Densest Packings of More than Three d -Spheres Are Nonplanar*

U. Schnell and J. M. Wills

Mathematisches Institut, Universität Siegen,
 D-57068 Siegen, Germany
 {schnell,wills}@mathematik.uni-siegen.de

Abstract. We prove that for a densest packing of more than three d -balls, $d \geq 3$, where the density is measured by parametric density, the convex hull of their centers is either linear (a sausage) or at least three-dimensional. This is also true for restrictions to lattice packings. These results support the general conjecture that densest sphere packings have extreme dimensions. The proofs require a Lagrange-type theorem from number theory and Minkowski's theory of mixed volumes.

1. Introduction and Results

Let B^d denote the unit ball in Euclidean d -space E^d , and let its volume be $V(B^d) = \kappa_d$. For $n \in \mathbb{N}$ let $C_n = \{c_1, \dots, c_n\} \subset E^d$ with $\|c_i - c_j\| \geq 2$ for $i \neq j$. Then $C_n + B^d$ is called a finite packing of n unit balls. Its density is given by the parametric density

$$\delta(B^d, C_n, \varrho) = n\kappa_d / V(\operatorname{conv} C_n + \varrho B^d),$$

where $\varrho > 0$ is a parameter. For brevity we write $P_n = \operatorname{conv} C_n$ and call the dimension of the packing $\dim P_n$. The densest packing of n unit balls and $\varrho > 0$ is given by

$$\max\{\delta(B^d, C_n, \varrho) \mid C_n + B^d \text{ packing}\}$$

or equivalently by

$$\min\{V(P_n + \varrho B^d) \mid C_n + B^d \text{ packing}\}. \quad (1)$$

* This research was partially supported by the Erwin Schrödinger Institute, Vienna.

The minimum in (1) is attained from a standard compactness argument, and we denote a P_n for which the minimum is attained by $P_{n,\varrho}^d$. The two basic problems are:

- (a) The maximum density or equivalently the minimum volume.
- (b) The shape of $P_{n,\varrho}^d$, in particular, $\dim P_{n,\varrho}^d$.

A. Thue in 1892 was the first who considered these problems in the special case $d = 2$, i.e., circle packings, and $\varrho = 1$. A list of prominent followers like Mahler, Segre, L. Fejes Tóth, Rogers, Zassenhaus, Bambah, Oler, Woods, Groemer, Graham et al. generalized and improved his partial results, all for $d = 2$ and $\varrho = 1$ (see the survey [GW]). Investigations for $\varrho = 1$ and $d \geq 3$ started after L. Fejes Tóth's famous sausage conjecture, which says that $\dim P_{n,\varrho}^d = 1$ for $d \geq 5$ and all $n \in \mathbb{N}$, and which is proved for all $d \geq 42$ [BH], [BHW1], [BHW2]. The general case $\varrho > 0$ was only considered after 1993 [Wi], [BHW1].

In all these papers (see also [BGW], [BG], [BS], and [BW]) either $\dim P_{n,\varrho}^d = 1$ or d (if $n > d$). This supports the general conjecture “densest sphere packings have extreme dimensions”, i.e., $\dim P_{n,\varrho}^d$ is either 1 or $\min(n - 1, d)$.

In this paper we consider the case $\dim P_{n,\varrho}^d = 2$ and solve it completely in

Theorem 1. *For $d \geq 3$, $n \geq 4$, and $\varrho > 0$ it follows that $\dim P_{n,\varrho}^d \neq 2$.*

Theorem 2. *The result is also true for restriction to lattice packings.*

It is not too difficult to prove Theorems 1 and 2 for sufficiently large n . The problem is to prove them for all n . Therefore the proofs of Theorems 1 and 2 require a theorem from number theory, which might be of interest by itself and is proved in Section 4.

Theorem 3. *For $n = 4$ and $n \geq 6$ there are integers $a > b \geq c \geq d \geq 0$ with*

$$n = a^2 - b^2 - c^2 - d^2 \quad (2)$$

and

$$3\sqrt{n} \geq 3a - b - c, \quad (3)$$

$$a \geq b + c + 1, \quad (4)$$

$$a \geq b + 2. \quad (5)$$

Remarks. (1) The title of this paper is a popular version of Theorem 1. With the following useful notation one gets another version of Theorem 1 (and 2). If $\{(n, \varrho) \mid \dim P_{n,\varrho}^d = i\}$, $i = 1, \dots, d$, denote the i -phases, then Theorem 1 says: for $d \geq 3$ and $n \geq 4$ the 2-phase is empty.

(2) Partial results of Theorems 1 and 2 have been shown before: For $\varrho = 1$ Theorem 1 was proved in [BG] (see also [BGW]), and hence for $\varrho \leq 1$. In [BG] it was even shown that for $\varrho \leq 1$ and $d \leq 10$ $\dim P_{n,\varrho}^d$ is either 1 or d (if $n > d$). Their methods correspond to our Lemma 2 for small ϱ , but they do not work for general ϱ . In [BW] Theorem 2 was shown for very special lattice packings. In [ABB] Theorem 2 is shown for large n .

