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Entering and Leaving j-Facets

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Abstract. Let S be a set of n points in d-space, no $i+1$ points on a common $(i-1)$-flat for $1 \le i \le d$. An oriented $(d-1)$-simplex spanned by d points in S is called a *j-facet of S* if there are exactly j points from S on the positive side of its affine hull. We show: (*) *For $j \le n/2 - 2$, the total number of $(\le j)$-facets (i.e. the number of i-facets with $0 \le i \le j$) in 3-space is maximized in convex position (where these numbers are known).* A large part of this presentation is a preparatory review of some basic properties of the collection of j-facets—some with their proofs—and of relations to well-established concepts and results from the theory of convex polytopes (h-vector, Dehn–Sommerville relations, Upper Bound Theorem, Generalized Lower Bound Theorem). The relations are established via a duality closely related to the Gale transform—similar to previous works by Lee, by Clarkson, and by Mulmuley.

A central definition is as follows. Given a directed line ℓ and a j-facet F of S, we say that ℓ *enters* F if ℓ intersects the relative interior of F in a single point, and if ℓ is directed from the positive to the negative side of F. One of the results reviewed is a tight upper bound of $\binom{j+d-1}{d-1}$ on the maximum number of j-facets entered by a directed line.

Based on these considerations, we also introduce a vector for a point relative to a point set, which—intuitively speaking—expresses "how interior" the point is relative to the point set. This concept allows us to show that statement (*) above is equivalent to the Generalized Lower Bound Theorem for d-polytopes with at most $d+4$ vertices.

1. Introduction

Let S be a set of n points in $\mathbb{R}^d$ *in general position*, i.e. no $i+1$ points on a common $(i-1)$-flat for $1 \le i \le d$. An oriented $(d-1)$-simplex spanned by d points in S is called a *j-facet of S* if it has exactly j points from S on the positive side of its affine hull; hence, $j \in \mathbb{Z}$ and $0 \le j \le n-d$. There is an obvious correspondence between 0-facets and facets of the convex hull of S.

The maximum possible number of j-facets of an n-point set in $\mathbb{R}^d$ has raised some interest, starting with first bounds in the plane by Lovász [12] and Erdős et al. [8] in the early seventies. The currently best upper bound in the plane is of the order $n\sqrt[3]{j+1}$ due to Dey [7]. Planar point sets where the number of j-facets is of the order $n \cdot e^{\Omega(\sqrt{\log(j+1)})}$ for $2j \le n-2$ are known due to a recent construction by Tóth [19]. We refer the reader to [3] and [2] for more references, also on the related problem of "k-sets," and on geometric algorithms where the number of j-facets occurs in the analysis (but see also [17] for very recent developments on the upper bound in three dimensions).

The emphasis of the first part of this paper is on the structure of the collection of j-facets, and on relations to more established concepts in the theory of convex polytopes that go beyond the observation that 0-facets are facets of the convex hull. To this end, we define that a *directed line* ℓ *enters* j*-facet* F if it intersects the relative interior of F in a single point, and if ℓ is directed from the positive to the negative side of F. If, instead, ℓ is directed from the negative to the positive side of F, then we say that ℓ *leaves* F.

Section 2 proves that no line can enter more than $\binom{j+d-1}{d-1}$ j-facets of a finite point set in $\mathbb{R}^d$. The proof mimics McMullen's proof of the bound on the entries of the h-vector of a simplicial convex polytope for the Upper Bound Theorem [13]. Section 3 makes this relation more explicit via a duality closely related to the Gale transform. (For example, this duality translates the Dehn–Sommerville relations to the fact that every directed line enters and leaves the same number of j-facets.) In slightly different settings—perhaps not as explicit, albeit essentially equivalent—such a relation has been worked out and exploited by Lee [10], Clarkson [5] and Mulmuley [15] (see also Remark 3 at the end of this paper).

An alternative proof of the bound on the number of j-facets entered by a line—by induction on the dimension—is given in Section 4. Based on the tools used in this proof, we also introduce a vector for a point relative to a point set, which expresses "how interior" the point is relative to the point set. This vector relates to the g-vector for convex polytopes, and we can employ the rich theory developed there [18], [14]. In particular, the Generalized Lower Bound Theorem appears useful in our setting.

Finally, in Section 5 we close with a conclusion for the overall number of $(\le j)$-facets (i.e. the total number of i-facets with $i \le j$) of n-point sets. We show that for $j \le n/2 - 2$, the number of $(\le j)$-facets in $\mathbb{R}^3$ is maximized in convex position where these numbers are known to be $2\left(\binom{j+2}{2}n - 2\binom{j+3}{3}\right)$ (this extends a corresponding result of Alon and Győri in the plane [1]). In fact, this statement can be shown to be equivalent to the Generalized Lower Bound Theorem for d-polytopes with at most $d+4$ vertices.

Conventions. For brevity we use $(a_i)_i$ for the sequence $(a_i)_{i=0}^{\infty} = (a_0, a_1, \ldots)$. Most of the sequences we introduce will be defined for all $i \in \mathbb{Z}$, mostly with $a_i = 0$ for $i < 0$. Similarly, $\sum_i a_i$ denotes $\sum_{i=0}^{\infty} a_i$. However, all the sequences $(a_i)_i$ we employ in such sums will vanish except for a finite number of terms.

The binomial coefficient $\binom{a}{b}$, $a, b \in \mathbb{Z}$, is defined to be 0 for $b < 0$ or $a < b$.

