Realizations of Regular Abstract Polyhedra of Types $\{3, 6\}$ and $\{6, 3\}$

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Abstract. This paper classifies and gives methods for computing the irreducible realizations of the abstract polyhedra corresponding to regular maps of type $\{3, 6\}$ and $\{6, 3\}$. A complete list of irreducible realizations is given for polyhedra of type $\{3, 6\}$.

1. Introduction

This paper addresses the problem of finding regular geometric structures that correspond to a certain class of abstract combinatorial polyhedra. For a concise history of the study of regular polyhedra, see Section 1 of [10]. Grünbaum [8], Danzer and Schulte [6], Coxeter and Moser [5], and McMullen [9], [10] have asked what combinatorial properties are common to all polyhedra and provided examples of “abstract” polyhedra. If we wish to visualize these objects, we need to realize them in some way. Coxeter and Moser’s *regular maps* in [5] are topological realizations of these abstract polyhedra. Coxeter’s *regular skew polyhedra* in [4] are polyhedra whose symmetry groups are flag transitive, but which may have nonplanar vertex figures or lie in four- or five-dimensional Euclidean spaces. If nonplanar faces are allowed, as Grünbaum suggests in [7], the list of possible skew polyhedra is, literally, endless. In [9] McMullen lays the foundations for a systematic study of geometric realizations of abstract polytopes.

Every regular abstract polyhedron $\mathcal{P}$ can be realized as one or more different regular skew polyhedra. It is simple to find high-dimensional or degenerate realizations of $\mathcal{P}$,

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but is more difficult to find interesting low-dimensional realizations. Low-dimensional realizations of $\mathcal{P}$ allow the use of visualization and other tools of low-dimensional geometry to study $\mathcal{P}$.

To this end, we start with high-dimensional realizations of the abstract polyhedra under consideration and decompose them into low dimensional realizations using techniques from representation theory. By Theorem 10 of [9], this provides a complete classification of the irreducible realizations of $\mathcal{P}$. Theorem 9 of [9] allows us to conclude that all possible realizations of the polyhedron can be built up from combinations of those irreducible realizations.

A list of the distinct irreducible realizations of $\{3, 6\}_{(b,0)}$ is given in Section 3; results for $\{3, 6\}_{(b,b)}$ are presented in the following section. There are infinitely many realizations of the abstract polyhedra $\{6, 3\}_{(b,0)}$ and $\{6, 3\}_{(b,b)}$. A listing of the distinct symmetry group actions on these realizations is given in Sections 5 and 6.

2. Definitions

An *abstract polyhedron* is a graded poset $\mathcal{P}$ with ranks $\{-1, 0, 1, 2, 3\}$ having the four properties listed below (see [11]). The rank zero elements of $\mathcal{P}$ are called *vertices* of $\mathcal{P}$; this set is denoted by $\mathcal{P}_0$. Elements of rank one are *edges*, denoted $\mathcal{P}_1$, and the rank two elements $\mathcal{P}_2$ are referred to as *faces* of $\mathcal{P}$. A *flag* of $\mathcal{P}$ is a maximal totally ordered subset of $\mathcal{P}$. We say two flags are *adjacent* when they differ by just one face.

1. $\mathcal{P}$ is a finite ranked poset with unique minimum and maximum elements whose ranks are -1 and 3 , respectively.
2. Each flag of $\mathcal{P}$ must contain five elements; in particular, each flag contains the -1 - and 3 -faces of $\mathcal{P}$.
3. The polyhedron $\mathcal{P}$ must be *strongly flag connected*; for every pair of flags φ and ψ there must be a chain of flags $\varphi = \varphi_0, \varphi_1, \dots, \varphi_k = \psi$ such that φ_i and φ_{i+1} are adjacent, with $\varphi \cap \psi \subset \varphi_i$ for each i .
4. For all i , if F is an i -face and G is an $i + 2$ -face of $\mathcal{P}$ with $F < G$, there are exactly two faces H such that $F < H < G$.

Abstract polyhedra are a special case of the more general *abstract polytopes* discussed in detail in [11], [10], and [9].

