F5), or (F7), or (F8). Let $S \ni q$ be an arbitrary surface transversal to Δ_0 and tangent to C_0 at q . Then the families Σ_μ and $\Sigma_\mu|_S$ are of the same type.*

Proposition 6 *Let $\Sigma_\mu = f_\mu + \Delta_\mu \in \mathcal{A}_{inv}^{n,n-1}$ be defined around a fixed point q and satisfy condition (F9). Let $S \ni q$ be an arbitrary surface transversal to Δ and such that C_0 is tangent of order 2 to S at q . Then Σ_μ and $\Sigma_\mu|_S$ are of the same type.*

Propositions 5 and 6 may be proved similarly to Proposition 4. The main point of the proof is the fact that the tangency of C_0 to S implies that for the parametrisation (6) of S , we additionally have

$$\frac{\partial y}{\partial s}(0) \neq 0, \quad \frac{\partial z_i}{\partial s}(0) = 0, \quad i = 2, \dots, n-1.$$

In the case of second order tangency we have also

$$\frac{\partial^2 z_i}{\partial s^2}(0) = 0, \quad i = 2, \dots, n-1.$$

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Arnold VI, Afrajmovich VS, Il'yashenko YuS, Shil'nikov LP (1999) Bifurcation theory and catastrophe theory, Springer, Berlin. MR MR1733750
2. Abed EH, Fu J-H (1986) Local feedback stabilization and bifurcation control. I. Hopf bifurcation, Syst Control Lett 7(1):11–17. MR MR833060 (87e:93085)
3. Abed EH, Fu J-H (1987) Local feedback stabilization and bifurcation control. II. Stationary bifurcation, Syst Control Lett 8(5):467–473. MR MR890085 (88e:93076)
4. Agrachev A, Zelenko I (2007) On feedback classification of control-affine systems with one and two-dimensional inputs. SIAM J Control Optim 46(4):1431–1460 (electronic). MR MR2346387 (2009e:93037)
5. Baitmann M (1978) Controllability regions on the plane. Diff Equat 14:407–417
6. Baitmann M (1978) Switching lines in the plane. Diff Equat 14:1093–1101
7. Bröcker Th, Lander L (1975) Differentiable germs and catastrophes, Cambridge University Press, Cambridge. MR MR0494220 (58 #13132)
8. Bonnard B (1991) Feedback equivalence for nonlinear systems and the time optimal control problem. SIAM J Control Optim 29(6):1300–1321. MR MR1132184 (93b:93030)
9. Bonnard B (1991) Quadratic control systems. Math Control Signals Syst. 4(2):139–160. MR MR1092633 (92d:93044)
10. Bressan A, Piccoli B (1998) A generic classification of time-optimal planar stabilizing feedbacks. SIAM J Control Optim 36(1):12–32 (electronic). MR MR1616525 (99c:93085)
11. Boscain U, Piccoli B (2004) Optimal syntheses for control systems on 2-D manifolds, mathematics and applications, vol 43, Springer, Berlin. MR MR2031058 (2005m:49002)
12. Čelikovský S, Nijmeijer H (1996) Equivalence of nonlinear systems to triangular form: the singular case, Syst Control Lett 27(3):135–144. MR MR1387097 (97b:93045)
13. Davydov AA (1988) Singularities of limit direction fields of two-dimensional control systems. Mat Sb (N.S.) 136(178), no. 4, 478–499, 590. MR MR965888 (89k:58209)
14. Davydov AA (1994) Qualitative theory of control systems, translations of mathematical monographs, vol 141, American Mathematical Society, Providence. MR MR1297761 (95h:93005)
15. Gardner RB, Shadwick WF, Wilkens GR (1989) Feedback equivalence and symmetries of Brunowski normal forms, dynamics and control of multibody systems (Brunswick, ME, 1988), Contemp. Math., vol 97, Amer. Math. Soc., Providence, pp 115–130. MR MR1021034 (91a:93012)
16. Gardner RB, Shadwick WF, Wilkens GR (1989) A geometric isomorphism with applications to closed loop controls, SIAM J Control Optim 27(6):1361–1368. MR MR1022432 (90i:93020)
17. Hunt LR, Su R (1981) Linear equivalents of nonlinear time varying systems, Proc. MTNS (Santa Monica, CA), pp 119–123
18. Iooss G, Joseph DD (1980) Elementary stability and bifurcation theory, Springer, New York. MR MR636256 (83e:34003)
19. Jakubczyk B (1990) Equivalence and invariants of nonlinear control systems, Nonlinear controllability and optimal control, Monogr. Textbooks Pure Appl. Math., vol 133, Dekker, New York, pp 177–218. MR MR1061386 (91c:93045)
20. Jakubczyk B (1998) Critical Hamiltonians and feedback invariants, Geometry of feedback and optimal control, Monogr. Textbooks Pure Appl. Math., vol 207, Dekker, New York, pp 219–256. MR MR1493015
21. Jakubczyk B (2001) Feedback invariants, critical trajectories and Hamiltonian formalism, Nonlinear control in the year 2000, vol 1 (Paris), Lecture Notes in Control and Inform. Sci., vol 258, Springer, London, pp 545–568. MR MR1806157 (2001j:93026)
22. Jakubczyk B (2006) Equivalence of deformations of functions germs (preprint)
23. Jakubczyk B, Respondek W (1980) On linearization of control systems. Bull Acad Polon Sci Sér Sci Math 28:(9–10):517–522 (1981). MR MR629027 (82k:93020)