(3) For packings of other convex bodies no such results can be expected (see [Gri] and [ABB]).

(4) Theorem 3 without (3)–(5) is a Lagrange-type theorem or more precisely a theorem on indefinite quadratic forms and is easy to prove even with $d = 0$. Moreover, it holds for all $n \in \mathbb{N}$. The restrictions (3)–(5) are needed in the proof of Theorem 1.

(5) Restrictions (3) and (5) are not true for $n \leq 3$ and $n = 5$. For $n \neq 5$ Theorem 2 is a corollary of Theorem 1. The case $n = 5$ requires extra considerations and is proved in Section 3.

2. Proof of Theorems 1 and 2 for $n \neq 5$

In the proof the constant

$$\varrho_d = \sqrt{3}\kappa_{d-2}/\kappa_{d-1} = O(\sqrt{d})$$

plays a central role. Note that $\varrho_d \geq \varrho_{d-1}$ since $\kappa_{d-2}^2 \geq \kappa_{d-1}\kappa_{d-3}$. We further use Steiner's formula for a convex body K ,

$$V(K + \varrho B^d) = \sum_{i=0}^d V_i(K) \kappa_{d-i} \varrho^{d-i}, \quad (6)$$

where V_i , $i = 0, 1, \dots, d$ denote the intrinsic volumes of K (see [SY]). In particular $V_0 = 1$ and V_d is the volume with respect to E^d . Instead of V_d we also write V if no confusion arises.

Lemma 1. *Theorem 1 is proved if, for any packing $C_n + B^d$ with $\text{conv } C_n = P_n$ and $\dim P_n = 2$,*

(A) *if S_n is a segment of length $2(n-1)$, then*

$$V_d(P_n + \varrho_d B^d) \geq V_d(S_n + \varrho_d B^d); \quad (7)$$

(B) *there is a packing $C'_n + B^d$ with $\text{conv } C'_n = Q_n$, $\dim Q_n = 3$,*

$$V_d(P_n + \varrho_d B^d) > V_d(Q_n + \varrho_d B^d), \quad (8)$$

and

$$V_1(P_n) > V_1(Q_n). \quad (9)$$

Proof. (i) For $0 \leq \varrho < \varrho_d$ we use (A): Steiner's formula and division by ϱ_d^{d-2} show that (7) is equivalent to

$$V_2(P_n)\kappa_{d-2} + V_1(P_n)\kappa_{d-1}\varrho_d + \kappa_d\varrho_d^2 \geq 2(n-1)\kappa_{d-1}\varrho_d + \kappa_d\varrho_d^2$$

or

$$b\varrho_d + c \geq 0$$

with $c = \kappa_{d-2}V_2(P_n) > 0$ and $b = \kappa_{d-1}(V_1(P_n) - 2(n-1))$. From $c > 0$ it follows that this inequality and hence (7) becomes a strict inequality if we replace ϱ_d by any $\varrho \in (0, \varrho_d)$.

(ii) For $\varrho \geq \varrho_d$ we use (B): Again Steiner's formula shows that (8) is equivalent to

$$\varrho_d \kappa_{d-2} V_2(P_n) + \varrho_d^2 V_1(P_n) \kappa_{d-1} \geq \kappa_{d-3} V_3(Q_n) + \kappa_{d-2} V_2(Q_n) \varrho_d + \kappa_{d-1} V_1(Q_n) \varrho_d^2$$

or

$$f(\varrho_d) = a\varrho_d^2 + b\varrho_d + c \geq 0,$$

with $a = \kappa_{d-1}(V_1(P_n) - V_1(Q_n))$, $b = \kappa_{d-2}(V_2(P_n) - V_2(Q_n))$, and $c = -\kappa_{d-3}V_3(Q_n) < 0$. By assumption we have $a > 0$. With $c < 0$ and $\varrho_d > 0$ it follows that $f'(\varrho_d) = 2a\varrho_d + b > 0$ and consequently $f'(\varrho) \geq 0$ and so $f(\varrho) \geq 0$, for $\varrho \geq \varrho_d$. So (8) and (9) imply that (8) is also true if ϱ_d is replaced by any $\varrho > \varrho_d$. $\square$

So the theorem is proved, if (A) and (B) are shown. (A) is shown in Lemma 2. (B) is more complicated. It requires Theorem 3 and the Lemmas 3–7.

