Hadwiger–Wills-Type Higher-Dimensional Generalizations of Pick’s Theorem*

K. Kołodziejczyk

Institute of Mathematics, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
kolodz@im.pwr.wroc.pl

Abstract. Pick’s famous area theorem has many generalizations and extensions including relatively recent work by Grünbaum and Shephard [3]. One of the generalizations is due to Hadwiger and Wills who considered nonproper lattice polygons having isolated points and one-dimensional parts. The aim of this note is to give generalizations of Hadwiger–Wills formula for nonproper lattice polyhedra in $\mathbb{R}^3$ and $\mathbb{R}^4$. The four-dimensional considerations indicate difficulties appearing in a search for an arbitrary-dimensional generalization of this type.

1. Introduction and Notation

Let $\mathbb{Z}^N$ denote the fundamental lattice of points with integer coordinates in $\mathbb{R}^N$. Elements of $\mathbb{Z}^N$ are called *lattice points*. We say that a simplex $\Delta \subset \mathbb{R}^N$ is a *lattice simplex* if all its vertices belong to $\mathbb{Z}^N$. A lattice simplex Δ is called *fundamental* if $\Delta \cap \mathbb{Z}^N$ consists only of the vertices of Δ (fundamental lattice 1-simplexes are simply called *fundamental lattice segments*). A set $P \subset \mathbb{R}^N$ is said to be a *polyhedron* if P is the underlying point set of a simplicial cell complex. A polyhedron P is called *lattice* or *integral* if all its vertices (0-simplexes) lie in $\mathbb{Z}^N$. Any lattice polyhedron P can be represented as the union

$$P = \bigcup_{i=1}^m \Delta_i, \quad (1)$$

where each Δ_i is a fundamental lattice simplex and $\Delta_j \setminus \Delta_k \neq \emptyset$ for $j \neq k$ (no Δ_j is contained in another simplex). Lattice polyhedra in $\mathbb{R}^2$ are called, as usual, *lattice*

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polygons. A lattice polyhedron P in $\mathbb{R}^N$ is called *proper* if every Δ_i in the union (1) is N -dimensional. An $(N - 1)$ -dimensional fundamental lattice simplex Δ is called *two-sided* in a lattice polyhedron P if $\text{relint } \Delta \cap \text{cl int } P = \emptyset$.

Since Reeve's work [10] a family of secondary lattices

$$\mathbb{Z}_n^N = \{x \in \mathbb{R}^N : nx \in \mathbb{Z}^N\}, \quad n \geq 2,$$

has been of great importance in problems dealing with lattice polyhedra, see [1], [2], and [8]. It will be convenient to use the following notation: $\mathbb{Z}_1^N = \mathbb{Z}^N$.

The central notion in this note is the family of *lattice point enumerators* G_n , $n \geq 1$, functionals that associate with a given subset S of $\mathbb{R}^N$ the number

$$G_n(S) = |S \cap \mathbb{Z}_n^N|, \quad n \geq 1.$$

Generalizing the famous Pick's theorem to nonproper lattice polygons, Hadwiger and Wills [4] (see also [1]) proved that the area, $A(P)$, of a lattice polygon P in $\mathbb{R}^2$ is given by the following formula:

$$2A(P) = 2G_1(P) - 2\chi(P) - |\Delta^1(P)|, \quad (2)$$

in which $\chi(P)$ is the *Euler characteristic* of P and $|\Delta^1(P)|$ denotes the number of fundamental lattice segments on ∂P with two-sided segments contributing twice to $|\Delta^1(P)|$. For proper lattice polygons (2) was also obtained by Varberg [12]. A remarkable generalization of Pick's theorem to a class of nonproper lattice polygons in $\mathbb{R}^2$ was obtained by Grünbaum and Shephard in [3]. Their introduction of oriented lattice polygons considerably enlarged the range of cases known in the wide literature dealing with Pick's theorem.