We may now define the notion of regularity for abstract polyhedra. Choose some base-flag Φ in $\mathcal{P}$. If there exist three automorphisms ρ_0, ρ_1 , and ρ_2 of $\mathcal{P}$ such that each ρ_i fixes all but the i -face of Φ , the automorphism group $\mathcal{A}(\mathcal{P})$ of $\mathcal{P}$ will be flag-transitive, and we say that $\mathcal{P}$ is a *regular abstract polyhedron* [11].

We are interested in the regular abstract polyhedra $\{3, 6\}_{(b,0)}$, $\{3, 6\}_{(b,b)}$, $\{6, 3\}_{(b,0)}$, and $\{6, 3\}_{(b,b)}$ described on p. 107 of [5]. Consider the tiling of the plane by triangles depicted in Fig. 1 and the translational symmetries labeled T_x and T_y . Taking the quotient of the plane by the symmetry group generated by $T_x^{b+c}T_y^c$ and $T_x^{-c}T_y^b$ yields a torus. The images of the vertices, edges, and faces of the original tiling on the torus describe the abstract polyhedron $\{3, 6\}_{(b,c)}$, which is regular exactly when $b = 0$ or $b = c$. The combinatorial duals of these polyhedra are also regular, denoted $\{6, 3\}_{(b,0)}$ and $\{6, 3\}_{(b,b)}$. More detailed descriptions of these objects are provided in the appropriate sections.

Fig. 1. Net for $\{3, 6\}_{(4,0)}$.

A *realization* of an abstract regular polyhedron is a collection of vertices V in some Euclidean space $\mathbb{R}^d$ which has the property that there is a surjection $\beta: \mathcal{P}_0 \mapsto V$ such that the action of each element of $\mathcal{A}(\mathcal{P})$ induces an isometry of V . (Note that since we assumed $\mathcal{P}$ is finite, the set V is also finite.) Since this isometry can be extended to the entire space (uniquely, if and only if $\text{Aff}(V) = \mathbb{R}^d$), this gives us a representation of the group $\mathcal{A}(\mathcal{P})$ on the vector space $\mathbb{R}^d$ [9, Theorem 6]. Note that the vertices of a realization need not be centered at the origin.

We say that the realization is *vertex faithful* or simply *faithful* if the map β is bijective. One example of a faithful realization of a polyhedron is the *simplex realization*. This is obtained by sending each vertex of $\mathcal{P}$ to one of the orthogonal unit basis vectors of $\mathbb{R}^n$, where $n = |\mathcal{P}_0|$ [9]. The corresponding representation of $\mathcal{A}(\mathcal{P})$ is called the *simplex representation*. The *trivial realization* is the nonfaithful realization in which all vertices of the realization coincide at a single point.

For convenience, we refer to appropriate segments joining elements of V as edges of the realization of $\mathcal{P}$, and consider the faces of the realization of $\mathcal{P}$ to be described by appropriate unions of edges of the realization. Note that these “faces” are not necessarily planar.

The edges of $\mathcal{P}$ are a special case of the more general notion of *diagonals* of $\mathcal{P}$. These are described by unordered pairs $\{v, w\}$ of vertices of $\mathcal{P}$. The *diagonal classes* $\{\Lambda_1, \dots, \Lambda_r\}$ of $\mathcal{P}$ are the equivalence classes of the diagonals under the action of $\mathcal{A}(\mathcal{P})$. The *diagonal vector* $\{\delta_1, \dots, \delta_r\}$ of a realization of $\mathcal{P}$ is given by squares of distances between vertices in diagonals of each class; $\delta_i = |\beta(v) - \beta(w)|^2$, where $\{v, w\} \in \Lambda_i$ [9]. Two nontrivial realizations $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ are said to be *equivalent* if their diagonal vectors are the same up to a scalar multiple; i.e., if they are equivalent up to scale.¹

Given two realizations $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ of a polyhedron $\mathcal{P}$ with vertex sets $V = \beta_P(\mathcal{P}_0)$ and $W = \beta_Q(\mathcal{P}_0)$, we obtain a third realization $P \# Q \subset \mathbb{R}^p \times \mathbb{R}^q$ called the *blend* $P \# Q$ by defining $\beta_{P \# Q}(p) = (\beta_P(p), \beta_Q(p))$. A realization is said to be *irreducible* or *pure* if it is not the blend of two nontrivial realizations. A realization of $\mathcal{P}$ is pure exactly when the representation of $\mathcal{A}(\mathcal{P})$ on $\text{Aff}(V)$ is irreducible [9, p. 47].