Lemma 2. *Let $n \geq 3$, let $C_n + B^d$ be a sphere packing, let $P_n = \text{conv } C_n$, and let $\dim P_n = 2$. Then:*

- (a) $V_d(P_n + \varrho_d B^d) \geq V_d(S_n + \varrho_d B^d)$.
- (b) *There is a P'_n with $\dim P'_n = 2$ such that $V(P'_n + \varrho B^d) < V(S_n + \varrho B^d)$ for $\varrho > \varrho_d$.*

Proof. (a) Let $\dim P_n = 2$. An elegant version of Groemer's theorem [Gro], formulated in terms of the Minkowski-sum (see [GW] and [Wi]), is

$$V_2\left(P_n + \frac{\sqrt{3}}{2}B^2\right) - V_2\left(S_n + \frac{\sqrt{3}}{2}B^2\right) \geq 0. \quad (10)$$

Then one gets, with Steiner's formula and (10),

$$\begin{aligned} & (\kappa_{d-2}\varrho^{d-2})^{-1}\{V_d(P_n + \varrho_d B^d) - V_d(S_n + \varrho_d B^d)\} \\ &= V_2(P_n) + V_1(P_n)\frac{\kappa_{d-1}}{\kappa_{d-2}}\varrho_d - 2(n-1)\frac{\kappa_{d-1}}{\kappa_{d-2}}\varrho_d \\ &= V_2(P_n) + V_1(P_n)\sqrt{3} - 2(n-1)\sqrt{3} \\ &= V_2(P_n) + V_1(P_n)2\left(\frac{\sqrt{3}}{2}\right) + \pi\left(\frac{\sqrt{3}}{2}\right)^2 \\ &\quad - 2(n-1)2\left(\frac{\sqrt{3}}{2}\right) - \pi\left(\frac{\sqrt{3}}{2}\right)^2 \\ &= V_2\left(P_n + \frac{\sqrt{3}}{2}B^2\right) - V_2\left(S_n + \frac{\sqrt{3}}{2}B^2\right) \geq 0 \end{aligned}$$

and (a) is shown.

(b) From Groemer's theorem it follows that in (10) equality holds if and only if C_n is a subset of the hexagonal lattice such that adjacent points of $C_n \cap \text{bd } P_n$ have distance 2. Wegner [We] proposed calling these $P_n = \text{conv } C_n$ Groemer polygons. Obviously for each $n \geq 3$ there is such a C'_n with $\dim P'_n = 2$ and hence $V_2(P'_n) > 0$ and $V_1(P'_n) < 2(n-1)$. Hence we get

$$V_2(P'_n) + (V_1(P'_n) - 2(n-1)) \frac{\kappa_{d-2}}{\kappa_{d-1}} Q_d = 0$$

and so

$$V_2(P'_n) + (V_1(P'_n) - 2(n-1)) \frac{\kappa_{d-2}}{\kappa_{d-1}} Q < 0 \quad \text{for } Q > Q_d. \quad \square$$

For the proof of (B) we first give a rather tight lower bound for planar packings.

Definition 1. For $n \geq 2$ let

$$g_n = \max\{V_1(\text{conv } C_n) : \dim C_n = 2, C_n + B^2 \text{ is a packing}\}.$$

Lemma 3. $g_n = n$ for $2 \leq n \leq 4$ and

$$g_n > -\pi\sqrt{3}/2 + \sqrt{3\pi^2/4 + 2\pi\sqrt{3}(n-1)} \quad \text{for } n \geq 5.$$

Proof. Let $P_n = \text{conv } C_n$. The cases $n = 2$ and $n = 3$ are clear. If $n = 4$, obviously $V_1(P_4) \geq 4$ if P_4 is a quadrangle. If P_4 is a triangle, then there is a vertex v with interior angle $\alpha \leq \pi/3$. From P_4 we cut off the triangle T with vertex v and adjacent edges of length 2. The third edge of T has length $\ell \leq 2$. The remaining polygon P' contains the convex hull P_3 of the remaining three centers of the circles. Hence $V_1(P_4) = V_1(T) + V_1(P') - V_1(T \cap P') \geq (2 + \ell/2) + 3 - \ell \geq 4$.

For $n \geq 5$ we conclude from (10) that $V(P_n + \sqrt{3}/2 B^2) \geq V(S_n + \sqrt{3}/2 B^2)$ and consequently $V_2(P_n) + \sqrt{3}V_1(P_n) \geq 2\sqrt{3}(n-1)$. From the isoperimetric inequality $V_2 \leq V_1^2/\pi$ it follows that $V_1(P_n)^2 + \pi\sqrt{3}V_1(P_n) > 2\pi\sqrt{3}(n-1)$ and so the inequality holds for $n \geq 5$. $\square$

The main part to prove (B) is the construction of dense three-dimensional packings of balls. For this the following functional for convex bodies, possibly degenerate, is a useful tool.