For a given lattice polyhedron P in $\mathbb{R}^N$ denote by B_n and I_n , $n \geq 1$, the numbers of points of the lattice $\mathbb{Z}_n^N$ on the boundary and in the interior of P , respectively. Thus

$$B_n = B_n(P) = |\mathbb{Z}_n^N \cap \partial P| \quad \text{and} \quad I_n = I_n(P) = |\mathbb{Z}_n^N \cap \text{int } P|.$$

Reeve [10], [11] was the first who proved a higher-dimensional Pick-type theorem. He showed that the volume, $V(P)$, of a proper lattice polyhedron P in $\mathbb{R}^3$ is given by

$$2n(n^2 - 1)V(P) = B_n - nB_1 + 2(I_n - nI_1) + (n - 1)[2\chi(P) - \chi(\partial P)], \quad n \geq 2. \quad (3)$$

Next Macdonald [9] generalized Pick's (and Reeve's) theorem to proper lattice polyhedra in $\mathbb{R}^N$ by establishing the following formula:

$$\begin{aligned} (N - 1)N! V(P) &= \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} (B_{N-k} + 2I_{N-k}) \\ &\quad + (-1)^{N-1} [2\chi(P) - \chi(\partial P)]. \end{aligned} \quad (4)$$

For more higher-dimensional Pick-type formulae the reader is referred to [5]–[7].

The aim of this note is to give Hadwiger–Wills-type higher-dimensional formulae applicable to nonproper lattice polyhedra in $\mathbb{R}^3$ and $\mathbb{R}^4$. In the next section we prove the following two theorems.

Theorem 1. *If P is any lattice polyhedron in $\mathbb{R}^3$, then*

$$12V(P) = 2G_2(P) - 4G_1(P) + 2\chi(P) - |\Delta^2(P)|, \quad (5)$$

where $|\Delta^2(P)|$ denotes the number of fundamental lattice triangles on ∂P and every two-sided triangle contributes to $|\Delta^2(P)|$ twice.

Theorem 2. *If P is any lattice polyhedron in $\mathbb{R}^4$, then*

$$72V(P) = 2G_3(P) - 6G_2(P) + 6G_1(P) - 2\chi(P) - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1), \quad (6)$$

where the sum extends over all three-dimensional fundamental lattice tetrahedra on ∂P , $|\Delta_2^3|$ denotes the number of lattice points from $\mathbb{Z}_2^4$ lying in the relative interior of a fundamental lattice tetrahedron Δ^3 from ∂P , and any two-sided tetrahedron contributes to the sum twice.

2. Proofs

In the proofs of both theorems we use several observations formulated below as lemmas.

Lemma 1. *If Δ^1 is a fundamental lattice segment in $\mathbb{R}^N$, then $|\mathbb{Z}_2^N \cap \text{relint } \Delta^1| = 1$ and $|\Delta^1 \cap \mathbb{Z}_2^N| = 3$.*

Proof. Of course the midpoint of Δ^1 belongs to $\mathbb{Z}_2^N \setminus \mathbb{Z}^N$ and $\mathbb{Z}^N \subset \mathbb{Z}_2^N$. Therefore $|\Delta^1 \cap \mathbb{Z}_2^N| \geq 3$. It is obvious that no additional points from $\mathbb{Z}_2^N$ can belong to Δ^1 . $\square$

Lemma 2. *If Δ^2 is a fundamental lattice triangle in $\mathbb{R}^N$, then $\mathbb{Z}_2^N \cap \text{relint } \Delta^2 = \emptyset$ and $|\Delta^2 \cap \mathbb{Z}_2^N| = 6$.*