2. Lines Entering j-Facets

Let $S \subseteq \mathbb{R}^d$ be a set of n points in general position. Let ℓ be a directed line disjoint from all convex hulls of $d-1$ points in S. For $j \in \mathbb{Z}$, let $\overline{h}_j = \overline{h}_j(\ell, S)$ denote the number of j-facets entered by line ℓ; hence, $\overline{h}_j = 0$ for $j < 0$ and for $j > n-d$.

Upper Bounds on the $\overline{h}_j$'s. We derive a number of simple facts. First observe that a directed line penetrates the convex hull of S at most once. This translates to

Fact 2.1. $\overline{h}_0 \leq 1$.

Next, we consider the sum $s^0 := \sum_j \overline{h}_j$. This sum denotes the overall number of $(d-1)$-simplices spanned by d points in S that are intersected by line ℓ. It is not too difficult to see that the sum $s^1 := \sum_j j\overline{h}_j$ denotes the number of d-simplices spanned by $d+1$ points in S that are intersected by line ℓ: Given a j-facet F entered by ℓ, there are exactly j d-simplices with facet F which are intersected by ℓ and where the last point of intersection is in F. Similarly, for $k \in \mathbb{Z}$, $s^k = s^k(\ell, S) := \sum_j \binom{j}{k}\overline{h}_j$ gives the number of $(k+d)$-element subsets of S whose convex hull is met by line ℓ; we have $s^k = 0$ for $k < 0$ and for $k > n-d$. Now observe that none of the values s^k changes if we move a point in S parallel to ℓ again in general position—the vector $(s^k)_k$ is invariant under such motions. On the other hand, we have the following inversion formula for sequences $(a_i)_i$ and $(b_j)_j$ of real numbers (proof omitted):

$$\forall i \geq 0, \quad a_i = \sum_j \binom{j}{i} b_j \qquad \Longleftrightarrow \qquad \forall j \geq 0, \quad b_j = \sum_i (-1)^{i+j} \binom{i}{j} a_i. \quad (1)$$

It asserts that $(s^k)_k$ determines $(\overline{h}_j)_j$. That is, the sequence $(\overline{h}_j)_j$ is also invariant under motions of points parallel to ℓ.

Fact 2.2. *If $p \in S$ is replaced by some other point p' again in general position on the line through p parallel to ℓ, then the sequence $(\overline{h}_j)_j$ does not change.*

In the next step we investigate the effect of removal of a point p in S, first the expected effect on the $\overline{h}$-sequence, if p is random. ($\mathbf{E}(X)$ denotes the expectation of random variable X.)

Fact 2.3. *For $j \in \mathbb{Z}$, $\mathbf{E}(\overline{h}_j(\ell, S\backslash\{p\})) = ((n-d-j)/n)\overline{h}_j + ((j+1)/n)\overline{h}_{j+1}$, where p is a random point chosen uniformly in S.*

Proof. For $0 \leq j \leq n-1-d$, a j-facet of $S\backslash\{p\}$ is either a j-facet of S with p one of the $n-d-j$ points on its negative side, or a $(j+1)$-facet of S with p one of the $j+1$ points on its positive side. For $j < 0$ and $j \geq n-d$ we get $\mathbf{E}(\overline{h}_j(\ell, S\backslash\{p\})) = 0$ as required. □

Fact 2.4. *For $j \in \mathbb{Z}$ and $p \in S$, $\overline{h}_j(\ell, S\backslash\{p\}) \leq \overline{h}_j$.*

Proof. For $0 \leq j \leq n-1-d$, Fact 2.2 allows us to move p so that it does not lie on the positive side of any $(j+1)$-facet of S entered by ℓ—without changing $\overline{h}_j$. Now the removal of p will not generate any new j-facets entered by ℓ. For $j < 0$ and $j \geq n-d$ the inequality is trivial. □

We have prepared all the ingredients for demonstrating the upper bounds for the $\overline{h}_j$'s. Facts 2.3 and 2.4 entail

$$\frac{n-d-j}{n}\overline{h}_j + \frac{j+1}{n}\overline{h}_{j+1} \leq \overline{h}_j,$$

for all j, and so

$$\overline{h}_{j+1} \leq \frac{j+d}{j+1}\overline{h}_j$$

for $j \geq 0$. Combined with Fact 2.1, this gives

$$\overline{h}_j \leq \binom{j+d-1}{j} = \binom{j+d-1}{d-1}$$

for $j \geq 0$.

Symmetry of $(\overline{h}_j)_{j\in\mathbb{Z}}$. We conclude this section by demonstrating the identity $\overline{h}_j = \overline{h}_{n-d-j}$. An $(n-d-j)$-facet entered by line ℓ corresponds to a j-facet left by ℓ by changing the orientation of the $(d-1)$-simplex. Hence, the identity claims that a directed line enters and leaves the same number of j-facets. The reader is encouraged to verify the relation via Fact 2.2, but we take a different path. First observe that

Fact 2.5. $\overline{h}_0 = \overline{h}_{n-d}$.