Definition 2. For a convex body $K \subset E^d$ let

$$S(K) = V_3(K) + \sqrt{3}V_2(K) + 3V_1(K).$$

Lemma 4. Let $n \geq 4$, $\dim Q_n = 3$, and $\dim P_n = 2$. Then (8) and (9) hold if

$$S(Q_n) \leq S(S_n) = 6(n-1) \quad (11)$$

and

$$V_1(Q_n) \leq g_n. \quad (12)$$

Proof. Multiplying (11) by $q_d/q_{d-1} \geq 1$ yields

$$\frac{q_d}{q_{d-1}} V_3(Q_n) + q_d \frac{\kappa_{d-2}}{\kappa_{d-3}} V_2(Q_n) + q_d^2 \frac{\kappa_{d-1}}{\kappa_{d-3}} V_1(Q_n) \leq q_d^2 \frac{\kappa_{d-1}}{\kappa_{d-3}} 2(n-1).$$

By $q_d \geq q_{d-1}$, Steiner's formula, and Lemma 2(a) we obtain

$$V_d(Q_n + q_d B^d) \leq V_d(S_n + q_d B^d) \leq V_d(P_n + q_d B^d).$$

So we have (8). Finally, (9) follows from (12), Definition 2, and Lemma 3. $\square$

Now we introduce 3-polytopes which induce packings of d -spheres denser than all one- and two-dimensional packings. These 3-polytopes are related to the planar Groemer polygons, i.e., convex hulls of subsets of the hexagonal Lagrange lattice such that all adjacent boundary lattice points have distance 2. In the following let L be the densest sphere packing lattice (called the Gauss lattice or face centered cubic lattice (fcc)). The Gauss lattice contains exactly four arrays of grids from the hexagonal Lagrange lattice.

Definition 3. An L -polytope P is called a Groemer polytope if $\dim P = 3$ and all its facets are contained in the densest hexagonal grids.

Remarks. Groemer polytopes have nice combinatorial and metrical properties. They have at most eight facets lying in the above-mentioned grids and with normals $\pm u_1, \dots, \pm u_4$, which are the medians of the eight orthants of E^3 . Elementary considerations show (see also [Grü]), for its f -vector (f_0, f_1, f_2) , $4 \leq f_0 \leq 12$, $6 \leq f_1 \leq 18$, and $4 \leq f_2 \leq 8$. All can be derived from the regular tetrahedron by truncation. Their edgelengths are integer multiples of 2.

An important property of Groemer polytopes is

Lemma 5. For any Groemer polytope P we have $S(P) \leq 6(n-1)$.

Proof. Let $\pm u_1, \dots, \pm u_4$ be the eight possible normals of Groemer polytopes and let $b_1, \dots, b_4$ be the number of lattice planes orthogonal to $u_1, \dots, u_4$ which have lattice points in P . Further, let $b = \max\{b_1, \dots, b_4\}$. We use induction on the number of lattice points in P . If $n \leq 6$, then P is the tetrahedron T or the regular octahedron O . Then we have $V_3(T) = 2/3\sqrt{2}$, $V_2(T) = 2\sqrt{3}$, $V_1(T) = 6/\pi \arccos(-1/3) = 3.649\dots$ (see, e.g., formulae (18) of [H]) and $V_3(O) = 8/3\sqrt{2}$, $V_2(O) = 4\sqrt{3}$, $V_1(O) = V_1(Z) - 2V_1(T) + 6 = 12/\pi \arccos(1/3) = 4.701\dots$, where Z is the fundamental cell of fcc with $V_1(Z) = 6$ and so the assertion follows. If $n \geq 7$, then $b \geq 3$ and we can find a lattice plane E orthogonal to a u_i such that $P_1 = P \cap E^+$ and $P_2 = P \cap E^-$ are Groemer polytopes with a smaller number of lattice points $n_1, n_2 < n$. Since $P_1 \cap P_2$ is a Groemer polygon and so $S(P_1 \cap P_2) = \sqrt{3}V_2(P_1 \cap P_2) + 3V_1(P_1 \cap P_2) = 6(n_1 + n_2 - n - 1)$, it follows by the induction hypothesis that

$$\begin{aligned} S(P) &= S(P_1) + S(P_2) - S(P_1 \cap P_2) \\ &\leq 6(n_1 - 1) + 6(n_2 - 1) - 6(n_1 + n_2 - n - 1) = 6(n - 1). \end{aligned} \quad \square$$

Remark. As the proof shows, even

$$\lambda V_3(P) + \sqrt{3}V_2(P) + 3V_1(P) \leq 6(n-1),$$

where $\lambda = (12 - 18/\pi \arccos(-1/3))3/4\sqrt{2} = 1.116\dots$

Theorem 3 is needed for the following crucial lemma.

Lemma 6. *For $n = 4$ and $n \geq 6$ there is a Groemer polytope Q_n with $L(Q_n) = n$ and $V_1(Q_n) \leq g_n$.*

Proof. (1) Construction of Q_n : Let kT , $k \in \mathbb{N} \cup \{0\}$, be the regular lattice tetrahedron of edglength $2k$ and with one vertex in the origin. In particular, $0 \cdot T$ is the origin. Then $L(kT) = \binom{k+3}{3}$ and $V_1(kT) = kV_1(T)$. Let $T_0 = \{0\}$ and

$$T_k = \text{cl}(k \cdot T \setminus (k-1) \cdot T) \quad \text{for } k \geq 1.$$