¹ In [9] McMullen considers any pair of realizations with different diagonal vectors inequivalent. Our definition of equivalence is slightly weaker.

Define the *Wythoff space* W_G of a representation G of $\mathcal{A}(\mathcal{P})$ to be the subspace of points fixed by the action of both ρ_1 and ρ_2 [9]. The dimension of the Wythoff space of an irreducible representation equals the multiplicity of that representation in the simplex representation.

From any representation G of $\mathcal{A}(\mathcal{P})$, we can obtain a realization of $\mathcal{P}$ by applying *Wythoff's construction*. Given a point $w \in W_G$, choose any β preimage v_0 of w , and define a set V of vertices and a map $\beta: \mathcal{P}_0 \rightarrow V$ by $\beta(v_0) = w, \beta(g(v_0)) = g(w)$. Given two points $p, q \in W_G$ this can yield two inequivalent realizations P and Q of $\mathcal{P}$ only when $\dim(W_G) > 1$. The linear combination $\lambda P + \mu Q$ is defined to be the realization determined by applying Wythoff's construction to $\lambda p + \mu q$ [9].

In the case that all the irreducible components G of the simplex representation of $\mathcal{P}$ have $w_G = 1$, we can enumerate the distinct irreducible realizations of the simplex realization of $\mathcal{P}$. By Theorems 9 and 10 of [9], we know we can reconstruct any realization of $\mathcal{P}$ by scaling and taking blends of these irreducible component realizations.

If $\overline{w}$ is the dimension of the Wythoff space of the simplex realization, then

$$\sum_G w_G^2 = \overline{w}, \quad \text{and} \quad \sum_G w_G d_G = |\mathcal{P}_0|,$$

where the sum is over the distinct irreducible orthogonal representations G of $\mathcal{A}(\mathcal{P})$ and d_G is the dimension of representation G [9, Theorem 17].²

In the case of polyhedra of type $\{3, 6\}$, we have $w_G = 1$ for each G . Using the fact that w_G equals the multiplicity of an irreducible representation G in the simplex realization, we conclude that these representations are irreducible and inequivalent. Our observations are confirmed by the fact that $\sum_G w_G d_G = |\mathcal{P}_0|$. Here, Wythoff's construction generates a complete list of irreducible realizations up to scalar multiples.

In the case of polyhedra of type $\{6, 3\}$, $w_G = 2$ does occur. We generate a list of component representations and determine that the list is complete and its elements are irreducible by comparison with the case $\{3, 6\}$ and by confirming $\sum_G w_G d_G = |\mathcal{P}_0|$. Because Wythoff's construction can produce inequivalent realizations from the same representation, it is impossible to provide a complete list of irreducible realizations.

3. Irreducible Realizations of $\{3, 6\}_{(b,0)}$

Figure 1 shows the *net* of the regular abstract polyhedron $\{3, 6\}_{(4,0)}$ and suggests a labeling of its vertices. If we identify opposite sides of the rhombus shown in the picture, the resulting figure is a torus subdivided into 32 triangular faces. The edges surrounding those faces form 48 edges, which meet at the 16 vertices. The -1 -face of $\{3, 6\}_{(4,0)}$ corresponds to the empty set, the 3-face corresponds to the entire torus, and the ordering on the poset is given by inclusion.

² McMullen does not consider the trivial realization, and so omits one one-dimensional realization from these sums.

The rhombus in Fig. 1 has four equilateral triangles along each edge. Identifying opposite sides of a rhombus with b equilateral triangles along an edge gives us the regular abstract polyhedron $\{3, 6\}_{(b,0)}$, which has $2b^2$ faces, $3b^2$ edges, and b^2 vertices [5].