For $j, k \in \mathbb{Z}$, define

$$\overline{h}_j^k := \sum_{i=0}^{n-d} \binom{i}{j} \cdot \overline{h}_i \cdot \binom{n-d-i}{k-j}.$$

$\overline{h}_j^k$ is the overall number of j-facets in $(k+d)$-element subsets of S entered by line ℓ, i.e. $\overline{h}_j^k = \sum_{Q\in\binom{S}{k+d}} \overline{h}_j(\ell, Q)$: For an i-facet of S to become a j-facet in a $(k+d)$-element subset of S, we have to select j from the i points on its positive side, $k-j$ from the $n-d-i$ points on its negative side, and all d points that span the i-facet. Because of Fact 2.5, we have $\overline{h}_0^k = \overline{h}_k^k$, and so

$$0 = \underbrace{\sum_{i=0}^{n-d} \overline{h}_i \cdot \binom{n-d-i}{k}}_{\overline{h}_0^k} - \underbrace{\sum_{i=0}^{n-d} \binom{i}{k} \cdot \overline{h}_i}_{\overline{h}_k^k} = \sum_{i=0}^{n-d} \binom{i}{k} \cdot (\overline{h}_{n-d-i} - \overline{h}_i).$$

The inversion formula (1) tells us that these identities determine the terms $(\overline{h}_{n-d-i} - \overline{h}_i)$. Thus $\overline{h}_{n-d-i} - \overline{h}_i = 0$ for all $0 \leq i \leq n-d$ is the unique solution.

This counting argument makes explicit that the symmetry of the sequence $(\overline{h}_j)_j$ is an immediate consequence of the fact that the number of 0-facets entered equals the number of 0-facets left (Fact 2.5); this number happens to be 0 or 1, which is not essential in our proof, though.

We summarize the findings of this section.

Theorem 1. *Let S be a set of n points in $\mathbb{R}^d$ in general position, and let ℓ be a directed line disjoint from all convex hulls of $d-1$ points in S. The numbers $\overline{h}_j$ of j-facets of S entered by ℓ satisfy*

(i) $\overline{h}_j = \overline{h}_{n-d-j}$ *for all* $j \in \mathbb{Z}$, *and*
(ii)

$$\overline{h}_j \le \min\left\{\binom{j+d-1}{d-1}, \binom{n-j-1}{d-1}\right\}$$

for $0 \le j \le n-d$, *and* $\overline{h}_j = 0$, *otherwise.*

The bound in (ii) is a consequence of (i) and $\overline{h}_j \le \binom{j+d-1}{d-1}$. We will see later that there are point sets and lines where this bound is attained for all j.

3. Convex Polytopes and h-Vectors

Let $\mathcal{S}$ be a finite multiset of points in $\mathbb{R}^d$. For $i \in \mathbb{Z}$, let $\tilde{f}_i = \tilde{f}_i(\mathcal{S})$ be the number of $(i+1)$-element subsets of $\mathcal{S}$ that are contained in a supporting hyperplane. For P a convex polytope and $i \in \mathbb{Z}$, let $f_i = f_i(P)$ be the number of i-faces of P, where we agree on $f_{-1} = 1$ and $f_d = 0$. If S is a set in general position (in particular, there are no multiple copies of the same point), then conv S is a simplicial polytope and $\tilde{f}_i(S) = f_i(\text{conv } S)$ for all $i \in \mathbb{Z}$.

The *h-vector* $(h_j)_{j=0}^d = (h_j(P))_{j=0}^d$ of a simplicial convex polytope P can be defined as the unique sequence of numbers satisfying (recall (1))

$$\forall i, \quad 0 \le i \le d, \qquad f_{i-1} = \sum_{j=0}^{d} \binom{j}{d-i} \cdot h_j,$$

see [20]. We skip here the more geometric equivalent description of the h-vector via shellings. Important properties of the h-vector of a simplicial n-vertex d-polytope are:

- The *Dehn–Sommerville Relations*:

$$\forall j, \quad 0 \le j \le d, \qquad h_j = h_{d-j}.$$

- The *Upper Bound Theorem* [13]:

$$\forall j, \quad 0 \le j \le d, \qquad h_j \le \min\left\{\binom{j+n-d-1}{n-d-1}, \binom{n-j-1}{n-d-1}\right\},$$

  and this bound is attained for all j for the convex hull of n points on the moment curve $\{(t^i)_{i=1}^d \mid t \in \mathbb{R}\}$.
- The *Generalized Lower Bound Theorem (GLBT)*:[1]

$$\forall j, \quad 1 \le j \le (d+1)/2, \qquad h_{j-1} \le h_j.$$

[1] Sometimes, the statement is presented for $j \le d/2$. However, for d odd and $j = (d+1)/2$, we have $h_{j-1} = h_j$ because of the Dehn–Sommerville Relations.

The only proof known for the GLBT goes via the g-theorem, which characterizes all possible h-vectors of simplicial d-polytopes [4], [18], [14].

Orthogonal Dual. We describe a duality between sequences of n points in $\mathbb{R}^d$ and $\mathbb{R}^{n-d-1}$ that is closely related to the Gale transform, see [9] and [20] (see the remark preceding Lemma 2). This allows us to relate the h-vector of simplicial convex polytopes to the $\overline{h}$-sequences we have considered in Section 2.

For integers $0 \leq d < n$, we call a matrix $A \in \mathbb{R}^{n\times d}$ *legal* if $A^\top \cdot \vec{1} = \vec{0}$ and if A has full rank d. We use $\vec{1}$ and $\vec{0}$ for vectors of all 1's and 0's, respectively, of appropriate dimension; here $\vec{1} = 1^n$ and $\vec{0} = 0^d$. We interpret matrix A as a sequence $S_A = (p_i)_{i=1}^n$ of n points in $\mathbb{R}^d$ in the obvious way: the ith row gives coordinates of p_i. The conditions for "legal" translate to the facts that the origin is the center of gravity of the points in S_A, and that there is no hyperplane containing all points in S_A—an assumption much weaker than general position!