T_k is a truncated lattice tetrahedron, which can also be defined as the convex hull of two regular lattice triangles of edglength $2k$ and $2(k-1)$ in two adjacent lattice planes. In particular $T_1 = T$. As L and V_1 are valuations (see [SY]), we have

$$\begin{aligned} L(T_k) &= L(kT) - L((k-1)T) + L(T_k \cap (k-1)T) \\ &= \binom{k+3}{3} - \binom{k+2}{3} + \binom{k+1}{2} = (k+1)^2 \end{aligned} \quad (13)$$

and

$$V_1(T_k) = V_1(kT) - V_1((k-1)T) + V_1(T_k \cap (k-1)T) = V_1(T) + 3(k-1). \quad (14)$$

For an integer $n \neq 5$ let a, b, c, d be such that $n = a^2 - b^2 - c^2 - d^2$ with $a-2 \geq b \geq c \geq d \geq 0$, $a \geq b+c+1$ and $3\sqrt{n} \geq 3a - (b+c+d)$ according to Theorem 3. We now consider the lattice polytopes T_{a-1}, T_b, T_c, T_d . From $a \geq 2$ it follows that $\dim T_{a-1} = 3$, whereas T_b, T_c, T_d might be lattice points if b, c , or $d = 0$. From T_{a-1} we truncate appropriate copies T'_b, T'_c, T'_d at its vertices. The closure of the remaining set we denote by Q_n and we have to show that it has the required properties. From $a \geq b+c+1$ it follows that the truncations do not overlap and from $a \geq b+2$ $\dim Q_n = 3$ follows. So by construction Q_n is a Groemer polytope. With the valuation property of L we get

$$\begin{aligned} L(Q_n) &= L(T_{a-1}) - L(T'_b) - L(T'_c) - L(T'_d) \\ &\quad + L(T_{a-1} \cap T'_b) + L(T_{a-1} \cap T'_c) + L(T_{a-1} \cap T'_d) \\ &= a^2 - (b+1)^2 - (c+1)^2 - (d+1)^2 + (2b+1) + (2c+1) + (2d+1) \\ &= a^2 - b^2 - c^2 - d^2 = n. \end{aligned}$$

(2) Calculation of $V_1(Q_n)$: From the valuation property of V_1 it follows that

$$\begin{aligned} V_1(Q_n) &= V_1(T_{a-1}) - V_1(T'_b) - V_1(T'_c) - V_1(T'_d) \\ &\quad + V_1(T_{a-1} \cap T'_b) + V_1(T_{a-1} \cap T'_c) + V_1(T_{a-1} \cap T'_d) \\ &= V_1(T) + 3(a-2) + \text{sign } b((2b+1) - V_1(T_b)) + \text{sign } c((2c+1) \\ &\quad - V_1(T_c)) + \text{sign } d((2d+1) - V_1(T_d)). \end{aligned}$$

The introduction of sign is useful to subsume the cases b, c , or $d = 0$. So, with (14),

$$V_1(Q_n) = 3a + V_1(T) - 6 + (4 - V_1(T))(\text{sign } b + \text{sign } c + \text{sign } d) - (b + c + d). \quad (15)$$

Obviously $(4 - V_1(T))(\text{sign } b + \text{sign } c + \text{sign } d) - d \leq 2(4 - V_1(T))$. With $V_1(T) = 3.649 \dots$ and with (3) in Theorem 3 it follows that

$$V_1(Q_n) \leq -1.649 + 3\sqrt{a} - (b + c + d) \leq -1.649 + 3\sqrt{n}.$$

Elementary considerations and Lemma 3 show that

$$-1.649 + 3\sqrt{n} \leq -2.7207 + \sqrt{10.8827n - 3.4806} \leq g_n \quad \text{for } n \geq 17.$$

For $n = 16$ we can choose $a = 4$ and $b = c = d = 0$ and from (15) and Lemma 3 we get $V_1(Q_{16}) < 9.65 < 10.34 \leq g_{16}$. Hence $V_1(Q_n) \leq g_n$ for $n \geq 16$. For the remaining $n = 4$ and $n \in [6, 15]$ the inequality is not tight enough and one has to calculate $V_1(Q_n)$ directly from (15). Finally the list in the Appendix shows that $V_1(Q_n) \leq g_n$ also for $n = 4$ and $6 \leq n \leq 15$. $\square$

Lemma 1(B) and hence Theorem 1 for $n = 4$ and $n \geq 6$ follows from lemmas 3–6. As Groemer polytopes are lattice polytopes, this also includes the proof of Theorem 2 for $n = 4$ and $n \geq 6$.

3. Proof of Theorems 1 and 2 for $n = 5$

The case $n = 5$ differs from all other $n \geq 4$ because there is no Groemer polytope for $n = 5$. The nonlattice case is solved in Lemma 7(a), the more complicated lattice case requires a 3-polytope for $d = 3$ and a 4-polytope (the regular 4-simplex) for $d \geq 4$ and is solved in Lemma 7(b) and (c).

Lemma 7.