Proof. Let $\Delta^2 = \text{conv}(x_1, x_2, x_3)$ be any fundamental lattice triangle. From Lemma 1 it follows that $|\Delta^2 \cap \mathbb{Z}_2^N| \geq 6$. We will show that there is no point from $\mathbb{Z}_2^N$ lying in the relative interior of Δ^2 . To this end denote by y_i the midpoint of the side of Δ^2 which is opposite to x_i . First we notice that there cannot be any such a point in $\text{conv}(x_i, y_j, y_k)$. If there were such a point, say z , then we would have $z = \alpha_1 x_i + \alpha_2 y_j + \alpha_3 y_k$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$. For the lattice point $z^* = 2z - x_i$ we would in turn have

$$\begin{aligned} z^* &= 2\alpha_1 x_i + 2\alpha_2 y_j + 2\alpha_3 y_k - x_i \\ &= 2\alpha_1 x_i + 2\alpha_2 \left(\frac{x_i + x_k}{2} \right) + \alpha_3 \left(\frac{x_i + x_j}{2} \right) - x_i \\ &= (2\alpha_1 + \alpha_2 + \alpha_3 - 1)x_i + \alpha_2 x_k + \alpha_3 x_j \\ &= \alpha_1 x_i + \alpha_2 x_k + \alpha_3 x_j. \end{aligned}$$

This means that z^* is a convex combination of x_1, x_2, x_3 and, therefore, belongs to Δ^2 . Since, obviously, z^* is a lattice point different from the vertices of Δ^2 this contradicts the fact that Δ^2 was fundamental.

Now we show that no point from $\mathbb{Z}_2^N$ (different from y_1, y_2, y_3) can lie in the triangle with vertices y_1, y_2, y_3 . Suppose to the contrary that there is a point $u \in \text{conv}(y_1, y_2, y_3)$. It means that

$$u = \beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3 = \beta_1 \left(\frac{x_2 + x_3}{2} \right) + \beta_2 \left(\frac{x_1 + x_3}{2} \right) + \beta_3 \left(\frac{x_1 + x_2}{2} \right),$$

where $\beta_1 + \beta_2 + \beta_3 = 1$ and $\beta_1, \beta_2, \beta_3 \geq 0$.

It is clear that $x_1 + x_2 + x_3 - 2u$ is a lattice point for which it holds that

$$\begin{aligned} x_1 + x_2 + x_3 - 2u &= x_1 + x_2 + x_3 - \beta_1(x_2 + x_3) - \beta_2(x_1 + x_3) - \beta_3(x_1 + x_2) \\ &= x_1(1 - \beta_2 - \beta_3) + x_2(1 - \beta_1 - \beta_3) + x_3(1 - \beta_1 - \beta_2) \\ &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3. \end{aligned}$$

This implies that the lattice point $x_1 + x_2 + x_3 - 2u$, being a convex combination of x_1, x_2, x_3 (and different from them), lies in Δ^2 and contradicts the fact that it is fundamental. $\square$

As an immediate consequence of Lemmas 1 and 2 we have the following fact.

Lemma 3. *The number of fundamental lattice segments on the boundary of a lattice polyhedron P in $\mathbb{R}^3$ is $B_2 - B_1$.*

Lemma 4. *The Euler characteristic of the boundary of a lattice polyhedron P in $\mathbb{R}^3$ is given by*

$$\chi(\partial P) = 2B_1 - B_2 + T,$$

where T denotes the number of fundamental lattice triangles on ∂P .

Proof. Take a lattice polyhedron P in $\mathbb{R}^3$ with a triangulation to fundamental lattice simplexes. By definition $\chi(\partial P)$ is equal to the number of 0-simplexes (which in this case is obviously B_1) minus the number of edges plus the number of triangles. Thus, using Lemma 3, we have

$$\chi(\partial P) = B_1 - (B_2 - B_1) + T = 2B_1 - B_2 + T$$

and the proof is complete. $\square$

Lemma 5. *If P is a proper lattice polyhedron in $\mathbb{R}^3$, then*

$$12V(P) = 2G_2 - 4G_1 + 2\chi(P) - T,$$

where T denotes the number of fundamental lattice triangles on ∂P .

Proof. In the case of $n = 2$ Reeve's formula reads

$$12V(P) = B_2 - 2B_1 + 2(I_2 - 2I_1) + 2\chi(P) - \chi(\partial P).$$

By substituting to it the expression for $\chi(\partial P)$ from Lemma 4 and using the obvious relation $G_i = B_i + I_i$ we easily obtain

$$12V(P) = 2G_2 - 4G_1 + 2\chi(P) - T$$

and the proof is complete. $\square$

Having Lemmas 1–5 we are now in a position to prove Theorem 1. However, since the structures of the proofs of Theorems 1 and 2 are similar we provide only the proof of the latter. To this end we need a few additional lemmas formulated and proved below. The first one is obvious.