Given legal matrices $A \in \mathbb{R}^{n\times d}$ and $B \in \mathbb{R}^{n\times(n-d-1)}$, we call B *an orthogonal dual of* A, in symbols $A \perp B$, if $A^\top \cdot B = 0^{d\times(n-d-1)}$. In other words, the columns of A are orthogonal to the columns of B. That is, the columns of A span a linear vector space of dimension d orthogonal to the linear space of dimension $n-d-1$ spanned by the columns of B, and both spaces are orthogonal to $\vec{1}$. Hence, given a legal matrix A, there is always an orthogonal dual B which is unique up to linear transformations. Clearly, $A \perp B \iff B \perp A$. (This convenient symmetry, enforced by the condition $A^\top \cdot \vec{1} = \vec{0}$, is the only difference to the standard Gale transform—apart from expository details.)

Lemma 2. *For $0 \leq d < n$, let $A \in \mathbb{R}^{n\times d}$ and $B \in \mathbb{R}^{n\times(n-d-1)}$ be legal matrices with $A \perp B$, and let $S_A = (p_i)_{i=1}^n$ and $S_B = (p_i^*)_{i=1}^n$. For some $I \subseteq \{1, 2, \ldots, n\}$, let $F := \{p_i \mid i \in I\}$ and $\overline{F^*} := \{p_i^* \mid i \notin I\}$.*

(i) *If F is contained in a supporting hyperplane of the points in S_A, then $\mathbf{0} \in \operatorname{conv} \overline{F^*}$.*
(ii) *If $\mathbf{0} \in \operatorname{conv} F$, then $\overline{F^*}$ is contained in a supporting hyperplane of the points in S_B.*

Proof. Let F lie in a supporting hyperplane. That is, there is a vector $v \in \mathbb{R}^{d+1}$, such that for $\lambda = (\lambda_i)_{i=1}^n := (A\vec{1}) \cdot v$, we have $\lambda \neq \vec{0}$, $\lambda_i \geq 0$ for all $1 \leq i \leq n$ and $\lambda_i = 0$ for $i \in I$. ($(A\vec{1})$ denotes the matrix A with an extra column of 1's.) Moreover,

$$B^\top \cdot \lambda = \underbrace{B^\top \cdot (A\vec{1})}_{0^{(n-d-1)\times(d+1)}} v = \vec{0}$$

which means that the origin is a positive linear (and thus convex) combination of points p_i^* with $i \notin I$.

For the reverse direction (ii), let $\lambda \in \mathbb{R}^n$ be a vector that witnesses the fact that $\mathbf{0} \in \operatorname{conv} F$. That is, $0 \leq \lambda \neq \vec{0}$, $A^\top\lambda = \vec{0}$, and $\lambda_i = 0$ for $p_i \notin F$; if $p_i \notin F$, then $i \notin I$. λ is orthogonal to the linear space spanned by the columns in A; consequently, it is in the linear space spanned by the columns of $(B\vec{1})$, and there is a vector v with $(B\vec{1}) \cdot v = \lambda$. Hence, v corresponds to a supporting hyperplane that contains all p_i^* with $\lambda_i = 0$. Since $\lambda_i = 0$ for $i \notin I$, the hyperplane contains all points in $\overline{F^*}$. □

f- and h-Vector under Orthogonal Duals. For $\mathcal{S}$ a finite multiset of points in $\mathbb{R}^d$, φ an i-flat, and $k \in \mathbb{Z}$, let $s^k = s^k(\varphi, \mathcal{S})$ denote the number of $(k + d + 1 - i)$-element subsets of $\mathcal{S}$ whose convex hull is intersected by φ. This generalizes our definition for lines from the previous section. We employ it here also for points (i.e. 0-flats).

Lemma 3. *For $0 \le d < n$, let $A \in \mathbb{R}^{n\times d}$ and $B \in \mathbb{R}^{n\times(n-d-1)}$ be legal matrices with $A \perp B$, and let $\mathcal{S} \subseteq \mathbb{R}^d$ and $\mathcal{S}^* \subseteq \mathbb{R}^{n-d-1}$ be the multisets of points in S_A and S_B, respectively. Then*

$$\tilde{f}_i(\mathcal{S}) = s^{d-i-1}(\mathbf{0}, \mathcal{S}^*) \quad \text{and} \quad \tilde{f}_i(\mathcal{S}^*) = s^{n-d-i-2}(\mathbf{0}, \mathcal{S}).$$

Proof. There is a bijection of $(i + 1)$-element subsets of $\mathcal{S}$ contained in supporting hyperplanes and $(n - (i + 1))$-element subsets of $\mathcal{S}^*$ that contain $\mathbf{0}$ in their convex hull, and $(d - i - 1) + (n - d - 1) + 1 = n - (i + 1)$; therefore, the left equality. The right equality follows from the symmetry of orthogonal duality. □

Theorem 4.