- (a) Let Q_5 be the bipyramid over the regular triangle. Then $S(Q_5) \leq 24$ and $V_1(Q_5) \leq g_5$.
- (b) Let Q'_5 be the pyramid over the square with edgelenhth 2. Then $V_3(Q_5 + \varrho_3 B^3) \leq V_3(S_5 + \varrho_3 B^3)$ and $V_1(Q_5) \leq g_5$.
- (c) Let $Q''_5 = T^4$ be the regular 4-simplex with edgelenhth 2. Then we have, for any planar packing P_5 , $V_d(Q''_5 + \varrho B^d) \leq V_d(P_5 + \varrho B^d)$ for $\varrho \geq \varrho_d$ and $d \geq 4$.

Proof. (a) From the valuation properties of V_i it follows that $V_3(Q_5) = 2V_3(T) = 4/3\sqrt{2}$, $V_2(Q_5) = 2V_2(T) - \sqrt{3} = 3\sqrt{3}$, and $V_1(Q_5) = 2V_1(T) - 3 < 4.299$. Hence $S(Q_5) = 23.779 < 24$ and $V_1(Q_5) < 4.41 < g_5$.

(b) By elementary calculation we obtain $V_3(Q_5) = (4\sqrt{2})/3$ and $V_2(Q_5) = 2(\sqrt{3} + 1)$. If O is the regular octahedron with edgelenhth 2, then $V_1(O) = 12/\pi \arccos(1/3) = 4.701 \dots$ and $V_1(O) = 2V_1(Q_5) - 4$. So $V_1(Q_5) = 2 + 6/\pi \arccos(1/3) = 4.35 \dots < 4.41 < g_5$ by Lemma 3. Further, with $\varrho_3 = 2\sqrt{3}/\pi$ and Steiner's formula we obtain

$$V_3(Q_5 + \varrho_3 B^3) - V_3(S_5 + \varrho_3 B^3) = V_3(Q_5) + 4\sqrt{3}/\pi \cdot V_2(Q_5) + 12/\pi \cdot (V_1(Q_5) - 8) = -0.0026 \dots < 0.$$

(c) The intrinsic volumes of T^4 are given by $V_4(T^4) = \sqrt{5}/6 < 0.38$, $V_3(T^4) = 5/3\sqrt{2} < 2.36$, $V_2(T^4) = 5\sqrt{3}/\pi \arccos(-1/4) < 5.03$, and $V_1(T^4) = 20/\pi^2 \int_{-\infty}^{\infty} e^{-x^2} (\int_{-\infty}^{x/\sqrt{2}} e^{-y^2} dy)^3 dx < 4.13$ (see, e.g., formulae (18)–(20) of [H]). For any planar P_5 and $\varrho \geq \varrho_d$ we get, from (10) and Lemma 3,

$$\begin{aligned} \kappa_{d-2} V_2(P_5) + \kappa_{d-1} V_1(P_5) \varrho &\geq \kappa_{d-2} (V_2(P_5) + \sqrt{3} V_1(P_5)) + (\varrho - \varrho_d) \kappa_{d-1} g_5 \\ &\geq \kappa_{d-2} 8\sqrt{3} + 4.41 \kappa_{d-1} \varrho - 4.41 \sqrt{3} \kappa_{d-2} \\ &\geq 6.21 \kappa_{d-2} + 4.41 \kappa_{d-1} \varrho. \end{aligned}$$

Hence by Steiner's formula we obtain, for $\varrho \geq \varrho_d \geq \varrho_{d-1} \geq \varrho_{d-2}$,

$$\begin{aligned} \varrho^{4-d} (V(T^4 + \varrho B^d) - V(P_5 + \varrho B^d)) &\leq 0.38 \kappa_{d-4} + 2.36 \varrho \kappa_{d-3} + (5.03 - 6.21) \varrho^2 \kappa_{d-2} + (4.13 - 4.41) \varrho^3 \kappa_{d-1} \\ &\leq 0.38 \kappa_{d-4} + 2.36 \varrho \kappa_{d-3} + \varrho^2 (-1.18 \kappa_{d-2} - 0.28 \varrho_d \kappa_{d-1}) \\ &\leq 0.38 \kappa_{d-4} + 2.36 \varrho \kappa_{d-3} + \varrho^2 \kappa_{d-2} (-1.18 - 0.28 \sqrt{3}) \\ &\leq 0.38 \kappa_{d-4} + \varrho (2.36 \kappa_{d-3} - 1.66 \varrho_{d-1} \kappa_{d-2}) \\ &\leq 0.38 \kappa_{d-4} + \varrho \kappa_{d-3} (2.36 - 1.66 \sqrt{3}) \leq 0.38 \kappa_{d-4} - 0.51 \varrho_{d-2} \kappa_{d-3} \\ &\leq \kappa_{d-4} (0.38 - 0.51 \sqrt{3}) < 0. \end{aligned} \quad \square$$

4. Proof of Theorem 3

The proof requires five steps.