Our goal is to describe the irreducible realizations of the polyhedra in this family. To this end, we decompose the simplex representation of $\{3, 6\}_{(b,0)}$ in $\mathbb{C}^{b^2}$ into its irreducible parts. We regard the vertices $v_{j,k}$, $0 \leq j, k < b$, as a basis for $\mathbb{C}^{b^2}$.

It is relatively simple to find a basis $\{f_{l,m}\} \subset \mathbb{C}^{b^2}$, each vector of which is an eigenvector of the elements T_x and T_y of $\mathcal{A}(\{3, 6\}_{(b,0)})$ indicated in Fig. 1, where we identify T_x and T_y with the matrices describing their action on $\mathbb{C}^{b^2}$. The orbits of these vectors under the action of $\mathcal{A}(\{3, 6\}_{(b,0)})$ are bases of the irreducible realization spaces of $\{3, 6\}_{(b,0)}$. A change of coordinates allows us to consider these as realizations over $\mathbb{R}$, and to apply the theorems mentioned in the previous section.

The abstract polyhedron $\{3, 6\}_{(b,0)}$ can also be constructed by starting with the tiling of the plane by equilateral triangles (denoted $\{3, 6\}$) and taking the quotient by the translations T_x^b and T_y^b [5]. In particular, we can say that $T_x^b = T_y^b = I$ when these elements of the automorphism group act on the simplex realization. Hence, the eigenvalues of T_x and T_y must be the b th roots of unity. This also suggests that simultaneous eigenvectors of T_x and T_y may be found using the b th roots of unity as coefficients in the $v_{j,k}$ basis.

If we use the labeling of the vertices of $\{3, 6\}_{(b,0)}$ indicated in Fig. 1, a simple calculation³ shows that the vectors $\{f_{l,m}\}_{0 \leq l, m < b}$ defined by

$$f_{l,m} = \sum_{0 \leq j, k < b} e^{(lj+mk)(2\pi i/b)} v_{j,k}$$

are simultaneous eigenvectors of T_x and T_y with eigenvalues $e^{-l(2\pi i/b)}$ and $e^{-m(2\pi i/b)}$, respectively.

We can use this information to compute the eigenvalues of the adjacency matrix A_1 of the edge graph of $\{3, 6\}_{(b,0)}$. There are six edges from each vertex, represented by T_x , $T_x T_y$, T_y , T_x^{-1} , $(T_x T_y)^{-1}$, and T_y^{-1} . Since we have a complete set of simultaneous eigenvectors for T_x and T_y , we know all of the eigenvectors and eigenvalues of A_1 . The same technique may be used to compute the eigenvalues of the matrices A_2 , A_3 , etc., defined by Bannai and Ito [1].

Theorem 1. *The adjacency matrix A_1 of the one-skeleton of the abstract polyhedron $\{3, 6\}_{(b,0)}$ is given by*

$$A_1 = T_x + T_x T_y + T_y + T_x^{-1} + (T_x T_y)^{-1} + T_y^{-1}.$$

The multiset of eigenvalues of this matrix is

$$\begin{aligned} & \left\{ 2 \cos \left(\frac{2\pi l}{b} \right) + 2 \cos \left(\frac{2\pi m}{b} \right) + 2 \cos \left(\frac{2\pi(l+m)}{b} \right) \mid 0 \leq l < b, 0 \leq m < b \right\} \\ & = \left\{ 8 \cos \left(\frac{\pi l}{b} \right) \cos \left(\frac{\pi m}{b} \right) \cos \left(\frac{\pi(l+m)}{b} \right) - 2 \mid 0 \leq l < b, 0 \leq m < b \right\}. \end{aligned}$$

³ $T_x(f_{l,m}) = \sum_{0 \leq j, k < b} e^{(lj+mk)(2\pi i/b)} v_{j+1,k} = e^{-(2\pi i/b)l} \sum_{j',k} e^{(lj'+mk)(2\pi i/b)} v_{j',k}.$

Fig. 2. Generators of the automorphism group of $\{3, 6\}_{(4,0)}$.

