Kalai's Squeezed Spheres Are Shellable*

C. W. Lee

Department of Mathematics, University of Kentucky,
Lexington, KY 40506, USA
lee@ms.uky.edu

Abstract. Kalai constructs a large collection of simplicial spheres to show that there is a substantial gap between the numbers of simplicial polytopes and of simplicial spheres. We prove here that Kalai's spheres are shellable.

1. Introduction

Let $c(d, n)$ (respectively, $s(d, n)$) be the number of combinatorial types of simplicial d -polytopes (respectively, simplicial $(d - 1)$ -spheres) with n labeled vertices. It is probably not surprising that there is a substantial gap between these two numbers. Kalai [5] proves that $\log s(d, n) \geq C_1(d)n^{[(d-1)/2]}$, whereas Goodman and Pollack [1], [2] demonstrate that $\log c(d, n) \leq d(d + 1)n \log n$. In particular, Kalai explicitly constructs a family of simplicial $(d - 1)$ -spheres with n labeled vertices. Each sphere is realized as the boundary of a shellable d -ball, so by Pachner's theorem [7] Kalai's spheres are all piecewise-linear. In this paper we prove that Kalai's spheres in fact lie in the still stricter subclass of shellable spheres. Hence the gap between the numbers of simplicial polytopes and spheres already occurs between simplicial polytopes and shellable spheres.

2. Terminology

We begin with notation and definitions, some of which follow Kalai [5]. For positive integer $n \geq 1$, let $[n]$ denote the set $\{1, \dots, n\}$. For nonnegative integer d , $[n]^{(d)}$ is the

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set of all subsets of $[n]$ of cardinality d . For $F \in [n]^{(d)}$ and $i \in [n] \cup \{0\}$, we write $F - i$ for $F \setminus \{i\}$ and $F + i$ for $F \cup \{i\}$. If $C \subseteq 2^{[n]}$, define $\text{cone}(C, 0) = \{F + 0 : F \in C\}$.

For $F, G \in [n]^{(d)}$, $F = \{a_1, \dots, a_d\}$, $G = \{b_1, \dots, b_d\}$, where $a_1 < \dots < a_d$ and $b_1 < \dots < b_d$, we say $F \leq_p G$ if $a_i \leq b_i$ for $i = 1, \dots, d$. Thus $\leq_p$ is a partial order on $[n]^{(d)}$. We also define the total reverse lexicographic order on $[n]^{(d)}$ by specifying that $F <_{\text{RL}} G$ if $\max(F \Delta G) \in G$.

Suppose F is a proper subset of $[n]$. Let $j = \min\{i \in [n] : i \notin F\}$ and $k = \max\{i \in [n] : i \notin F\}$. Then $\{1, \dots, j-1\}$ is the *left* set of F (possibly empty) and $\{k+1, \dots, n\}$ is the *right* set of F (also possibly empty). If $j, k \notin F$ but $i \in F$ for all $i = j+1, \dots, k-1$, then $\{j+1, \dots, k-1\}$ is a *middle* set of F .

In the special case that $F = [n]$, we say that both the left set and the right sets equal F , and that F has no middle sets.

Consider some element $j \in F \subseteq [n]$. Define $\ell(F, j) = \max\{i \in \mathbf{Z} : i < j, i \notin F\}$ and $L(F, j) = (F - j) + \ell(F, j)$. Thus $L(F, j)$ is the set resulting from “pushing” j and its immediate predecessors one step to the left. Define $r(F, j) = \min\{i \in \mathbf{Z} : i > j, i \notin F\}$ and $R(F, j) = (F - j) + r(F, j)$. So $R(F, j)$ is the set resulting from “pushing” j and its immediate successors one step to the right.

Let d be a positive odd integer and let n be a positive integer. Take $F_d(n)$ to be the collection of all members of $[n]^{(d+1)}$ having even cardinality left, middle, and right sets. So sets in $F_d(n)$ consist of unions of $(d+1)/2$ disjoint pairs of adjacent elements. For $F \in F_d(n)$, when we write $F = \{a_1, \dots, a_{d+1}\}$ we implicitly assume that the elements have been indexed so that $a_1 < \dots < a_{d+1}$.

For even positive integer d , put $F_d(n) = \text{cone}(F_{d-1}(n), 0)$.