Lemma 6. *If Δ^1 is a fundamental lattice segment in $\mathbb{R}^N$, then $|\mathbb{Z}_3^N \cap \text{relint } \Delta^1| = 2$ and $|\Delta^1 \cap \mathbb{Z}_3^N| = 4$.*

Lemma 7. *If Δ^2 is a fundamental lattice triangle in $\mathbb{R}^N$, then $|\mathbb{Z}_3^N \cap \text{relint } \Delta^2| = 1$ and $|\Delta^2 \cap \mathbb{Z}_3^N| = 10$.*

Proof. Let $\Delta^2 = \text{conv}(x_1, x_2, x_3)$ be a fundamental lattice triangle in $\mathbb{R}^N$. It is obvious that $c = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3$ belongs to $\mathbb{Z}_3^N \cap \text{relint } \Delta^2$. This and Lemma 6 imply that $|\Delta^2 \cap \mathbb{Z}_3^N| \geq 10$. We show that no other points from $\mathbb{Z}_3^N$ can belong to Δ^2 . We do it by checking that such points cannot lie in some smaller triangles forming a partition of Δ^2 . For this purpose denote by y_j^k and y_j^i two points from $\mathbb{Z}_3^N$ lying on the side of Δ^2 which is opposite to the vertex x_j . We assume that y_j^k is the point which is closer to x_k and, consequently, y_j^i is closer to x_i .

First we notice that there cannot be any point from $\mathbb{Z}_3^N$ in $\text{conv}(x_i, y_k^i, y_j^i)$. If there were such a point, say z , then we would have $z = \alpha_1 x_i + \alpha_2 y_k^i + \alpha_3 y_j^i$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$ where $y_k^i = 2/3x_i + 1/3x_j$ and $y_j^i = 2/3x_i + 1/3x_k$. The point $z^* = x_i + 3(z - x_i) = 3z - 2x_i$ would be, of course, a lattice point. We would have for it

$$\begin{aligned} z^* &= 3z - 2x_i = 3\alpha_1 x_i + 3\alpha_2(2/3x_i + 1/3x_j) + 3\alpha_3(2/3x_i + 1/3x_k) - 2x_i \\ &= (3\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2)x_i + \alpha_2 x_j + \alpha_3 x_k \\ &= \alpha_1 x_j + \alpha_2 x_k + \alpha_3 x_i. \end{aligned}$$

This would mean that the lattice point z^* , apparently different from the vertices of Δ^2 , lies in Δ^2 and would contradict the fact that Δ^2 is fundamental.

Now we show that points from $\mathbb{Z}_3^N$, different from the vertices, cannot lie in the triangles $\text{conv}(c, y_j^k, y_j^i)$ and $\text{conv}(c, y_k^i, y_j^i)$ either. If there is $u \in \text{conv}(c, y_j^k, y_j^i)$, then

$$\begin{aligned} x_j + 3(u - c) &= x_j + 3(\alpha_1 c + \alpha_2 y_j^k + \alpha_3 y_j^i) - 3c \\ &= x_j + 3\alpha_1(1/3x_i + 1/3x_j + 1/3x_k) + 3\alpha_2(2/3x_k + 1/3x_i) \\ &\quad + 3\alpha_3(2/3x_i + 1/3x_k) - 3(1/3x_i + 1/3x_j + 1/3x_k) \\ &= (1 + \alpha_1 - 1)x_j + (\alpha_1 + \alpha_2 + 2\alpha_3 - 1)x_i + (\alpha_1 + 2\alpha_2 + \alpha_3 - 1)x_k \\ &= \alpha_1 x_j + \alpha_2 x_k + \alpha_3 x_i, \end{aligned}$$

which again implies that the lattice point $x_j + 3(u - c)$ belongs to Δ^2 , a contradiction.