The transformations T_x and T_y originate in translational symmetries of $\{3, 6\}$. Figure 2 indicates a convenient choice of the generating “reflections” ρ_0 , ρ_1 , and ρ_2 of $\mathcal{A}(\{3, 6\}_{(4,0)})$. These automorphisms act as follows:

$$\rho_0: v_{j,k} \mapsto v_{1-k,1-j}, \quad \rho_1: v_{j,k} \mapsto v_{j,j-k}, \quad \text{and} \quad \rho_2: v_{j,k} \mapsto v_{k,j}.$$

Note that $T_x = \rho_1 \rho_0 (\rho_1 \rho_2)^2$ and $T_y = \rho_2 \rho_1 \rho_0 \rho_1 \rho_2 \rho_1$.

We know that $f_{l,m}$ is an eigenvector of T_x . Using the above description of the action of the ρ_i , we see that

$$\rho_0(f_{l,m}) = e^{(l+m)(2\pi i/b)} f_{-m,-l}, \quad \rho_1(f_{l,m}) = f_{l+m,-m}, \quad \text{and} \quad \rho_2(f_{l,m}) = f_{m,l}. \quad (1)$$

A subspace of $\mathbb{C}^{b^2}$ is invariant under the action of $\mathcal{A}(\{3, 6\}_{(b,0)})$ exactly when it is invariant under the action of the ρ_i . Since $\rho_0 = \rho_1 T_x (\rho_2 \rho_1)^2$ and $T_y = \rho_2^{-1} T_x \rho_2$, it is equivalent to require that the subspace be invariant under the action of T_x , ρ_1 , and ρ_2 . In particular, the spaces spanned by the orbits of the vectors $f_{l,m}$ under the action of ρ_1 and ρ_2 will be fixed by $\mathcal{A}(\{3, 6\}_{(b,0)})$. For arbitrary l and m , this orbit is

$$\left\{ \begin{array}{cccc} f_{l,m}, & f_{-l,-m}, & f_{l+m,-m}, & f_{-l-m,m}, \\ f_{-l,l+m}, & f_{l,-l-m}, & f_{m,-l-m}, & f_{-m,l+m}, \\ f_{-l-m,l}, & f_{l+m,-l}, & f_{-m,-l}, & f_{m,l}. \end{array} \right\} \quad (2)$$

For each choice of l and m these vectors span a space $V_{l,m}$ on which the representation of $\mathcal{A}(\{3, 6\}_{(b,0)})$ is irreducible; we prove this by the method outlined in Section 2. Note that for some values of l and m , $\dim(V_{l,m}) < 12$.

The Wythoff subspace $W_{l,m}$ of $V_{l,m}$ is spanned by the sum of the vectors of (2) and has dimension $w_{l,m} = 1$. This fact continues to hold once we have converted to representations over $\mathbb{R}$, so we need only show that the number of different $V_{l,m}$ equals $\bar{w}$ to see that $\sum_G w_G^2 = \bar{w}$. However, because all the $W_{l,m}$ are one-dimensional, $\bar{w}$ is just the number of diagonal classes of $\{3, 6\}_{(b,0)}$ [9]. We now describe a bijection between diagonal classes and choices of l and m which yield distinct representation spaces $V_{l,m}$.

To determine the different diagonal classes, fix $v_{0,0}$ and study its relationship to the other $v_{j,k}$. If $(v_{0,0}, v_{j,k})$ is in diagonal class Λ and $v_{j',k'}$ is sent to $v_{j,k}$ by some action of ρ_1 and ρ_2 , then $(v_{0,0}, v_{j',k'}) \in \Lambda$. We can classify the different diagonal classes by the vertices in the smallest region that completely covers $\{3, 6\}_{(b,0)}$ when acted on by ρ_1 and ρ_2 . Such a region is shown in Fig. 3.

Fig. 3. Fundamental region for $\{3, 6\}_{(6,0)}$.