(i) First we prove that (2) holds without the restrictions (4), (3), and (5): Let $n \in \mathbb{N}$ be given and $q \in \mathbb{N}$ such that $q^2 < n \leq (q+1)^2$. Then $(q+1)^2 - n = b^2 + c^2 + d^2$ with some $b \geq c \geq d \geq 0$ if and only if $(q+1)^2 - n \neq 4^\alpha m$ with some $\alpha \geq 0$ and $m \equiv 7 \pmod{8}$, as is known from Lagrange's theorem (see [HW]). However, at least one of the four numbers $(q+i)^2 - n$, $i = 1, 2, 3, 4$, is a sum of three squares b^2, c^2, d^2 , since $\{(q+i)^2 : i = 1, 2, 3, 4\}$ contains all the quadratic residues $0, 1$, and $4 \pmod{8}$ and no translation of $\{0, 1, 4\}$ is contained in $\{0, 4, 7\} \pmod{8}$. With $a = q+i$ it follows that $n = a^2 - b^2 - c^2 - d^2$. $\sqrt{n} \leq a \leq \sqrt{n} + 4$ follows from the construction.

(ii) Next we prove that in the proof of (2) we can choose

$$\begin{aligned} i &\leq 2 & \text{for } n &\leq 255 & \text{and } n &\neq 93 & \text{and} \\ i &= 1 & \text{for } n &\leq 64 & \text{and } n &\notin \{18, 29, 42, 57\}. \end{aligned}$$

Assume that $n \leq 255$ (and so $q \leq 15$) and $k_1 = (q+1)^2 - n \leq 2q \leq 30$ and $k_2 = (q+2)^2 - n \leq 4q+3 \leq 63$ cannot be represented as a sum of three squares and thus $k_1, k_2 \in \{7, 15, 23, 31, 39, 47, 55, 63\} \cup \{28, 60\}$.

If $k_1 = 28$, then $q \in \{14, 15\}$ and $k_2 = 31 + 2q \in \{59, 61\}$, which is a contradiction and so $k_1 \in \{7, 15, 23\}$. From $k_2 = k_1 + (2q+3)$ it follows that $k_2 \in \{28, 60\}$. If $k_2 = 60$, then $k_1 = 57 - 2q \geq 27$, which is a contradiction. So $k_2 = 28$. From $4q+3 \geq 28$ it

follows that $q \geq 6$. With $k_1 = 25 - 2q \leq 13$ it follows that $k_1 = 7$, $q = 9$ and so $n = 93$ is the only exception.

The second statement is an immediate consequence of the above characterization of integers which are sums of three squares.

(iii) Proof of (3): From $b + c = \sqrt{b^2 + c^2 + 2bc} \geq \sqrt{b^2 + c^2 + d^2} = \sqrt{a^2 - n}$ it follows that it suffices to prove $3(a - \sqrt{n}) \leq \sqrt{a^2 - n}$ or equivalently $9(a - \sqrt{n}) \leq a + \sqrt{n}$ or $a \leq 5/4\sqrt{n}$. With $a \leq \sqrt{n} + i$ this is satisfied for $\sqrt{n}/4 \geq i$ or $n \geq 16i^2$.

In general the worst case is $i = 4$ and hence (3) holds for $n \geq 256$.

For $64 \leq n \leq 143$ and $n \neq 93$ we have $i \leq 2$ and so (3) holds for $n \geq 64$. For $n = 93$ we use the representation $93 = 12^2 - 5^2 - 5^2 - 1^2$.

For $16 \leq n \leq 63$ and $n \notin \{18, 29, 42, 57\}$ we have $i = 1$ and so (3) holds for $n \geq 16$. For the exceptions we use the representations $18 = 6^2 - 3^2 - 3^2$, $29 = 7^2 - 4^2 - 2^2$, $42 = 8^2 - 3^2 - 3^2 - 2^2$ and $57 = 9^2 - 4^2 - 2^2 - 2^2$.

For $n \leq 15$ we use the representations given in the Appendix.

(iv) Proof of (4): We have $b + c \leq \sqrt{2}\sqrt{b^2 + c^2} \leq \sqrt{2}\sqrt{a^2 - n}$ and so it is sufficient to prove $2(a^2 - n) \leq (a - 1)^2$. With $a = q + i$ this is equivalent to $q^2 \geq 2(i + 1)q + i^2 + 2i - 3$.

In general the worst case is $i = 4$ with $q^2 \geq 10q + 21$, which is satisfied for $q \geq 12$ and hence (4) holds for $n \geq 144$.

For $n \leq 143$ and $n \neq 93$ the worst case is $i = 2$ with $q^2 \geq 6q + 5$, which is satisfied for $q \geq 7$ and hence for $n \geq 49$. For $n = 93$ we use the representation $93 = 12^2 - 5^2 - 5^2 - 1^2$.

For $n \leq 48$ and $n \notin \{18, 29, 42\}$ we have $i = 1$ with $q^2 \geq 4q$, which is satisfied for $q \geq 4$ and hence for $n \geq 16$. For the exceptions we use the representations $18 = 6^2 - 3^2 - 3^2$, $29 = 7^2 - 4^2 - 2^2$ and $42 = 8^2 - 3^2 - 3^2 - 2^2$.

