Cyclic Product Theorems for Polygons (I) Constructions using Circles

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Abstract. If a point U_i is chosen on each edge of a plane n -gon P , then the product of the n signed ratios in which the points U_i divide the edges is called a *cyclic product* for P . The basic problem is to find geometrical constructions such that, for every n -gon P , the corresponding cyclic products either take constant values or satisfy simple relations. Many straight-line constructions are known. Here we describe some constructions which also involve circles.

Introduction

Let $P = [V_0, \dots, V_{n-1}]$ be an n -gon in the Euclidean plane, and let k be a positive integer ($k < n$). Also suppose that, for $i = 0, \dots, n-1$, some geometric construction leads to a point U_i on the edge or diagonal $V_i V_{i+k}$ of P . Then we shall use the term *cyclic product* (for P) to mean an expression of the form $\prod (\|V_i U_i\| / \|U_i V_{i+k}\|)$. Here and throughout, when considering n -gons, the subscripts are to be taken modulo n and the product is from $i = 0$ to $i = n-1$. The double modulus signs show that *signed* or *directed* lengths of the indicated segments are to be used. Thus, assuming the (collinear) points V_i , U_i , V_{i+k} are distinct, $\|V_i U_i\| / \|U_i V_{i+k}\|$ is positive if and only if U_i lies between V_i and V_{i+k} .

Familiar examples of cyclic products which take a constant value are given by Menelaus' and Ceva's theorems [6, Chapter 8]. In Menelaus' theorem, U_i is the point of intersection of an arbitrary fixed line with $V_i V_{i+1}$ and then $\prod (\|V_i U_i\| / \|U_i V_{i+1}\|) = (-1)^n$; in Ceva's theorem, if $n = 2s + 1$ is odd and the lines $V_i U_{i+s}$ are concurrent, then $\prod (\|V_i U_i\| / \|U_i V_{i+1}\|) = 1$.

Dozens of other examples of cyclic products for n -gons are known (see [4], [5] and references given there). In all of these the points U_i are determined by straight-line constructions (as with Menelaus' and Ceva's theorems). Here we shall be concerned with a situation in which the points U_i are determined by constructions involving circles. Some of our results were conjectured or proved by Branko Grünbaum and appear in

his (unpublished) lecture notes [3]. Details and acknowledgements appear in the final section of this paper.

Our cyclic products are what Howard Eves calls *h-expressions* [1, p. 247]. These are clearly affine invariants, but Eves proves the remarkable fact that they are also projective invariants. The simplest example of an *h-expression* is the cross-ratio of four points on a line.

In this paper we restrict ourselves to cases where the value of a cyclic product, or the product of two cyclic products, takes a fixed numerical value. Clearly, there will exist other relations involving more than two such products, but to search for these would make the investigation unwieldy. In fact, although we have made a serious effort to find *all* relations involving one or two cyclic products, it is almost certain that others exist, especially if one admits more and more complicated constructions.

To avoid repetition we shall now introduce some notation that is used throughout. The letter P will always represent an n -gon with vertices $V_0, V_1, \dots, V_{n-1}$. If X, Y are distinct points, then XY is the line through X and Y , $[X, Y]$ is the closed line segment with end points X and Y , and $|XY|$ will be used to mean the unsigned length of this segment so that $\pm|XY| = \|XY\|$. Similarly, for three non-collinear points X, Y, Z , we shall use $|XYZ|$ and $\|XYZ\|$ to mean the unsigned and signed area of the triangle $[X, Y, Z]$. We recall that the signed area of a simple polygon is positive if its vertices are named in a counterclockwise direction, and negative if named in a clockwise direction.

The polygons we consider are quite general: vertices may be collinear or coincide, edges may intersect at relative interior points or overlap, etc. We make only two requirements. The first is that no three consecutive vertices of a polygon P are collinear, and the second depends on the construction under consideration, namely, that everything is well defined. Hence, in particular whenever a line is defined as the join of two points we shall suppose those points are distinct; when a point is defined as the intersection of two lines, then those two lines must not be parallel. When a circle is defined as passing through three points, then those points must not be collinear. Finally, in every cyclic product that occurs we shall suppose that none of the denominators vanish. It would, of course, be possible to specify, in each theorem, the exact restrictions that must hold, but as they are essentially obvious, to do so would be merely tedious and needlessly complicate the exposition without increasing its clarity.

The twelve theorems given here are divided into four types, but this is a matter of convenience only, and there is no doubt that other equally good classifications could be found.