3. Squeezed Balls and Spheres

For positive d consider a nonempty subcollection I of $F_d(n)$ that is an initial set of $F_d(n)$ with respect to the partial order $\leq_p$. Equivalently, for $F \in I$, $L(F, j) \in I$ whenever $L(F, j) \in F_d(n)$. Kalai [5] proves that the simplicial d -complex $B(I) = \{G : G \subseteq F \text{ for some } F \in I\}$ with facets in I is topologically a d -ball. He calls such simplicial complexes *squeezed balls* and proves that they are shellable. Kalai observes that squeezed d -balls $B(I)$ for even d are just those simplicial complexes of the form $\text{cone}(B(J), 0)$, where J is an initial set of $F_{d-1}(n)$.

The boundary $S(I) = \partial B(I)$ of a squeezed ball $B(I)$ is topologically a $(d-1)$ -sphere, and is called a *squeezed sphere*. The facets of $S(I)$ are those subsets of $B(I)$ of cardinality d that are contained in exactly one facet of $B(I)$.

Kalai proves that the reverse lexicographic ordering of the facets of a squeezed ball $B(I)$ is a shelling order. By examining this shelling, one can readily characterize the facets of $S(I)$.

Proposition 1. *Suppose d is odd and $B(I)$ is a squeezed d -ball. Let $F = \{a_1, \dots, a_{d+1}\} \in I$ be a facet of $B(I)$ and $a_i \in F$. Then $F - a_i$ is a facet of $S(I)$ if and only if*

1. i is even and a_i is in the left set of F , or
2. i is odd and $R(F, a_i) \notin I$.

Table 1. Facets of $B(I)$.

	1	2	3	4	5	6	7	8
G_1	1	1^3	1	1^2	1	1^1		
G_2	1	1^5	1	1^4		1	1	
G_3	1	1^6		1	1	1	1	
G_4		1^{20}	1	1^{19}	1	1^{18}	1	
G_5	1^{17}	1^8	1	1^7			1^{11}	1
G_6	1^{16}	1^9		1	1		1^{12}	1
G_7	1^{15}	1^{10}			1^{14}	1	1^{13}	1

Suppose d is even and $B(I) = \text{cone}(B(J), 0)$ is a squeezed d -ball. Then a subset $F \in B(I)$ of cardinality d is a facet of $S(I)$ if and only if

- 1. F is a facet of $B(J)$, or
- 2. $F = G + 0$ where G is a facet of $S(J)$.

4. Example

Tables 1 and 2 show an example of a squeezed 5-ball $B(I)$ with eight vertices, and the squeezed 4-sphere $S(I)$ that is its boundary. In Table 1 we list the characteristic vectors of the members G_i of I in reverse lexicographic order. The superscripts indicate elements that may be deleted from members of I to obtain facets of $S(I)$, and the corresponding

Table 2. Facets of $S(I)$.

	1	2	3	4	5	6	7	8
H_1	1	1	1	1	1			
H_2	1	1	1		1	1^*		
H_3	1		1	1^*	1	1^*		
H_4	1	1	1			1	1^*	
H_5	1		1	1^*		1	1^*	
H_6	1			1	1^*	1	1^*	
H_7	1	1	1				1	1^*
H_8	1		1	1^*			1	1^*
H_9	1			1	1^*		1	1^*
H_{10}	1				1	1^*	1	1^*
H_{11}	1	1^*	1	1^*				1^*
H_{12}	1	1^*		1	1^*			1^*
H_{13}	1	1^*			1	1^*		1^*
H_{14}	1	1^*				1^*	1^*	1^*
H_{15}		1^*			1^*	1	1^*	1
H_{16}		1^*		1^*	1		1^*	1
H_{17}		1^*	1^*				1^*	1
H_{18}		1	1^*	1	1^*		1^*	
H_{19}		1	1^*		1^*	1^*	1^*	
H_{20}			1^*	1^*	1^*	1^*	1^*	

ordering of the facets of $S(I)$ is a shelling order. In Table 2 we list the characteristic vectors of the facets H_i of $S(I)$ in a shelling order. For each facet, the elements of subset X (defined in the next section in connection with the proof of the shelling order) are marked by asterisks.