For $n \leq 15$ we use the representations as given in the Appendix.

(v) Proof of (5): If $a < b + 2$, then by the assumptions $a = b + 1$. From (4) it follows that $c = d = 0$ and so $n = a^2 - b^2 = 2a - 1$. Consequently $2a - 1 = n > (a - 4)^2$ or equivalently $a^2 - 10a + 17 < 0$, which is only true for $a \in [3, 7]$. $a = 3$ yields the exception $n = 5$. The other cases are solved by the representations in the Appendix. $\square$

Appendix

The following list gives representations $n = a^2 - b^2 - c^2 - d^2$ as needed in Theorem 3 and Lemma 6 for $n = 4$ and $6 \leq n \leq 15$. With (15) and Lemma 3 this proves $V_1(Q_n) < g_n$.

$4 = 2^2$	$V_1(Q_4) = 3.65$	$g_4 = 4$
$6 = 3^2 - 1^2 - 1^2 - 1^2$	$V_1(Q_6) = 4.65$	$g_6 \geq 5.14$
$7 = 3^2 - 1^2 - 1^2$	$V_1(Q_7) = 5.32$	$g_7 \geq 5.81$
$8 = 3^2 - 1^2$	$V_1(Q_8) = 5.98$	$g_8 \geq 6.42$
$9 = 3^2$	$V_1(Q_9) = 6.65$	$g_9 \geq 7.00$
$10 = 4^2 - 2^2 - 1^2 - 1^2$	$V_1(Q_{10}) = 6.65$	$g_{10} \geq 7.54$
$11 = 4^2 - 2^2 - 1^2$	$V_1(Q_{11}) = 7.32$	$g_{11} \geq 8.06$
$12 = 4^2 - 2^2$	$V_1(Q_{12}) = 7.98$	$g_{12} \geq 8.55$

$$\begin{array}{lll}
 13 = 4^2 - 1^2 - 1^2 - 1^2 & V_1(Q_{13}) = 7.65 & g_{13} \geq 9.03 \\
 14 = 4^2 - 1^2 - 1^2 & V_1(Q_{14}) = 8.32 & g_{14} \geq 9.48 \\
 15 = 4^2 - 1^2 & V_1(Q_{15}) = 8.98 & g_{15} \geq 9.92
 \end{array}$$

References

- [ABB] V. Arhelger, U. Betke, K. Böröczky, Jr., Large finite lattice packings, preprint.
- [BG] U. Betke, P. Gritzmann, Über L. Fejes Tóth's Wurstvermutung in kleinen Dimensionen, *Acta Math. Hungar.* **43** (1984), 299–307.
- [BGW] U. Betke, P. Gritzmann, J.M. Wills, Slices of L. Fejes Tóth's sausage conjecture, *Mathematika* **29** (1982), 194–201.
- [BH] U. Betke, M. Henk, Finite packings of spheres, *Discrete Comput. Geom.* **19** (1998), 197–227.
- [BHW1] U. Betke, M. Henk, J.M. Wills, Finite and infinite packings, *J. Reine Angew. Math.* **453** (1994), 165–191.
- [BHW2] U. Betke, M. Henk, J.M. Wills, Sausages are good packings, *Discrete Comput. Geom.* **13** (1995), 297–311.
- [BS] K. Böröczky, Jr., U. Schnell, Asymptotic shape of finite packings, *Canad. J. Math.* **50**(1) (1998), 16–28.
- [BW] K. Böröczky, Jr., J.M. Wills, Finite sphere packings and critical radii, *Beiträge Algebra Geom.* **38** (1997), 193–211.
- [Gri] P. Gritzmann, Finite Packungen und Überdeckungen, Habilitationsschrift, Siegen (1984).
- [GW] P. Gritzmann, J.M. Wills, Lattice points, Chapter 3.2 in: *Handbook of Convex Geometry*, P.M. Gruber, J.M. Wills, eds., North-Holland, Amsterdam (1993).
- [Gro] H. Groemer, Über die Einlagerung von Kreisen in einen konvexen Bereich, *Math. Z.* **73** (1960), 285–294.
- [Grü] B. Grünbaum, *Convex Polytopes*, Wiley, London (1967).
- [H] H. Hadwiger, Gitterpunktzahl im Simplex und Wills'sche Vermutung, *Math. Ann.* **239** (1979), 271–288.
- [HW] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford (3rd edn.) (1954).
- [SY] J.R. Sangwine-Yager, Mixed volumes, Chapter 1.2 in: *Handbook of Convex Geometry*, P.M. Gruber, J.M. Wills, eds., North-Holland, Amsterdam (1993).
- [We] G. Wegner, Über endliche Kreispackungen in der Ebene, *Studia Sci. Math. Hungar.* **21** (1986), 1–28.
- [Wi] J.M. Wills, Finite sphere packings and sphere coverings, *Rend. Sem. Mat. Messina Ser. II* **2** (1993), 91–97.

Received November 27, 1998, and in revised form January 4, 1999. Online publication May 16, 2000.