Finally, if there were a point from $\mathbb{Z}_3^N$, say w , in $\text{conv}(c, y_k^i, y_j^i)$, then—proceeding similarly as above—we would show that the lattice point $x_i + 3(c - w)$ would belong to Δ^2 giving again a contradiction. This completes the proof of Lemma 7. $\square$

Suppose that we have a triangulation of a lattice polyhedron P in $\mathbb{R}^4$ into fundamental lattice simplexes. On the boundary of P there can be zero-, one-, two- and three-dimensional lattice simplexes. The points from $\mathbb{Z}^4$ can appear only at vertices of the simplexes. Points from $\mathbb{Z}_2^4$ can lie in the relative interiors on one- and three-dimensional simplexes. Reeve's fundamental lattice tetrahedra in $\mathbb{R}^3$ having vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, r)$, $r \geq 1$, show that—in contrast to one-, and two-dimensional simplexes—the three-dimensional fundamental lattice simplexes can contain different number of points from $\mathbb{Z}_2^4$ in its relative interiors. For an n -dimensional fundamental lattice simplex Δ^n denote by $|\Delta_k^n|$ the number of lattice points from $\mathbb{Z}_k^n$ in its relative interior.

Lemma 8. *If Δ^3 is a fundamental lattice tetrahedron, then $|\Delta_3^3| = 4|\Delta_2^3|$.*

Proof. Let Δ^3 be a three-dimensional fundamental lattice tetrahedron. First we assume that Δ^3 lies in $\mathbb{R}^3$. To prove Lemma 8 it is enough to show that $I_3(\Delta^3) = 4I_2(\Delta^3)$. To this end we use Reeve's formula (3) twice: once when $n = 2$ and the other time when $n = 3$. Notice that from Lemmas 2 and 7 it follows that $B_2(\Delta^3) = 10$ and $B_3(\Delta^3) = 20$, respectively. Of course $B_1(\Delta^3) = 4$, $I_1(\Delta^3) = 0$, and $2\chi(\Delta^3) - \chi(\partial\Delta^3) = 0$. Thus we have

$$12V(\Delta^3) = 10 - 8 + 2I_2(\Delta^3) \quad \text{and} \quad 48V(\Delta^3) = 20 - 12 + 2I_3(\Delta^3).$$

Comparison of the two formulae easily gets $I_3(\Delta^3) = 4I_2(\Delta^3)$ and establishes the case.

In the general case the proof is almost identical with the change that instead of Reeve's formula (3) we use its simple modification applicable to lattice polyhedra with vertices in the sublattice $\mathbb{L}$ generated by Δ^3 , see p. 779 of [1] for details. We remark that both Lemmas 2 and 7 remain true for simplexes with vertices at $\mathbb{L}$. The proof is complete. $\square$

Lemma 9. *The number K of fundamental lattice segments on the boundary of a lattice polyhedron P in $\mathbb{R}^4$ is given by*

$$K = B_2 - B_1 - \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|.$$

Proof. We count all points from $\mathbb{Z}_2^4$ in the relative interiors of different simplexes lying on the boundary of a lattice polyhedron P in $\mathbb{R}^4$. We have exactly B_1 lattice points at vertices of P . By Lemma 1 there is exactly one point from $\mathbb{Z}_2^4$ in the relative interior of each fundamental lattice segment; so together we have K such points. Lemma 2 implies that points from $\mathbb{Z}_2^4$ do not lie in the relative interiors of fundamental lattice triangles, but they can lie in the relative interiors of three-dimensional fundamental lattice simplexes, and each simplex Δ^3 can contain a different number of $|\Delta_2^3|$ points. Thus we have

$$B_2 = B_1 + K + \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|.$$

Solving now the above for K gives

$$K = B_2 - B_1 - \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|$$

and ends the proof. $\square$

Lemma 10. *The number T of fundamental lattice triangles on the boundary of a lattice polyhedron P in $\mathbb{R}^4$ is given by*

$$T = B_3 - 2B_2 + B_1 - 2 \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|.$$