24. Jakubczyk B, Respondek W (1989) Feedback equivalence of planar systems and stabilizability, *Robust control of linear systems and nonlinear control* (Amsterdam, 1989), *Progr. Systems Control Theory*, vol 4, Birkhäuser Boston, Boston, MA, pp 447–456. MR MR1115416 (92g:93079)
25. Jakubczyk B, Respondek W (1991) Feedback classification of analytic control systems in the plane, *Analysis of controlled dynamical systems* (Lyon, 1990), *Progr. Systems Control Theory*, vol 8, Birkhäuser Boston, Boston, MA, pp 263–273. MR MR1132000 (93b:93027)
26. Jakubczyk B, Respondek W (2004) On bifurcations of quadratic planar control systems. In: *Proceedings of the NOLCOS'04* (Stuttgart)
27. Jakubczyk B, Respondek W (2005) Phase portraits of control systems on the plane and their bifurcations. *Control Cybern* 34:819–847
28. Jakubczyk B, Respondek W (2006) Bifurcations of 1-parameter families of control-affine systems in the plane. *SIAM J Control Optim* 44(6):2038–2062
29. Kang W (1995) Quadratic normal forms of nonlinear control systems with uncontrollable linearization. In: *Proceedings of the 34th CDC* (New Orleans)
30. Kang W (1996) Extended controller form and invariants of nonlinear control systems with a single input. *J Math Syst Estim Control* 6(1):27–51. MR MR1361291 (96h:93018)
31. Kang W (1998) Bifurcation and normal form of nonlinear control systems. I and II. *SIAM J Control Optim* 36(1):193–212 and 213–232 (electronic). MR MR1616557 (99c:93054) and MR1616561 (99c:93055)
32. Kang W, Krener AJ (1992) Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems. *SIAM J Control Optim* 30(6):1319–1337. MR MR1185625 (93k:93054)
33. Krener AJ, Kang W, Chang DE (2004) Control bifurcations. *IEEE Trans Automat Control* 49(8):1231–1246. MR MR2078236 (2005e:34123)
34. Kuznetsov YA (1998) *Elements of applied bifurcation theory*, applied mathematical sciences, vol 112, Springer, New York. MR MR1711790 (2000f:37060)
35. Martinet J (1982) *Singularities of smooth functions and maps*, London Mathematical Society Lecture Note Series, vol 58, Cambridge University Press, Cambridge. MR MR671585 (83i:58018)
36. Respondek W (1998) Feedback classification of nonlinear control systems on $\mathbf{R}^2$ and $\mathbf{R}^3$, *Geometry of feedback and optimal control*, Monogr. Textbooks Pure Appl. Math., vol 207, Dekker, New York, pp 347–381. MR MR1493018
37. Respondek W (1998) Orbital feedback linearization of single-input control systems. In: *Proceedings of the NOLCOS'98* (Enschede)
38. Respondek W (2003) Transforming a single input system to a p -normal form via feedback. In: *Proceedings of the 42th IEEE conference on decision and control* (Maui, Hawaii), pp 1574–1579
39. Respondek W, Tall IA (2006) Feedback equivalence of nonlinear control systems: a survey on formal approach, *Normal Forms, Bifurcations, Chaos in automatic: from theory towards applications*. Taylor and Francis, pp 137–262
40. Rupniewski MW (2007) Local bifurcations of control affine systems in the plane. *J Dyn Control Sys* 13:135–159. MR MR2288946 (2007m:93017)
41. Respondek W, Zhitomirskii M (1995) Feedback classification of nonlinear control systems on 3-manifolds. *Math. Control Signals Syst* 8(4):299–333. MR MR1403291 (97i:93051)
42. Sampei M, Furuta K (1986) On time scaling for nonlinear systems: application to linearization. *IEEE Trans Aut Contr* 31:459–462
43. Sussmann HJ (1987) The structure of time-optimal trajectories for single-input systems in the plane: the C^∞ nonsingular case. *SIAM J Control Optim* 25(2):433–465. MR MR877071 (88e:49052)
44. Sussmann HJ (1987) The structure of time-optimal trajectories for single-input systems in the plane: the general real analytic case, *SIAM J Control Optim* 25(4):868–904. MR MR893988 (88h:93045)
45. Tall IA (2005) Feedback classification of multi-input nonlinear control systems. *SIAM J Control Optim* 43(6):2049–2070 (electronic). MR MR2179477
46. Tall IA, Respondek W (2002) Feedback classification of nonlinear single-input control systems with controllable linearization: normal forms, canonical forms, and invariants. *SIAM J Control Optim* 41(5):1498–1531 (electronic). MR MR1971960 (2004g:93036)
47. Zhitomirskii M, Respondek W (1998) Simple germs of corank one affine distributions, *Singularities Symposium—Łojasiewicz* 70 (Kraków, 1996; Warsaw, 1996), Banach Center Publ., vol 44, Polish Acad. Sci., Warsaw, pp 269–276. MR MR1677328 (2000b:58005)