We finish this Introduction with two remarks. The first of these enables us, in effect, to nearly double the number of theorems in the paper. If V_i, V_{i+k}, U_i, W_i are four collinear points such that $\|V_i U_i\|/\|U_i V_{i+k}\| = -\|V_i W_i\|/\|W_i V_{i+k}\|$, then these four points are said to form a *harmonic range*, and W_i is said to be the *harmonic conjugate* of U_i with respect to V_i and V_{i+k} . Clearly, $\prod(\|V_i U_i\|/\|U_i V_{i+k}\|) = (-1)^n \prod(\|V_i W_i\|/\|W_i V_{i+k}\|)$ and so each relation for cyclic products involving the points U_i can be translated into another involving the points W_i . In some cases (such as the theorems of Section 2) these harmonic conjugates arise in a very natural way; in others (such as those in Section 1) they do not do so, and in such cases we make no further mention of cyclic product relations derived using harmonic conjugates in this manner.

The second remark is that many cyclic product relations involving one such product can be transformed into one involving two products by a process we call *asymmetrisation*.

For example if a construction involving vertices V_{i-r}, V_i, V_{i+r} of the polygon P leads to a relation involving one cyclic product, then sometimes we can find a relation involving two cyclic products by using the same construction twice, once with V_{i-r}, V_i, V_{i+s} and once with V_{i-s}, V_i, V_{i+r} . When this is possible we give a brief statement; the proofs are minor modifications of the symmetric case.

1. Circle Constructions of the First Kind

Each of the four theorems in this section use either a fixed circle, or one that is defined by three points. All the theorems depend on what is known as the *power* of a point with respect to a circle Q . We recall the definition. Let Q be a given circle and X_0 be any given point. Let m be a (variable) line through X_0 which intersects Q in the points Y and Z . Then the *power* $p(X_0, Q)$ of X_0 with respect to Q is defined by $p(X_0, Q) = \|X_0Y\| \cdot \|X_0Z\|$. The most important property of the power is that its value depends only on the point X_0 and the circle Q , and *not* on the line m used to define it. If X_0 lies inside the circle Q , then $p(X_0, Q)$ is negative. If X_0 lies outside Q , then $p(X_0, Q)$ is positive—it is the square of the length of a tangent from X_0 to Q . Notice that in this case m must, of course, be chosen so as to intersect Q .

Theorem 1.1. *Let P be a given n -gon and Q be a fixed circle. Suppose that, for each i , the side V_iV_{i+1} of P meets Q in the points U_i and W_i . Then*

$$\prod \frac{\|V_iU_i\|}{\|U_iV_{i+1}\|} \cdot \prod \frac{\|V_iW_i\|}{\|W_iV_{i+1}\|} = 1.$$

Proof. See Fig. 1 for an example with $n = 5$.

$$\prod \frac{\|V_iU_i\|}{\|U_iV_{i+1}\|} \cdot \prod \frac{\|V_iW_i\|}{\|W_iV_{i+1}\|} = \prod \frac{\|V_iU_i\| \cdot \|V_iW_i\|}{\|U_iV_{i+1}\| \cdot \|W_iV_{i+1}\|} = \prod \frac{p(V_i, Q)}{p(V_{i+1}, Q)} = 1.$$

The last equality follows since the terms in the numerator of the product are identical with those in the denominator, and so complete cancellation takes place. $\square$

Fig. 1

Note that, in the above, the points U_i and W_i can be interchanged without affecting the result.

Theorem 1.1 is a classical theorem in projective geometry due to Carnot, see p. 299 of [1], who stated it for triangles ($n = 3$) and Q a proper conic. The latter is not a significant generalisation since by Eve's theorem, mentioned above, the cyclic products are projective invariants and so the statement of the theorem remains true if a projective transformation is applied to the figure. The n -gonal form of the theorem is also mentioned by Eves [1, Exercise 6.2-16]. It is included here since it clearly belongs to the same "family" of theorems as the remaining theorems of this section.

It would appear, at first sight, that we can generalise Theorem 1.1 by considering the diagonal $V_i V_{i+k}$ (where k is a suitably chosen integer) instead of the side $V_i V_{i+1}$. However, this generalisation is illusory—if k is prime to n it is equivalent to applying the original theorem to the "stellated" polygon $[V_0, V_k, V_{2k}, \dots, V_{(n-1)k}]$. If k is not prime to n it is equivalent to applying the theorem to a suitable set of h polygons, where $h = \text{HCF}(k, n) (= \text{GCD}(k, n))$. Similar remarks apply to all the other theorems in this section.