Proof. B_3 sums up points from $\mathbb{Z}_3^4$ lying at vertices of P , the relative interiors of fundamental lattice: segments, triangles, and three-dimensional simplexes. In the relative interior of each fundamental lattice segment there are, by Lemma 6, exactly two such points. By Lemma 7 there is exactly one such point in the relative interior of any fundamental lattice triangle. Lemma 8 implies that every fundamental lattice three-dimensional simplex Δ^3 has $4|\Delta_2^3|$ points from $\mathbb{Z}_3^4$ in its relative interior. Therefore we have

$$B_3 = B_1 + 2K + T + \sum_{\Delta^3 \subset \partial P} 4|\Delta_2^3|.$$

Applying Lemma 9 we get

$$\begin{aligned} B_3 &= B_1 + 2 \left(B_2 - B_1 - \sum_{\Delta^3 \subset \partial P} |\Delta_2^3| \right) + T + 4 \sum_{\Delta^3 \subset \partial P} |\Delta_2^3| \\ &= 2B_2 - B_1 + T + 2 \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|. \end{aligned}$$

Solving the above for T gives

$$T = B_3 - 2B_2 + B_1 - 2 \sum_{\Delta^3 \subset \partial P} |\Delta_2^3|$$

and completes the proof. $\square$

Using Lemmas 9 and 10 one can check that the following fact is true.

Lemma 11. *The Euler characteristic of the boundary of a lattice polyhedron P in $\mathbb{R}^4$ is given by*

$$\chi(\partial P) = 3B_1 - 3B_2 + B_3 - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1).$$

Lemma 12. *If P is a proper lattice polyhedron in $\mathbb{R}^4$, then its volume is given by*

$$72V(P) = 2G_3(P) - 6G_2(P) + 6G_1(P) - 2\chi(P) - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1).$$

Proof. Use the case $N = 4$ of (4) and substitute to it the expression for $\chi(\partial P)$ from Lemma 11. In this way we get the desired formula. $\square$

Proof of Theorem 2. By Lemma 12 Theorem 2 holds for any proper lattice polyhedron P in $\mathbb{R}^4$. It is clear that it is enough to show that the validity of (6) for P implies its validity for $P^* = P \cup P_1$, where P_1 is either an isolated point, or a fundamental lattice segment, or a fundamental lattice triangle, or a two-sided fundamental lattice tetrahedron. This leads us to consider the four cases below. Using relations listed in either of the cases it can be checked that (6) remains true for P^* . Since calculations in all cases are similar we check them only in the last two cases.

- P_1 is an isolated lattice point. In this case we obviously have $G_1(P^*) = G_1(P) + 1$, $G_2(P^*) = G_2(P) + 1$, $G_3(P^*) = G_3(P) + 1$, $\chi(P^*) = \chi(P) + 1$, $\Delta^3(P^*) = \Delta^3(P)$, and $V(P^*) = V(P)$.
- P_1 is a fundamental lattice segment. Denote by a , $0 \leq a \leq 2$, the number of lattice points that are not common to P and P_1 . The following relations are clear: $G_1(P^*) = G_1(P) + a$, $G_2(P^*) = G_2(P) + a + 1$, $G_3(P^*) = G_3(P) + a + 2$, $\chi(P^*) = \chi(P) + a - 1$, $\Delta^3(P^*) = \Delta^3(P)$, and $V(P^*) = V(P)$.
- P_1 is a fundamental lattice triangle with a , $0 \leq a \leq 3$, and b , $0 \leq b \leq 3$, vertices and edges, respectively, that are not common to P and P_1 . In this case, as is easy to check, we have $G_1(P^*) = G_1(P) + a$, $G_2(P^*) = G_2(P) + a + b$, $G_3(P^*) = G_3(P) + a + 2b + 1$, $\chi(P^*) = \chi(P) + a - b + 1$, $\Delta^3(P^*) = \Delta^3(P)$, and $V(P^*) = V(P)$. Using the relations we get

$$\begin{aligned}
 & 2G_3(P^*) - 6G_2(P^*) + 6G_1(P^*) - 2\chi(P^*) - \sum_{\Delta^3 \subset \partial P^*} (|\Delta_2^3| + 1) \\
 &= 2(G_3(P) + a + 2b + 1) - 6(G_2(P) + a + b) + 6(G_1(P) + a) \\
 &\quad - 2(\chi(P) + a - b + 1) - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1) \\
 &= 2G_3(P) - 6G_2(P) + 6G_1(P) - 2\chi(P) - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1) \\
 &= 72V(P) \\
 &= 72V(P^*),
 \end{aligned}$$

which shows that (6) is valid for P^* .