(i) *If $(h_j)_{j=0}^d$ is the h-vector of a simplicial n-vertex d-polytope, then there is a set S of n points in general position in $\mathbb{R}^{n-d}$, and a line ℓ disjoint from all convex hulls of $(n - d) - 1$ points in S, such that $\overline{h}_j(\ell, S) = h_j$ for $0 \le j \le d$.*
(ii) *Let S be a set of n points in general position in $\mathbb{R}^d$, and let ℓ be a line disjoint from all convex hulls of $d - 1$ points in S. If ℓ intersects the convex hull of S, then there is a simplicial m-vertex $(n - d)$-polytope P with $m \le n$ and $h_j(P) = \overline{h}_j(\ell, S)$ for $0 \le j \le n - d$.*

Proof. Let P be a simplicial n-vertex d-polytope, and let V be the set of vertices of P. Since P is simplicial, a small perturbation of the vertex set of P that does not change its f-vector allows us to assume that $V \cup \{c\}$, c the centroid of V, is a set of $n + 1$ points in general position in $\mathbb{R}^d$. Moreover, a translation of P allows us to assume that the origin is the centroid of V. Let $A \in \mathbb{R}^{n\times d}$ be a matrix which has the coordinates of the points in V in its rows. Now consider an orthogonal dual $B \in \mathbb{R}^{n\times(n-d-1)}$ of A, and let T be the multiset of points in S_B. The general position of $V \cup \{\mathbf{0}\}$ implies that $T \cup \{\mathbf{0}\}$ is a *set* of $n + 1$ points in general position (argument omitted). We have $f_i(P) = \tilde{f}_i(V) = s^{d-i-1}(\mathbf{0}, T)$. Now we lift $T \subseteq \mathbb{R}^{n-d-1}$ to a set $S \subseteq \mathbb{R}^{n-d}$ by adding to each point in T an $(n - d)$th coordinate, such that S is in general position in $\mathbb{R}^{n-d}$ (random coordinates uniform from $[0, 1)$ will do with probability 1). Let ℓ denote the x_{n-d}-axis directed toward $x_{n-d} = +\infty$. Obviously, $s^{d-i-1}(\mathbf{0}, T) = s^{d-i-1}(\ell, S)$, and so, according to the relation between $s^k(\ell, S)$ and $\overline{h}_j = \overline{h}_j(\ell, S)$ we had derived in Section 2,

$$\sum_{j=0}^{d} \binom{j}{d-i-1} \cdot h_j = \tilde{f}_i(V) = s^{d-i-1}(\ell, S) = \sum_{j=0}^{d} \binom{j}{d-i-1} \cdot \overline{h}_j \tag{2}$$

for $-1 \le i \le d - 1$. Equation (2) implies $h_j = \overline{h}_j$ for $0 \le j \le d$ via (1), and we have completed the proof of statement (i).

For the proof of (ii), let $S \subseteq \mathbb{R}^d$ and ℓ as in the claimed statement, with $\ell \cap \operatorname{conv} S \neq \emptyset$. A suitable projection and perturbation gives a set $T \subseteq \mathbb{R}^{d-1}$ and $x \in \mathbb{R}^{d-1}$ such that $T \cup \{x\}$ is in general position, $x \in \operatorname{conv} T$ and $s^k(x, T) = s^k(\ell, S)$ for all $k \in \mathbb{Z}$. Let c be the centroid of T. We first assume that $c = x$. Then we apply a translation which maps $c = x$ to the origin $\mathbf{0}$. Now we apply the orthogonal dual construction as in (i) which gives us a set V of points in $\mathbb{R}^{n-(d-1)-1} = \mathbb{R}^{n-d}$. $P = \operatorname{conv} V$ is the requested $(n-d)$-polytope with at most n vertices (employ an identity similar to (2)). If $c \neq x$, then there is a hyperplane H normal to $c - x$ and disjoint from $\operatorname{conv} T$, such that we can apply a projective transformation π which makes H the hyperplane at infinity with $\pi(x)$ the centroid of $\pi(T)$ and $s^k(x, T) = s^k(\pi(x), \pi(T))$ for all $k \in \mathbb{Z}$ (detailed argument omitted). Now we can proceed as before to show (ii). □

The theorem shows that not only the proof of Theorem 1 mimics McMullen's proof of the Upper Bound Theorem—the statements are actually equivalent to the Dehn–Sommerville Relations and the Upper Bound Theorem. The fact that the Upper Bound Theorem is tight for points on the moment curve implies that the bounds in Theorem 1 are tight. We do not give a proof of the Generalized Lower Bound Theorem in the "j-facet setting", but we will shortly interpret and use it in this setting.

4. Lines Entering j-Facets up to a Point

Alternative Proof for the Bounds on the $\overline{h}_j$'s. Let S' be a set of n points in $\mathbb{R}^{d+1}$ in general position (it is $(d+1)$-space now!). Let ℓ be a directed line parallel to the x_{d+1}-axis and disjoint from all convex hulls of d points in S'. For $j \in \mathbb{Z}$, let $\overline{h}_j = \overline{h}_j(\ell, S')$.

Let S be the orthogonal projection of S' to the hyperplane $x_{d+1} = 0$ and let x be the projection of ℓ. That is, by removing the last coordinate, we can consider $S \cup \{x\}$ as a set of points in $\mathbb{R}^d$. A small perturbation of S' that does not change the $\overline{h}_j$'s allows us to assume that $S \cup \{x\}$ is in general position.

We choose a directed line λ in $\mathbb{R}^d$ through x that is disjoint from all convex hulls of $d-1$ points in S. For $i \in \mathbb{Z}$, we let $\hat{h}_i = \hat{h}_i(x, \lambda, S)$ be the number of i-facets of S entered by λ before x (i.e. with x on the negative side).