For the next three theorems we require the following lemma:

Lemma 1. *Let the circle Q have the equation*

$$f(x, y) = x^2 + y^2 - 2ax - 2by + c = 0$$

(where $a^2 + b^2 > c$ and the coefficient of $x^2 + y^2$ is equal to 1). If X_0 is the point (x_0, y_0) , then $p(X_0, Q) = f(x_0, y_0)$.

Proof. See Fig. 2. If the radius of Q is r and the distance of X_0 from the centre O of Q is s , then, by considering the case where m is tangent to Q (when X_0 is outside the circle) or m is perpendicular to OX_0 (when X_0 is inside the circle), we see immediately that $p(X_0, Q) = s^2 - r^2$. However, as O is the point (a, b) and the radius of the circle is $\sqrt{(a^2 + b^2 - c)}$, it follows that

$$p(X_0, Q) = (x_0 - a)^2 + (y_0 - b)^2 - (a^2 + b^2 - c) = f(x_0, y_0)$$

as required. □

The lemma gives a simple method of calculating the power $p(X_0, Q)$ of the point $X_0(x_0, y_0)$ with respect to the circle $Q(X_1, X_2, X_3)$ through the three points $X_1(x_1, y_1)$, $X_2(x_2, y_2)$, $X_3(x_3, y_3)$. The equation of $Q(X_1, X_2, X_3)$ is

$$D(X, X_1, X_2, X_3) = \det \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

In the expansion of the determinant, the coefficient of $x^2 + y^2$ is

$$\det \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{or} \quad 2\|X_1 X_2 X_3\|.$$

(a)

(b)

Fig. 2

Hence

$$p(X_0, Q(X_1, X_2, X_3)) = \frac{D(X_0, X_1, X_2, X_3)}{2\|X_1X_2X_3\|}.$$

This leads at once to an interesting identity which we have not seen in the literature, namely,

$$\begin{aligned} p(X_0, Q(X_1, X_2, X_3)) \cdot \|X_1X_2X_3\| \\ &= p(X_1, Q(X_2, X_3, X_0)) \cdot \|X_2X_3X_0\| \\ &= p(X_2, Q(X_3, X_0, X_1)) \cdot \|X_3X_0X_1\| \\ &= p(X_3, Q(X_0, X_1, X_2)) \cdot \|X_0X_1X_2\|. \end{aligned}$$

Fig. 3

Theorem 1.2. Let P be a given n -gon, and X, Y be two fixed points. For each $i = 0, \dots, n-1$, define Q_i as the (unique) circle through V_{i+1}, X and Y , and suppose that the diagonal $V_i V_{i+2}$ of P meets Q_i in the points U_i and W_i . Then

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+2}\|} \cdot \prod \frac{\|V_i W_i\|}{\|W_i V_{i+2}\|} = (-1)^n.$$

Proof. See Fig. 3 for an example with $n = 5$.

$$\frac{\|V_i U_i\|}{\|U_i V_{i+2}\|} \cdot \frac{\|V_i W_i\|}{\|W_i V_{i+2}\|} = \frac{p(V_i, Q_i)}{p(V_{i+2}, Q_i)} = \frac{D(V_i, V_{i+1}, X, Y) \cdot \|V_{i+1} XY\|}{D(V_{i+2}, V_{i+1}, X, Y) \cdot \|V_{i+1} XY\|} \quad (1)$$

and so

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+2}\|} \cdot \prod \frac{\|V_i W_i\|}{\|W_i V_{i+2}\|} = (-1)^n \prod \frac{D(V_i, V_{i+1}, X, Y)}{D(V_{i+1}, V_{i+2}, X, Y)} = (-1)^n.$$

The last equality follows since all the terms in the product cancel and so the theorem is established. $\square$

Theorem 1.3. Let P be a given n -gon ($n \geq 4$) and let X be a fixed point. For each $i = 0, \dots, n-1$ let Q_i be the (unique) circle through X, V_{i+1}, V_{i+2} , and let U_i, W_i be the two points in which the diagonal $V_i V_{i+3}$ of P meets Q_i . Then

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+3}\|} \cdot \prod \frac{\|V_i W_i\|}{\|W_i V_{i+3}\|} = 1.$$