A bijection between the spaces $V_{l,m}$ and the vertices $v_{j,k}$ shown in this fundamental region is given by $\Omega: f_{l,m} \mapsto v_{l+m,l}$, where the subscripts are interpreted modulo b . Although the roles of ρ_1 and ρ_2 are reversed ($\Omega(\rho_2(f_{l,m})) = \rho_1(\Omega(f_{l,m}))$ and $\Omega(\rho_1(f_{l,m})) = \rho_2(\Omega(f_{l,m}))$), the orbit of $f_{l,m}$ under the action of ρ_1 and ρ_2 will be of the same order as that of $v_{l+m,l} = \Omega(f_{l,m})$. Since a representative vertex from each orbit appears in Fig. 3, the number of distinct $V_{l,m}$ is the same as the number of vertices appearing in Fig. 3. The dimension of $V_{l,m}$ is the order of the orbit of $v_{l+m,l}$; Table 1 summarizes this information.

To this point we have been working with representations of $\mathcal{A}(\{3, 6\}_{(b,0)})$ over $\mathbb{C}$. In order to apply the formula $\sum w_G^2 = \overline{w}$, we must consider representations over $\mathbb{R}$. The chief obstacle to this is the fact that ρ_0 acts on the $f_{l,m}$ with complex coefficients. We circumvent this by breaking the action down into its real and imaginary parts. Notice that the coefficient of $\rho_0(f_{l,m})$ is the complex conjugate of $\rho_0(f_{-l,-m})$; in particular, $\rho_0(f_{l,m} + f_{-l,-m}) = a(f_{m,l} + f_{-m,-l}) - b(if_{m,l} - if_{-m,-l})$ for some $a, b \in \mathbb{R}$. If we change the basis to

$$\left\{ \begin{array}{ll} f_{l,m} + f_{-l,-m}, & if_{l,m} - if_{-l,-m}, \\ f_{l+m,-m} + f_{-l-m,m}, & if_{l+m,-m} - if_{-l-m,m}, \\ f_{-l,l+m} + f_{l,-l-m}, & if_{-l,l+m} - if_{l,-l-m}, \\ f_{m,-l-m} + f_{-m,l+m}, & if_{m,-l-m} - if_{-m,l+m}, \\ f_{-l-m,l} + f_{l+m,-l}, & if_{-l-m,l} - if_{l+m,-l}, \\ f_{m,l} + f_{-m,-l}, & if_{m,l} - if_{-m,-l}, \end{array} \right\} \quad (3)$$

Table 1. Number and dimension of realizations of $\{3, 6\}_{(b,0)}$.

$b \bmod 6$	Dimension over $\mathbb{R}$					Total
	1	2	3	6	12	
0	1	1	1	$b - 3$	$(b^2 - 6b + 12)/12$	$(b^2 + 6b + 12)/12$
1	1	0	0	$b - 1$	$(b^2 - 6b + 5)/12$	$(b^2 + 6b + 5)/12$
2	1	0	1	$b - 2$	$(b^2 - 6b + 8)/12$	$(b^2 + 6b + 8)/12$
3	1	1	0	$b - 2$	$(b^2 - 6b + 9)/12$	$(b^2 + 6b + 9)/12$
4	1	0	1	$b - 2$	$(b^2 - 6b + 8)/12$	$(b^2 + 6b + 8)/12$
5	1	0	0	$b - 1$	$(b^2 - 6b + 5)/12$	$(b^2 + 6b + 5)/12$

the ρ_i act on the basis vectors with real coefficients.⁴ Each of our original complex representations breaks down into equivalent “real” and “imaginary” representations of $\mathcal{A}(\{3, 6\}_{(b,0)})$ over $\mathbb{R}$. Because the pairs are equivalent, we know that those corresponding to inequivalent $V_{l,m}$ are also inequivalent. The dimension of the Wythoff space of each of these real representations is still one. Hence this representation corresponds to a unique realization of $\{3, 6\}_{(b,0)}$ which we call $R_{l,m}$.

We may now use the results in Table 1 to confirm that $\sum_G w_G^2 = \bar{w}$ and $\sum_G w_G d_G = b^2$, and so conclude that the $R_{l,m}$ enumerated by the vertices shown in Fig. 3 are all of the irreducible orthogonal realizations of $\{3, 6\}_{(b,0)}$ up to scaling. Furthermore, we can express the action of T_x and T_y on the basis described above, which allows us to calculate locations of the vertices of the $R_{l,m}$.