5. Shelling Squeezed Spheres

We first prove that squeezed d -spheres are shellable when d is odd by giving an explicit total order for the facets, which we then show is a shelling order. Assume $B(I)$ is a squeezed d -ball for odd d . Let $F, G \in I$, where $F = \{a_1, \dots, a_{d+1}\}$ and $G = \{b_1, \dots, b_{d+1}\}$. Suppose $F - a_k$ and $G - b_\ell$ are facets of $S(I)$. Set $r = \min\{j: a_i = b_i \text{ for all } i \geq j\}$ (taking $r = d + 2$ if $a_{d+1} \neq b_{d+1}$). Define $(G - b_\ell) <_S (F - a_k)$ if one of the following conditions holds. In each case, an instance will be given from the facets of $S(I)$ in the above example.

1. k is odd and ℓ is even. ($H_1 <_S H_{11}$.)
2. k and ℓ are both even and $G <_{\text{RL}} F$. ($H_1 <_S H_5$.)
3. k and ℓ are both even, $G = F$, and $\ell > k$. ($H_1 <_S H_2$.)
4. k and ℓ are both odd, $k = \ell \geq r$, and $G <_{\text{RL}} F$. ($H_{11} <_S H_{12}$.)
5. k and ℓ are both odd and $\ell > k \geq r$. ($H_{13} <_S H_{14}$.)
6. k and ℓ are both odd and $\ell \geq r > k$. ($H_{12} <_S H_{14}$.)
7. k and ℓ are both odd, $k, \ell < r$, and $F <_{\text{RL}} G$. ($H_{11} <_S H_{18}$.)

Theorem 1. *If d is odd and $S(I)$ is a squeezed $(d - 1)$ -sphere, then $<_S$ is a shelling order of its facets.*

Proof. It suffices to show for each facet of $S(I)$ there is a subset X with the properties (a) for each $a_i \in X$ there is a preceding facet (under $<_S$) of $S(I)$ containing $X - a_i$ and (b) X itself is not contained in any preceding facet. We illustrate the various cases that arise by instances from the above example.

Assume that $F = \{a_1, \dots, a_{d+1}\} \in I$ and $F - a_k$ is a facet of $S(I)$.

Suppose first that k is even. Let $X = \{a_i: i > k \text{ and } i \text{ is even}\}$. Take any $a_i \in X$. Assume that a_i is in the left set of F (e.g., $F = G_2$, $a_k = 2$, $a_i = 4$). Then $F - a_i$ is a preceding facet (by condition 3) of $S(I)$ that contains $(F - a_k) - a_i$. Assume that a_i is not in the left set of F (e.g., $F = G_2$, $a_k = 2$, $a_i = 7$). Then $L(F, a_i) \in I$ and $L(F, a_i) - a_k$ is a preceding facet (by condition 2) of $S(I)$ that contains $(F - a_k) - a_i$. Hence X satisfies (a). Now assume that $G - b_\ell$ is a preceding facet of $S(I)$ that contains X . Then $(G - b_\ell) <_S (F - a_k)$ via condition 2 or 3. Condition 2 is impossible, since $G <_{\text{RL}} F$ implies G cannot contain X . However, condition 3 is also impossible, because $F = G$ and $\ell > k$ implies $b_\ell \in X$. Therefore X satisfies (b).

For the remainder of the proof assume that k is odd. Note that $F - a_k$ has exactly one odd middle or right set (the elements of F immediately following a_k). Call this set T . Let $X = X_1 \cup X_2 \cup X_3$ where $X_1 = \{a_i: i < k \text{ and } i \text{ is even}\}$, $X_2 = \{a_i: a_i \in T \text{ and } i \text{ is even}\}$, and $X_3 = \{a_i: i > k \text{ and } i \text{ is odd}\}$. Choose any $a_i \in X$. In each of the following cases we will find a facet H of $S(I)$ that precedes $F - a_k$ and contains $(F - a_k) - a_i$.

Case 1: Assume $a_i \in X_1$ and a_i is in the left set of F . (For example, $F = G_5$, $a_k = 7$, $a_i = 2$.) Take $H = F - a_i$, which is a preceding facet by condition 1.

Case 2: Assume $a_i \in X_1$ and a_i is not in the left set of F . Let $F_1 = L(F, a_i)$, $a = \ell(F, a_i)$, $F_2 = R(F_1, a_k)$, and $F_3 = R(F, a_k)$. Note that $F_3 \notin I$ since $F - a_k$ is a facet of $S(I)$. If $F_2 \notin I$ (e.g., $F = G_6$, $a_k = 7$, $a_i = 5$), then take $H = F_1 - a_k$, which is a preceding facet by condition 4. If, on the other hand, $F_2 \in I$ (e.g., $F = G_4$, $a_k = 6$, $a_i = 5$), then $R(F_2, a) = F_3 \notin I$, so $H = F_2 - a$ is a preceding facet by condition 7.