Proof. The proof is similar to that of Theorem 1.2 using

$$\frac{\|V_i U_i\|}{\|U_i V_{i+3}\|} \cdot \frac{\|V_i W_i\|}{\|W_i V_{i+3}\|} = \frac{p(V_i, Q_i)}{p(V_{i+3}, Q_i)} = \frac{D(V_i, V_{i+1}, V_{i+2}, X) \cdot \|V_{i+1} V_{i+2} X\|}{D(V_{i+3}, V_{i+1}, V_{i+2}, X) \cdot \|V_{i+1} V_{i+2} X\|}$$

instead of (1). $\square$

Theorem 1.4. *Let P be a given n -gon ($n \geq 5$) and for each $i = 0, \dots, n-1$ let Q_i be the (unique) circle through the three vertices $V_{i+1}, V_{i+2}, V_{i+3}$ of P . Suppose the diagonal $V_i V_{i+4}$ of P meets Q_i in U_i and W_i . Then*

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+4}\|} \cdot \prod \frac{\|V_i W_i\|}{\|W_i V_{i+4}\|} = (-1)^n.$$

Proof. The proof is similar to that of Theorem 1.2 using

$$\frac{\|V_i U_i\|}{\|U_i V_{i+4}\|} \cdot \frac{\|V_i W_i\|}{\|W_i V_{i+4}\|} = \frac{p(V_i, Q_i)}{p(V_{i+4}, Q_i)} = \frac{D(V_i, V_{i+1}, V_{i+2}, V_{i+3}) \cdot \|V_{i+1} V_{i+2} V_{i+3}\|}{D(V_{i+4}, V_{i+1}, V_{i+2}, V_{i+3}) \cdot \|V_{i+1} V_{i+2} V_{i+3}\|}$$

instead of (1). $\square$

If, in each of the above theorems, the points U_i and W_i are not defined because the edge or diagonal $V_i V_{i+k}$ does not meet the circle Q_i , then instead of $(\|V_i U_i\| \cdot \|V_i W_i\|) / (\|U_i V_{i+k}\| \cdot \|W_i V_{i+k}\|)$ we may write t_i^2 / t_{i+k}^2 where t_i and t_{i+k} are the lengths of the tangents from V_i and V_{i+k} to Q_i . This follows since both expressions are equal to $p(V_i, Q_i) / p(V_{i+k}, Q_i)$. However, if this is done, the theorems become assertions about powers of vertices with respect to the circles instead of statements about cyclic products.

2. Circle Constructions of the Second Kind

The constructions here, like those of the previous section, involve circles drawn through three points. Here we also use tangents to the circles at one of the points.

Lemma 2. *If Q is the circumcircle of the triangle $[A, B, C]$ and the tangent to Q at A meets the line BC in D , then*

$$\frac{\|BD\|}{\|DC\|} = -\frac{|AB|^2}{|AC|^2}.$$

Proof. See Fig. 4. By elementary geometry the triangles $[B, A, D]$ and $[A, C, D]$ are similar, so $|BD|/|AD| = |AB|/|AC|$ and $|AD|/|DC| = |AB|/|AC|$. Multiplying these equations together we obtain $|BD|/|DC| = |BA|^2/|AC|^2$, and then the statement of the lemma follows from the definition of directed lengths. $\square$

Theorem 2.1. *Let P be a given n -gon and X be a fixed point. For each $i = 0, \dots, n-1$ let Q_i be the (unique) circle through the three points V_i, V_{i+1}, X . Let the tangent to Q_i*

Fig. 4

at X meet the side $V_i V_{i+1}$ of P in U_i . Then

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = (-1)^n.$$

In Fig. 5(a) we show the construction of the point U_i , and in Fig. 5(b) the construction of all the points U_i ($i = 0, 1, \dots, 4$) for a pentagon. Many of the subsequent diagrams will follow the same arrangement. For each theorem we show the basic construction, and, in some cases, the complete construction for a pentagon. Where the complete construction is not shown this is because, even when n is as small as 5, the resulting diagram is so complicated as to be unintelligible.

Proof. By Lemma 2, with X, V_i, V_{i+1} written in place of A, B, C , respectively, we see that $D = U_i$ and hence

$$\frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = -\frac{|X V_i|^2}{|X V_{i+1}|^2}. \quad (2)$$

Thus,

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = (-1)^n \prod \frac{|X V_i|^2}{|X V_{i+1}|^2} = (-1)^n.$$

In the last product complete cancellation occurs and so the theorem is proved. $\square$

If, with the above notation, Y_i is the point of contact of the second tangent from U_i to Q_i and the line $X Y_i$ meets the side $V_i V_{i+1}$ of P in W_i , then W_i is the harmonic conjugate of U_i with respect to V_i, V_{i+1} and so, by the remark in the Introduction, $\prod (\|V_i W_i\| / \|W_i V_{i+1}\|) = 1$.