Given a block diagonal matrix M with matrices $\{M_1, M_2, \dots, M_k\}$ on its diagonal, we write $M = (M_1 | M_2 | \dots | M_k)$. If we let $R(\theta)$ be the rotation matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, we can concisely describe the irreducible realizations of $\{3, 6\}_{(b,0)}$ as follows:

Theorem 2. *The vertices of a generic irreducible realization $R_{l,m}$ of $\{3, 6\}_{(b,0)}$ are given by the orbit of the point $p = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$ under the action of the matrices:*

$$T_x = (R(\varphi) | R(\varphi + \psi) | R(-\varphi) | R(\psi) | R(-\varphi - \psi) | R(\psi))$$

and

$$T_y = (R(\psi) | R(-\psi) | R(\varphi + \psi) | R(-\varphi - \psi) | R(\varphi) | R(\varphi)),$$

where $\varphi = -2\pi l/b$ and $\psi = -2\pi m/b$. Here $\beta(v_{j,k}) = T_x^j T_y^k(p)$.

The matrices of the ρ_i are easily calculated from (1) and (3). In the generic case, each ρ_i acts by fixing some six-dimensional subspace and reflecting its complement through the origin.

Contemplating the descriptions of T_x and T_y , we see that the vertices of $R_{l,m}$ lie on the surface of a 6-torus in $\mathbb{R}^{12}$. Since $\{3, 6\}_{(b,0)}$ is topologically a 2-torus, they lie on an embedding of T^2 in T^6 . Although this 12-dimensional figure is difficult to visualize, we can describe the lower-dimensional realizations of $\{3, 6\}_{(b,0)}$.

The realization $R_{0,0}$ is the trivial one. When $l = m = b/3$, $R_{l,m}$ is two-dimensional with vertices lying at the corners of a triangle. When $l = m = b/2$, $R_{l,m}$ is three-dimensional, $T_x = (-1 | 1 | -1)$, $T_y = (-1 | -1 | 1)$ and the vertices of the realization lie at the corners of a regular tetrahedron centered at the origin with one vertex on the line $x = y = z$.

The realizations $R_{l,0}$ ($l \neq 0, b/2$) are six-dimensional, with

$$T_x = (R(\varphi) | R(-\varphi) | I_2) \quad \text{and} \quad T_y = (I_2 | R(\varphi) | R(-\varphi)).$$

They can be visualized as follows. Consider the action of the parametric equation $P_1: (x, y, z) \rightarrow (\sin(x), \cos(x), \sin(y), \cos(y), \sin(z), \cos(z))$ on the cube of edge length 2π shown in Fig. 4; opposite faces of the cube shown there are identified

⁴ Note that for some values of l and m , these vectors are not linearly independent and the representation has dimension less than 12.

Fig. 4. The faces of $\{3, 6\}_{(4,0)}$ embedded in a cube.

to form a 3-torus. Opposite edges of the two shaded triangles are identified to form a 2-torus embedded in that 3-torus. The shaded triangles can be subdivided into b^2 smaller triangles; in the example shown, $b = 4$. The images of the vertices, edges and faces of this subdivision under the action of P_1 correspond to those of the realization $R_{1,0}$ of $\{3, 6\}_{b,0}$ (see [2]). For l greater than one, P_1 is replaced by $P_l: (x, y, z) \rightarrow (\sin(lx), \cos(lx), \sin(ly), \cos(ly), \sin(lz), \cos(lz))$.

When realization $R_{l,l}$ is six-dimensional, $T_x = (R(\varphi)|R(2\varphi)|R(-\varphi))$ and $T_y = (R(\varphi)|R(-\varphi)|R(2\varphi))$. They can be visualized as above, by retriangulating the shaded planes of Fig. 4 so that the edges of the triangular faces run from vertex to face center of the cube, rather than from vertex to vertex.

Note that the realization $R_{l,0}$ is faithful exactly when T_x and T_y have order b , which occurs whenever l and b are relatively prime. Similar arguments lead to a characterization of all faithful realizations of $\{3, 6\}_{(b,0)}$.