We want to argue that

$$\overline{h}_j - \overline{h}_{j-1} = \hat{h}_j - \hat{h}_{n-d-j}, \tag{3}$$

for all $j \in \mathbb{Z}$. Before we proceed with this argument, note that $\hat{h}_i \leq \overline{h}_i(\lambda, S)$. If we know that $\overline{h}_i(\lambda, S) \leq \binom{i+d-1}{d-1}$, then from (3) it follows that

$$\overline{h}_j = \sum_{i=0}^{j} (\hat{h}_i - \hat{h}_{n-d-i}) \leq \sum_{i=0}^{j} \hat{h}_i \leq \sum_{i=0}^{j} \binom{i+d-1}{d-1} = \binom{j+(d+1)-1}{(d+1)-1}$$

and we have an inductive proof of the upper bound $\binom{j+d-1}{d-1}$ starting in dimension $d = 1$.

So why does (3) hold? We count the number $s^k = s^k(x, S)$ of $(k+d+1)$-element subsets of S whose convex hulls contain x, or, equivalently, the number $s^k(\ell, S')$ of

$(k + (d + 1))$-element subsets of S' whose convex hull is intersected by ℓ. For $k \in \mathbb{Z}$,

$$s^k(\ell, S') = \sum_i \underbrace{\binom{i}{k}}_{\binom{i+1}{k+1}-\binom{i}{k+1}} \cdot \overline{h}_i = \sum_i \binom{i}{k+1} \cdot (\overline{h}_{i-1} - \overline{h}_i), \tag{4}$$

where the first equality was derived in Section 2.

We develop the numbers s^k "directly" in $\mathbb{R}^d$ in the set S. To this end we observe a point ξ moving on λ toward x. As ξ enters an i-facet of S, it exits $\binom{i}{k+1}$ convex hulls of $k + d + 1$ points, and it penetrates $\binom{n-d-i}{k+1}$ convex hulls of $k + d + 1$ points in S'. This shows that

$$s^k(x, S) = \sum_{i=0}^{n-d} \hat{h}_i \cdot \left(\binom{n-d-i}{k+1} - \binom{i}{k+1}\right) = \sum_i \binom{i}{k+1} \cdot (\hat{h}_{n-d-i} - \hat{h}_i) \tag{5}$$

for $k \in \mathbb{Z}$. $s^k(\ell, S') = s^k(x, S)$, (4), and (5) imply (3) via (1).

Apart from the alternative proof of the upper bounds for the $\overline{h}_j$'s, we want to point out two implications of (3). First, the difference $\hat{h}_j - \hat{h}_{n-d-j}$ does not depend on the choice of line λ through x. Second, since we know from the GLBT that $\overline{h}_j \geq \overline{h}_{j-1}$ for $2j \leq (n - (d+1)) + 1 = n - d$, we can conclude that $\hat{h}_j - \hat{h}_{n-d-j} \geq 0$ for $2j \leq n - d$. In other words, the GLBT says that for $2j \leq n - d$, we can never leave more j-facets than we enter j-facets as we move along a line starting at a point outside the convex hull of S.

The g-Values of a Point Relative to S. Let S be a set of n points in $\mathbb{R}^d$ in general position, let x be a point not in S such that $S \cup \{x\}$ is in general position, and let λ be a directed line through x which is disjoint from all convex hulls of $d - 1$ points in S. We define

$$g_j = g_j(x, S) := \hat{h}_j(x, \lambda, S) - \hat{h}_{n-d-j}(x, \lambda, S)$$

for $0 \leq j \leq n - d$ (see Fig. 1). Recall that g_j does not depend on the choice of λ.

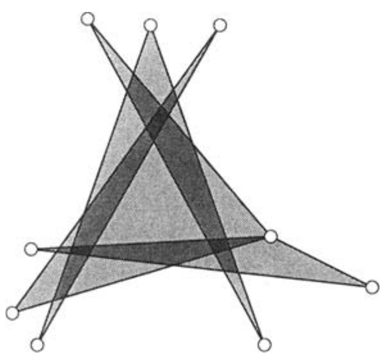

Fig. 1. The function $g_3(x, S)$ for a set S of nine points in the plane. Darker shading indicates larger $g_3(x, S)$ for points x in that area.

Lemma 5.

(i) $g_j = -g_{n-d-j}$ *for* $0 \le j \le n-d$.
(ii) *For* $n-d$ *even*, $g_{(n-d)/2} = 0$.
(iii) $g_j \ge 0$ *for* $0 \le 2j \le n-d$.
(iv) $s^k(x, S) = -\sum_{i=0}^{n-d} \binom{i}{k+1} g_i(x, S)$ *for all* $k \in \mathbb{Z}$.
(v) $x \notin \operatorname{conv} S$ *iff* $g_j(x, S) = 0$ *for all* $j, 0 \le j \le n-d$.

Recall that (iii) is equivalent to the GLBT for simplicial $(n-d-1)$-polytopes with at most n vertices. While this statement seems to be difficult to prove, the reader is encouraged to verify it for $j < (n-d)/d$ via centerpoints (see [10]): Given $S \subseteq \mathbb{R}^d$, a point $c \in \mathbb{R}^d$ is called a *centerpoint* if every hyperplane containing c has at most $d|S|/(d+1)$ points from S on either side. Such a centerpoint exists for every finite point set.