Theorem 2.2. Let P be a given n -gon and for $i = 0, \dots, n-1$ let Q_i be the (unique) circle through the three points V_i, V_{i+1}, V_{i+2} . Let the tangent to Q_i at V_{i+1} meet the line

Fig. 5

$V_i V_{i+2}$ of P in U_i . Then

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+2}\|} = (-1)^n.$$

Proof. The proof is similar to that of Theorem 2.1. Here we write V_i , V_{i+1} , V_{i+2} in place of B , A and C in Lemma 2, so that $D = U_i$, and in place of (2) we have

$$\frac{\|V_i U_i\|}{\|U_i V_{i+2}\|} = -\frac{|V_{i+1} V_i|^2}{|V_{i+1} V_{i+2}|^2}.$$

Taking the product from $i = 0$ to $i = n - 1$ yields the statement of the theorem. □

In the case $n = 3$ of Theorem 2.2, which is well known, the points U_1, U_2, U_3 are collinear by the converse of Menelaus' theorem.

If, with the above notation, Y_i is the point of contact of the second tangent from U_i to Q_i and the line $V_{i+1}Y_i$ meets the diagonal V_iV_{i+2} of P in W_i , then W_i is the harmonic conjugate of U_i with respect to V_i, V_{i+2} and so, by the remark in the Introduction, $\prod (\|V_iW_i\|/\|W_iV_{i+2}\|) = 1$.

Theorem 2.3. *Let P be an n -gon and X be a fixed point. For each $i = 0, \dots, n-1$ let Q_i be the (unique) circle through the points V_i, V_{i+1}, X . Let the tangent to Q_i at V_i meet the line XV_{i+1} in U_{i+1} and the tangent to Q_i at V_{i+1} meet the line XV_i at W_i . Then*

$$\prod \frac{\|V_{i+1}U_{i+1}\|}{\|U_{i+1}X\|} \cdot \prod \frac{\|XW_i\|}{\|W_iV_i\|} = 1.$$

Proof. See Fig. 6 for an example with $n = 5$. The proof is similar to that of Theorem 2.1 except that we apply Lemma 2 twice, firstly with V_i, V_{i+1}, X in place of A, B, C of Lemma 2 (so that $D = U_{i+1}$) and secondly with V_{i+1}, V_i, X in place of A, B, C (so $D = W_i$). We obtain, in place of (2),

$$\frac{\|V_{i+1}U_{i+1}\|}{\|U_{i+1}X\|} = -\frac{|V_iV_{i+1}|^2}{|V_iX|^2}$$

and

$$\frac{\|V_iW_i\|}{\|W_iX\|} = -\frac{|V_{i+1}V_i|^2}{|V_{i+1}X|^2}.$$

Take the quotient of these, and then the product from $i = 0$ to $i = n-1$ yields the statement in the theorem. $\square$

Theorem 2.2 has an asymmetric form:

Theorem 2.4. *Let P be an n -gon and s, t be positive integers, $s < n/2, t < n/2, s \neq t$. For each $i = 0, \dots, n-1$ let Q_i be the (unique) circle through the points V_{i-s}, V_i, V_{i+t} and R_i be the (unique) circle through V_{i-t}, V_i, V_{i+s} . Let the tangent to Q_i at V_i meet the line $V_{i-s}V_{i+t}$ in U_i and the tangent to R_i at V_i meet the line $V_{i-t}V_{i+s}$ at W_i . Then*

$$\prod \frac{\|V_{i-s}U_i\|}{\|U_iV_{i+t}\|} \cdot \prod \frac{\|V_{i-t}W_i\|}{\|W_iV_{i+s}\|} = 1.$$

Proof. See Fig. 7 for the construction of the points U_i and W_i . The proof is similar to that of Theorem 2.1 except that we apply Lemma 2 twice, firstly with V_i, V_{i-s}, V_{i+t} in place of A, B, C (so that $D = U_i$) and secondly with V_i, V_{i-t}, V_{i+s} in place of A, B, C , respectively (so that $D = W_i$). Taking the product of the two equations so obtained and then the product from $i = 0$ to $i = n-1$ yields the statement in the theorem. $\square$

(a)

(b)

Fig. 6

Fig. 7

3. Circle Constructions of the Third Kind

Each of the two constructions described in this section involve circles which touch an edge of the polygon at one of the two vertices on that edge and satisfy one extra condition. More precisely, they touch the edge of a polygon P at one of its vertices and either pass through another vertex or touch another edge of P .