Case 3: Assume $a_i \in X_2$. (Examples: $F = G_5$, $a_k = 1$, $a_i = 2$; $F = G_4$, $a_k = 6$, $a_i = 7$; $F = G_7$, $a_k = 5$, $a_i = 6$.) Let $a < a'$ be the smallest two elements of $[n]$ not in $F \setminus \{a_k, a_i\}$. Then $F_1 = (F \setminus \{a_k, a_i\}) \cup \{a, a'\} \in I$. Take $H = F_1 - a'$, which is a preceding facet by condition 1.

Case 4: Assume $a_i \in X_3$. Let $F_1 = R(F, a_i)$, $F_2 = R(F, a_k)$, and $F_3 = R(F_2, a_i) = R(F_1, a_k)$. Note that $F_2 \notin I$ since $F - a_k$ is a facet of $S(I)$. Hence F_3 is also not in I , since $F_2 \leq_p F_3$. If $F_1 \notin I$ (e.g., $F = G_6$, $a_k = 1$, $a_i = 7$), then take $H = F - a_i$, which is a preceding facet by condition 5. If, on the other hand, $F_1 \in I$ (e.g., $F = G_6$, $a_k = 1$, $a_i = 4$), then take $H = F_1 - a_k$, which is a preceding facet by condition 7.

At this point we have verified that X satisfies (a). Now assume $G - b_\ell$ is a preceding facet of $S(I)$ that contains X . Then $(G - b_\ell) <_S (F - a_k)$ via condition 1, 4, 5, 6, or 7. The cardinality of $G \setminus X$ equals the number of odd left, middle, and right sets of X , which severely restricts the positioning of the elements of $G \setminus X$ so that $G \in F_d(n)$: each element of $G \setminus X$ must be adjacent to one of the odd sets of X . Condition 1 would require the left set of G to be larger than the left set of F , with insufficient remaining elements to ensure $G \in F_d(n)$. Condition 4 requires that G must agree with F from a_k onward, and yet $G <_{\text{RL}} F$, which is impossible if G is to contain X . Condition 5 also requires that G must agree with F from a_k onward. However, then ℓ odd implies $b_\ell \in X$, which is a contradiction. Condition 6 requires G to disagree with F at some point on or after a_k but to agree from b_ℓ onward, but then $b_\ell \in X$. Condition 7 requires $F <_{\text{RL}} G$ and that G disagree with F on or after a_k . However, if G is to contain X , it must agree with F from a_k onward, another contradiction. Therefore X satisfies (b), completing the proof. $\square$

We now have everything necessary to establish the main result.

Corollary 1. *Squeezed spheres are shellable.*

Proof. Assume $S(I)$ is a squeezed $(d - 1)$ -sphere. If d is odd, then $S(I)$ is shellable by the previous theorem. If d is even, then there is a squeezed $(d - 1)$ -ball $B(J)$ such that $B(I) = \text{cone}(B(J), 0)$. Then $S(I)$ can be shelled by first shelling the facets of $B(J)$ (in reverse lexicographic order) and then shelling the facets of $S(I)$ that contain 0 in the order induced by the shelling of $S(J)$. $\square$

6. Comments

1. The ordering $<_S$ was chosen in such a way to coordinate well with the reverse lexicographic shelling of $B(I)$ and the accompanying changes of $S(I)$ and its h -vector by bistellar operations, as the reader might see from some experimentation. However, the technique of the proof suffers from being quite specific to the problem at hand, and does not readily suggest its applicability to more general structures.

2. Hebble [4] shows that the dual graphs of all squeezed 2-spheres and squeezed 3-spheres are Hamiltonian. As a consequence, for every simple 4-polytope P there is a Hamiltonian simple 4-polytope Q with the same f -vector. (So far no non-Hamiltonian simple d -polytope, $d \geq 4$, has been found.)

3. Lickorish [6] explains how to create nonshellable spheres, and Hachimori and Ziegler [3] have recently proved the even stronger result that nonconstructible d -spheres can be found for every $d \geq 3$. We know now that Kalai's piecewise-linear spheres are shellable and thus that there is a large gap between the numbers of simplicial polytopes and of shellable spheres; one might well suspect that there is also a large difference between the numbers of shellable spheres and of piecewise-linear spheres. Perhaps a way can be found to construct large numbers of them to estimate this difference.

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