In the next section we use

$$\Gamma_j = \Gamma_j(S) := \sum_{p \in S} g_j(p, S \setminus \{p\})$$

for $0 \le j \le (n-1)-d$, and

$$\Sigma^k = \Sigma^k(S) := -\sum_{i=0}^{(n-1)-d} \binom{i}{k+1} \Gamma_i \tag{6}$$

for $k \in \mathbb{Z}$. We record the immediate implications of Lemma 5 to the introduced values.

Lemma 6.

(i) *For* $0 \le j \le (n-1)-d$, $\Gamma_j = -\Gamma_{(n-1)-d-j}$.
(ii) *For* $(n-1)-d$ *even*, $\Gamma_{(n-d-1)/2} = 0$.
(iii) $\Gamma_j \ge 0$ *for* $2j \le (n-1)-d$.
(iv) *For all* $k \in \mathbb{Z}$, Σ^k *is the number of pairs* (p, Q), $Q \in \binom{S}{k+d+1}$, $p \in S \setminus Q$, *with* $p \in \operatorname{conv} Q$.
(v) *S is in convex position iff* $\Gamma_j = 0$ *for all* $j, 0 \le j \le (n-1)-d$.

5. A Conclusion

Given a set S of n points in $\mathbb{R}^d$ in general position, we denote by $e_j = e_j(S)$ the number of j-facets of S and we set $E_j = E_j(S) := \sum_{i \le j} e_i(S)$. We show a tight upper bound on E_j in 3-space for $2j \le n-4$. Two simple facts we will need below: $e_j = e_{n-d-j}$ and $E_{n-d} = 2\binom{n}{d}$.

First, we count the number of 0-facets of $(k+d)$-element subsets of S, i.e. $e_0^k := \sum_{Q \in \binom{S}{k+d}} e_0(Q)$, in terms of the E_j's:

$$e_0^k = \sum_j \binom{n-d-j}{k} \underbrace{e_j}_{e_{n-d-j}} = \sum_{j=0}^{n-d} \binom{j}{k} \underbrace{e_j}_{E_j - E_{j-1}}$$

$$= -\underbrace{\binom{0}{k}E_{-1}}_{0} + \sum_{j=0}^{n-d-1} \underbrace{\left(\binom{j}{k} - \binom{j+1}{k}\right)}_{-\binom{j}{k-1}} E_j + \binom{n-d}{k}\underbrace{E_{n-d}}_{2\binom{n}{d}}$$

$$= 2\overbrace{\binom{n}{k+d}\binom{k+d}{d}}^{\binom{n}{d}\binom{n-d}{k}} - \sum_{j=0}^{n-d-1}\binom{j}{k-1}E_j.$$

Second, we count the number of vertices of the convex hulls of $(k+d)$-element subsets of S, i.e. $f_0^k := \sum_{Q\in\binom{S}{k+d}} f_0(\text{conv } Q)$:

$$f_0^k + \Sigma^{k-2} = (k+d)\binom{n}{k+d}$$

since every pair (p, Q), $Q \in \binom{S}{k+d-1}$, $p \in S\backslash Q$, contributes either one to f_0^k (if $p \notin \text{conv } Q$) or one to Σ^{k-2} (if $p \in \text{conv } Q$). We substitute Σ^{k-2} according to (6):

$$f_0^k = (k+d)\binom{n}{k+d} + \sum_{j=0}^{n-d-1}\binom{j}{k-1}\Gamma_j.$$

In the plane, $e_0^k = f_0^k$ yields

$$\sum_{j=0}^{n-3}\binom{j}{k-1}(E_j+\Gamma_j) = \binom{n}{k+2}\underbrace{\left(2\binom{k+2}{2} - (k+2)\right)}_{k(k+2)}.$$

This equality is satisfied for and only for $E_j + \Gamma_j = (j+1)n$. In 3-space, Euler's Relation gives $e_0^k = 2f_0^k - 4\binom{n}{k+3}$ and

$$\sum_{j=0}^{n-4}\binom{j}{k-1}(E_j+2\Gamma_j) = \binom{n}{k+3}\underbrace{\left(2\binom{k+3}{3} - 2(k+3)+4\right)}_{k(k+1)(k+5)/3}.$$

Here, $E_j + 2\Gamma_j = 2(\binom{j+2}{2}n - 2\binom{j+3}{3})$ constitutes the unique solution.

Lemma 7.

(i) *In the plane,* $E_{n-2} = 2\binom{n}{2}$ *and* $E_j = (j+1)n - \Gamma_j$ *for* $0 \le j \le n-3$.
(ii) *In 3-space,* $E_{n-3} = 2\binom{n}{3}$ *and* $E_j = 2(\binom{j+2}{2}n - 2\binom{j+3}{3} - \Gamma_j)$ *for* $0 \le j \le n-4$.

Lemma 6(iii) and (v) provide

Corollary 8.

(i) *In the plane,* $E_j \le (j+1)n$ *for* $0 \le 2j \le n-3$ *with equality for* S *in convex position.*

(ii) *In 3-space,* $E_j \le 2(\binom{j+2}{2}n - 2\binom{j+3}{3})$ *for* $0 \le 2j \le n-4$ *with equality for S in convex position.*

Bound (i) has been previously established in [1] and [16]. Bound (ii) was known for $j \le n/4 - 2$ [2]. The restriction of "$2j \le n - d - 1$" is a crucial threshold for exact E_j-bounds, since, for $n - d$ even, $E_{(n-d)/2} = \binom{n}{d} + e_{(n-d)/2}/2$.