Theorem 3.1. *Let P be a given n -gon ($n \geq 4$) and for each $i = 0, \dots, n-1$ let Q_i be the (unique) circle which touches the side $[V_i, V_{i+1}]$ of P at the vertex V_i and passes through V_{i+2} . Also, let R_i be the (unique) circle which touches the same side $[V_i, V_{i+1}]$ at the vertex V_{i+1} and passes through V_{i-1} . Let the second intersection of Q_i with $V_{i+1}V_{i+2}$ be denoted by U_i and the second intersection of R_i with V_iV_{i-1} be denoted by W_i . Then*

$$\prod \frac{\|V_{i+1}U_i\|}{\|U_iV_{i+2}\|} \cdot \prod \frac{\|V_{i-1}W_i\|}{\|W_iV_i\|} = (-1)^n.$$

Proof. See Fig. 8. Assume that the positive direction on each edge $[V_i, V_{i+1}]$ is $\overrightarrow{V_iV_{i+1}}$ and for each $i = 0, \dots, n-1$ write $k_i = \|V_iV_{i+1}\| > 0$. Then by properties of the power $p(V_{i+1}, Q_i)$ we see $\|V_{i+1}U_i\| \cdot \|V_{i+1}V_{i+2}\| = |V_{i+1}V_i|^2 = k_i^2$ so that $\|V_{i+1}U_i\| = k_i^2/k_{i+1}$. But,

$$\|U_iV_{i+2}\| = \|U_iV_{i+1}\| + \|V_{i+1}V_{i+2}\| = -\frac{k_i^2}{k_{i+1}} + k_{i+1} = \frac{k_{i+1}^2 - k_i^2}{k_{i+1}},$$

and so $\|V_{i+1}U_i\|/\|U_iV_{i+2}\| = k_i^2/(k_{i+1}^2 - k_i^2)$. In a similar way, $\|V_iW_i\|/\|W_iV_{i-1}\| = k_i^2/(k_{i-1}^2 - k_i^2)$, and therefore,

$$\prod \frac{\|V_{i+1}U_i\|}{\|U_iV_{i+2}\|} \cdot \prod \frac{\|V_{i-1}W_i\|}{\|W_iV_i\|} = \prod \frac{k_{i-1}^2 - k_i^2}{k_{i+1}^2 - k_i^2} = (-1)^n \prod \frac{k_{i-1}^2 - k_i^2}{k_i^2 - k_{i+1}^2} = (-1)^n$$

since all the terms in the last product cancel. $\square$

Theorem 3.2. *Let P be a given n -gon ($n \geq 4$) and for each $i = 0, \dots, n-1$ let Q_i be the (unique) circle which touches the side V_i, V_{i+1} of P at the vertex V_i and also touches the side $V_{i+1}V_{i+2}$ (on the same side of V_iV_{i+1} as V_{i+2}) at the point U_i . Let R_i be the (unique) circle which touches the same side V_i, V_{i+1} at the vertex V_{i+1} and also touches*

Fig. 8

the side $V_i V_{i-1}$ (on the same side of $V_i V_{i+1}$ as V_{i-1}) at the point W_i . Then

$$\prod \frac{\|V_{i+1}U_i\|}{\|U_i V_{i+2}\|} \cdot \prod \frac{\|V_{i-1}W_i\|}{\|W_i V_i\|} = (-1)^n.$$

Proof. See Fig. 9 for an example with $n = 5$. Define k_i as in the proof of Theorem 3.1. Since tangents from an external point to a circle are equal, and, by the definition of U_i , $\|V_{i+1}U_i\| > 0$, so $\|V_{i+1}U_i\| = k_i$. Also $\|V_{i+1}U_i\| + \|U_i V_{i+2}\| = \|V_{i+1}V_{i+2}\| = k_{i+1}$. Hence $\|U_i V_{i+2}\| = k_{i+1} - k_i$, and we obtain $\|V_{i+1}U_i\|/\|U_i V_{i+2}\| = k_i/(k_{i+1} - k_i)$. In a similar way, $\|V_{i-1}W_i\|/\|W_i V_i\| = (k_{i-1} - k_i)/k_i$, and so

$$\prod \frac{\|V_{i+1}U_i\|}{\|U_i V_{i+2}\|} \cdot \prod \frac{\|V_{i-1}W_i\|}{\|W_i V_i\|} = \prod \left(-\frac{k_{i-1} - k_i}{k_i - k_{i+1}} \right) = (-1)^n$$