- $P_1 = \delta^3$ is a two-sided fundamental lattice tetrahedron. Similarly as above, denote by a , $0 \leq a \leq 4$, b , $0 \leq b \leq 6$, and c , $0 \leq c \leq 4$, the number of vertices, edges, and faces, respectively, that are not common to P and P_1 . It is easy to check that in this case we have the following relations $G_1(P^*) = G_1(P) + a$, $G_2(P^*) = G_2(P) + a + b + |\delta_2^3|$, $G_3(P^*) = G_3(P) + a + 2b + c + 4|\delta_2^3|$, $\chi(P^*) = \chi(P) + a - b + c - 1$, $\Delta^3(P^*) = \Delta^3(P) \cup \delta^3$, and $V(P^*) = V(P)$. Now using the above relations we obtain

$$\begin{aligned}
 & 2G_3(P^*) - 6G_2(P^*) + 6G_1(P^*) - 2\chi(P^*) - \sum_{\Delta^3 \subset \partial P^*} (|\Delta_2^3| + 1) \\
 &= 2(G_3(P) + a + 2b + c + 4|\delta_2^3|) - 6(G_2(P) + a + b + |\delta_2^3|) \\
 &\quad + 6(G_1(P) + a) - 2(\chi(P) + a - b + c - 1)
 \end{aligned}$$

$$\begin{aligned}
 & - \left[\sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1) + 2(|\delta_2^3| + 1) \right] \\
 & = 2G_3(P) - 6G_2(P) + 6G_1(P) - 2\chi(P) - \sum_{\Delta^3 \subset \partial P} (|\Delta_2^3| + 1) \\
 & = 72V(P) \\
 & = 72V(P^*).
 \end{aligned}$$

This means that (6) is true for P^* and Theorem 2 is established. $\square$

3. Remarks

1. Let $|\Delta_0^1| = 0$. Using this convention one can see that (2), (5), and (6) can be written in the following uniform way:

$$\begin{aligned}
 (N-1)N! V(P) &= 2 \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} G_{N-k} + (-1)^{N-1} 2\chi(P) \\
 &\quad - \sum_{\Delta^{N-1} \subset \partial P} (|\Delta_{N-2}^{N-1}| + 1),
 \end{aligned}$$

where $N \leq 4$ and every two-sided Δ^{N-1} contributes twice to the sum.

2. The results of the note remain valid in the more general case; namely, when instead of $\mathbb{Z}^3$ and $\mathbb{Z}^4$ we take arbitrary lattices in $\mathbb{R}^3$ and $\mathbb{R}^4$.

3. The main and common task in the proofs of Theorems 1 and 2 is summing up points from $\mathbb{Z}_k^N$ in the relative interiors of different lattice simplexes. Keeping in mind Reeve’s fundamental lattice tetrahedra one can realize that the task of summing up lattice points in higher dimensions is getting more and more complicated. For example, for a lattice polyhedron P in $\mathbb{R}^5$ we have

$$B_2 = B_1 + K + \sum_{\Delta^3 \subset \partial P} |\Delta_2^3| + \sum_{\Delta^4 \subset \partial P} |\Delta_2^4|$$

and

$$B_3 = B_1 + 2K + T + \sum_{\Delta^3 \subset \partial P} |\Delta_3^3| + \sum_{\Delta^4 \subset \partial P} |\Delta_3^4|.$$

This indicates some of the difficulties appearing in a search for an arbitrary-dimensional generalization of this type.

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