For constant dimension d, an asymptotic bound of the order $n^{\lfloor d/2 \rfloor}(j+1)^{\lceil d/2 \rceil}$—asymptotically tight for points on the moment curve—is known [6].

Remark 1. We write GLBT(d, n) for the statement of the Generalized Lower Bound Theorem for simplicial d-polytopes with at most n vertices. We have seen that GLBT$(d, d+3)$ implies Corollary 8(i), and GLBT$(d, d+4)$ implies part (ii) of that corollary. In fact, one can show now that GLBT$(d, d+3)$ is equivalent to (i) and GLBT$(d, d+4)$ is equivalent to (ii). That is, [1] and [16] have shown GLBT$(d, d+3)$.

The argument proceeds as follows. Suppose we have $d+4$ points in general position in $\mathbb{R}^d$, whose convex hull violates GLBT$(d, d+4)$. By the duality described in Section 3 this corresponds to a set of $n = d+4$ points in $\mathbb{R}^4$ and a directed line ℓ such that $\overline{h}_{j-1} > \overline{h}_j$ for some $2j \le d+1 = n-3$. Now we project this point set parallel to ℓ to obtain a three-dimensional n-point set S with a point x with $g_j(x, S) < 0$. Note that we can project S to a sphere centered at x without changing $g_j(x)$: clearly, such a projection will not change $s^k(x)$, $k \in \mathbb{Z}$, and so, due to Lemma 5(iv), it will not change the $g_j(x)$'s. Let S' be this projected set together with x, i.e. $|S'| = n+1$. Since all points in S' apart from x are extreme, we have $\Gamma_j(S') = g_j(x, S' \backslash \{x\}) < 0$, where $2j \le n-3 = |S'| - 4$. Now Lemma 7 infers the fact that S' has more $(\le j)$-facets than a set of $n+1$ points in convex position.

Remark 2. It is not clear how the bounds in Corollary 8 generalize to higher dimensions. All we can claim at this point (without providing the proof here) is that *if* the number of $(\le j)$-facets in 4-space is maximized in convex position for $2j \le n-5$, then it is maximized for points on the moment curve, or, more generally, by the vertex sets of neighborly polytopes (where these numbers are known).

Remark 3. We have mentioned relations to other papers in the Introduction. In Lee's contribution [10] the duality is worked out, and a winding number is introduced, equivalent to the g_j-values of a point we defined here. Also a proof of GLBT$(d, d+3)$ in this dual setting is presented.

In [5] Clarkson presents a nice probabilistic proof for an upper bound of $\binom{j+d-1}{d-1}$ for the number of so-called local minima in j-levels of arrangements of hyperplanes in d-space. This translates to the bounds for the number of j-facets entered by a line (by polar duality). He uses LP-duality to show that this way he gave a new proof of the Upper Bound Theorem.

Finally, Mulmuley considers in [15] so-called h-matrices of bounded k-complexes of arrangements of hyperplanes. "Our" h- and $\overline{h}$-vector appears in such an h-matrix as the first row and column. Again, properties are derived similar to the Upper Bound Theorem and Dehn–Sommerville Relations.

One difference between our setting and the ones (related by polar duality) in [5] and [15] is that they have to add extra objects in order to ensure boundedness—an issue that never occurs in our scenario.

Remark 4. A *k-set* of a finite set S in $\mathbb{R}^d$ is a subset K of S that can be separated from $S \backslash K$ by a hyperplane. By the relation between k-sets and j-facets mentioned in Theorem 3 of [2], Corollary 8 implies that for $k \leq n/2 - 1$ the number of $(\leq k)$-sets of n-point sets in $\mathbb{R}^3$ is maximized in convex position.

Remark 5. We refer to a paper by Linhart [11], since he proves the same bound for a similar problem. We briefly translate his setting to a scenario comparable with ours. We are given a set S' of $n+2$ points in general position in $\mathbb{R}^d$. Let x and y be two distinct points in S', and $S := S' \backslash \{x, y\}$. For $0 \leq j \leq n+1-d$, we denote by $\hat{e}_j$ the number of j-facets of $S \cup \{x\}$ incident to x and with y on its positive side; $\hat{E}_j := \sum_{i=0}^{j} \hat{e}_j$. Then Linhart proves that *for all* $0 \leq j \leq n-(d-1)$ we have $\hat{E}_j \leq (j+1)n$ if $d=3$, we have $\hat{E}_j \leq 2\left(\binom{j+2}{2}n - 2\binom{j+3}{3}\right)$ if $d=4$, and we have $\hat{E}_j \leq n\left(\binom{j+2}{2}(n-1) - 2\binom{j+3}{3}\right)/2$ if $d=5$.

So how does this relate to our problem of counting all j-facets? If x can be separated from S by a hyperplane H, then we can consider S'', the set of intersections of the segments $\overline{xp}$, $p \in S$, with the hyperplane H. Clearly, there is a bijection between the j-facets of S incident to x on one hand, and the j-facets of S'' in H on the other hand. That is, on one hand, the bound we obtained here for $(\leq j)$-facets in 3-space implies Linhart's bound in 4-space only when x is separable; on the other hand, we are not restricted to j-facets containing a specific point y. Hence the results are incomparable. It explains why Linhart's bounds are valid for all j, while this cannot be the case for our problem.

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