since all the terms in the last product cancel. $\square$

4. Circle Constructions of the Fourth Kind

In Sections 1 and 2 we considered constructions in which the circles were determined by the condition that they pass through three points (fixed points or vertices of the given polygon P). We now consider constructions in which the circles are determined by the condition that they touch three lines (either fixed lines or sides of the polygon P). In this case the situation is more complicated; whereas there is a unique circle through three given points, there are, in general, four circles which touch three given lines (namely, the incircle and three excircles of the triangle formed by the three lines). This has the important consequence that we need to specify, in each of the theorems, exactly which circles are to be chosen. As we shall see there are many ways to do this, and as a result, each construction leads to many cyclic product relations.

Theorem 4.1. *Let m be a fixed line and P be a given n -gon. For each $i = 0, 1, \dots, n-1$ let F_i be the point in which the side $V_{i-1}V_i$ of P meets m . Let Q_i be a circle, centre O_i , such that*

- (i) Q_i touches the sides $V_{i-1}V_i$, $V_i V_{i+1}$ of P and also the line m ,
- (ii) the points O_i , F_{i+1} , O_{i+1} are collinear, and
- (iii) Q_i touches the line m at the point U_i ,

then

$$\prod \frac{\|F_i U_i\|}{\|U_i F_{i+1}\|} = (-1)^n. \quad (3)$$

Proof. See Fig. 10 for an example with $n = 5$. The points O_{i-1} , F_i , O_i lie on one of the two bisectors of the angle between m and the line $V_{i-1}V_i$ at F_i (because both the circles Q_{i-1} and Q_i touch the lines m and $V_{i-1}V_i$). There are two such bisectors at F_i and we may choose either. Temporarily call these *a-bisectors* and *b-bisectors* (see Fig. 11). The

Fig. 9

Fig. 10

two kinds are distinguished by the fact that if one moves in a counterclockwise direction about F_i , starting from the positive direction of m , then one meets the a -bisector *before* the line $V_{i-1}V_i$ but the b -bisector *after* this line. Define

$$e_i = \begin{cases} 1 & \text{if we choose the } a\text{-bisector at } F_i, \\ -1 & \text{if we choose the } b\text{-bisector at } F_i. \end{cases}$$

As O_i , the centre of Q_i , lies on the bisectors at both F_i and F_{i+1} there are four possibilities

Fig. 11

according to which bisectors are chosen. These are shown in Fig. 12. With angles θ_i and θ_{i+1} as marked, and $r_i > 0$ as the radius of Q_i , in every case

$$\frac{\|F_i U_i\|}{\|U_i F_{i+1}\|} = -\frac{e_i e_{i+1} r_i \cot \theta_i}{r_i \cot \theta_{i+1}}$$

and so

$$\prod \frac{\|F_i U_i\|}{\|U_i F_{i+1}\|} = (-1)^n \prod e_i e_{i+1} \prod \frac{\cot \theta_i}{\cot \theta_{i+1}} = (-1)^n.$$

Fig. 12

There are 2^n possible sets of bisectors that may be chosen at the points F_i . Each such set determines a sequence of circles Q_i that satisfy the conditions of the theorem. We thus arrive at 2^n sets of points U_i , each of which leads to a cyclic product relation. Hence, in general, (3) represents 2^n such relations. $\square$

Theorem 4.2. *Let P be a given n -gon, and for each $i = 0, 1, \dots, n-1$ let Q_i be a circle, centre O_i , such that*

- (i) Q_i touches the sides $V_{i-1}V_i$, V_iV_{i+1} , $V_{i+1}V_{i+2}$ of P ,
- (ii) the points O_i , V_{i+1} , O_{i+1} are collinear, and
- (iii) Q_i touches the side V_iV_{i+1} of P at the point U_i .

Then

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = 1. \quad (4)$$

Proof. See Fig. 13 for examples with $n = 5$. In Fig. 13(a) the polygon is convex, and in Fig. 13(b) it is self-intersecting. The points O_{i-1} , O_i lie on one of the two bisectors of the angle between $V_{i-1}V_i$ and V_iV_{i+1} at V_i (because both the circles Q_{i-1} and Q_i touch the lines $V_{i-1}V_i$ and V_iV_{i+1}). There are two such bisectors at V_i and we may choose either. There is an *internal bisector*, which separates the points V_{i-1} , V_{i+1} , and an *external bisector*, which does not separate these two points. Define

$$e_i = \begin{cases} 1 & \text{if } O_i V_i O_{i-1} \text{ is the internal bisector at } V_i, \\ -1 & \text{if } O_i V_i O_{i-1} \text{ is the external bisector at } V_i. \end{cases}$$

Also define

$$f_i = \begin{cases} 1 & \text{if } [V_i, V_{i+1}] \text{ is a locally convex edge of } P, \text{ that is, } V_{i-1} \text{ and } V_{i+2} \text{ lie on the same side of the line } V_i V_{i+1}, \\ -1 & \text{if } [V_i, V_{i+1}] \text{ is an inflectional edge of } P, \text{ that is, } V_{i-1} \text{ and } V_{i+2} \text{ lie on opposite sides of the line } V_i V_{i+1}. \end{cases}$$

There are essentially six combinations of internal/external bisectors and locally convex/inflectional edges. Three of these are shown in Fig. 14, and the reader will have no difficulty in drawing the other three cases. With angles θ_i and θ_{i+1} as marked, and $r_i > 0$ as the radius of Q_i , in every case

$$\frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = \frac{e_i e_{i+1} f_i r_i \cot \theta_i}{r_i \cot \theta_{i+1}}$$

and so

$$\prod \frac{\|V_i U_i\|}{\|U_i V_{i+1}\|} = \prod e_i e_{i+1} \prod f_i \prod \frac{\cot \theta_i}{\cot \theta_{i+1}}.$$

Now $\prod e_i e_{i+1} = \prod e_i^2 = 1$ and $\prod f_i = 1$ since the number of inflectional edges in a polygon must be even. This was proved by Fabricius-Bjerre [2]. In the last product all the terms cancel and so (4) is proved. $\square$

Fig. 13

Fig. 14

As in the previous theorem there are 2^n possible sets of bisectors, one at each of the vertices V_i . Each such set determines a sequence of circles Q_i that satisfy the conditions of the theorem, since O_i , the centre of Q_i , is the point of intersection of the bisectors at V_i and V_{i+1} . We thus arrive at 2^n sets of points U_i , each of which leads to a cyclic product relation. Hence, in general, (4) represents 2^n such relations.

Both Theorems 4.1 and 4.2 can be generalised in the following way. Instead of a sequence of n circles Q_i consider a sequence of $2n$ circles $Q_0, \dots, Q_{2n-1}$ with the property that, for $i = 0, 1, \dots, n-1$, in Theorem 4.1 the triples of points O_i, F_{i+1}, O_{i+1} and $O_{i+n}, F_{i+1}, O_{i+n+1}$ (modulo $2n$) are collinear, and in Theorem 4.2 the triples of points O_i, V_{i+1}, O_{i+1} and $O_{i+n}, V_{i+1}, O_{i+n+1}$ (modulo $2n$) are collinear. Defining the points U_i ($i = 0, \dots, 2n-1$) as before as the points of contact of these circles (with either m or $V_i V_{i+1}$) (3) and (4) continue to hold except that our cyclic products now have $2n$ terms. Similar generalisations to sequences of $3n, 4n, \dots$, circles are clearly also possible.

Special cases of the above theorems occur when all the sides of P touch a circle Q (in Theorem 4.2) or all the sides of P and the line m touch a circle Q (in Theorem 4.1). Then one possible sequence of circles Q_i consists of the circle Q repeated n times. Condition (ii) becomes redundant, but the proofs still hold and both theorems are true in this case. If $n = 3$ there are eight sequences of circles which satisfy the conditions of Theorem 4.2. Of these, four degenerate into a single circle (the incircle or three excircles of the triangle P). In this case the lines $V_0 U_1, V_1 U_2$ and $V_2 U_0$ are concurrent and then $\prod (\|V_i U_i\| / \|U_i V_{i+1}\|) = 1$ is a consequence of Ceva's theorem.

5. Conclusion

As stated in the Introduction, Branko Grünbaum proved and conjectured some of the results given here in his lecture notes [3]. To be precise, he proved Carnot's Theorem 1.1 using the idea of the power of a point with respect to a circle, as we did, and he conjectured Theorems 1.2–1.4. By the term “conjecture” we mean that he stated the results and then verified them numerically by computing the cyclic products for a large number of n -gons. He also proved Lemma 2 analytically (in contrast to the geometric method used here) and deduced Theorems 2.1 and 2.2 from it. All the other theorems in this paper are new.

All our theorems and proofs can be understood with a minimum of geometric knowledge. It is therefore surprising they were not discovered many years, or even centuries ago.

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