4. Abstract Polyhedra of Type $\{3, 6\}_{(b,b)}$

Identifying opposite sides of the rhombus shown in Fig. 5 describes the abstract regular polyhedron $\{3, 6\}_{(2,2)}$, which has $3b^2$ vertices and $6b^2$ faces. In general, $\{3, 6\}_{(b,b)}$ is

Fig. 5. A net for $\{3, 6\}_{(2,2)}$.

Table 2. Number and dimension of realizations of $\{3, 6\}_{(b,b)}$.

b	Dimension over $\mathbb{R}$					Total
	1	2	3	6	12	
Even	1	1	1	$2b - 3$	$(3b^2 - 12b + 12)/12$	$(3b^2 + 12b + 12)/12$
Odd	1	1	0	$2b - 2$	$(3b^2 - 12b + 9)/12$	$(3b^2 + 12b + 9)/12$

obtained by identifying points of $\{3, 6\}$ that are equivalent under the action of the symmetry group $\langle T_x^{3b}, T_y^{3b}, (T_x^2 T_y)^b \rangle$, where T_x and T_y transform $\{3, 6\}$ as was described in Section 2 [5].

There is a natural map from $\{3, 6\}_{(b,b)}$ to $\{3, 6\}_{(b,0)}$ given by making the identification $T_x^b = T_y^b = 1$. In other words, $\{3, 6\}_{(b,b)}$ *collapses* onto $\{3, 6\}_{(b,0)}$. So, every realization of $\{3, 6\}_{(b,0)}$ gives rise to a realization of $\{3, 6\}_{(b,b)}$ [9].

The chief difference between these two families of abstract polyhedra is the existence of a group element $T_x^2 T_y$ with order lower than that of T_x and T_y . If we construct a set of simultaneous eigenvectors of T_x and $T_x^2 T_y$ analogous to the $\{f_{l,m}\}$ of the previous section, repeating the calculations of that section yields the enumeration of irreducible realizations presented in Table 2.

The vertex sets of these realizations can be derived from the vertex sets of realizations of $\{3, 6\}_{(3b,0)}$, which collapses onto $\{3, 6\}_{(b,b)}$. For example, the six-dimensional realizations of $\{3, 6\}_{(b,b)}$ correspond to the realizations $R_{l,0}$ and $R_{l,l}$ described in Section 3.

5. Abstract Polyhedra of Type $\{6, 3\}_{(b,0)}$

The combinatorial dual of the abstract polyhedron $\{3, 6\}_{(b,0)}$ is $\{6, 3\}_{(b,0)}$ [5]; identifying opposite sides of the dotted rhombus shown in Fig. 6 yields the polyhedron $\{6, 3\}_{(3,0)}$. Since $\{6, 3\}_{(b,0)}$ has $2b^2$ vertices, the simplex representation of its automorphism group is over the space $\mathbb{C}^{2b^2}$.

Fig. 6. A net for $\{6, 3\}_{(3,0)}$.

Once again the polyhedron has “translational” symmetries T_x and T_y ; in terms of the generating reflections, $T_x = \rho_0 \rho_1 \rho_2 \rho_1 \rho_0 \rho_1$ and $T_y = \rho_1 \rho_0 \rho_1 \rho_0 \rho_1 \rho_2$. It is evident from Fig. 6 that the edge graph of $\{6, 3\}_{(b,0)}$ is bipartite. The symmetries ρ_2 , T_x , and T_y permute the vertices labeled $\{v_{j,k}\}_{0 \leq j,k < b}$ in exactly the same way that ρ_1 , T_x , and T_y permuted the vertices of $\{3, 6\}_{(b,0)}$ in Section 3. The vertices $\{w_{j,k}\}_{0 \leq j,k < b}$ of the other half of the graph form a separate orbit under the action of T_x and T_y .

The simultaneous eigenvectors of the action of T_x and T_y on $\mathbb{C}^{2b^2}$ are $\{f_{l,m}, g_{l,m}\}$ where $0 \leq l, m < b$, and $f_{l,m}$ and $g_{l,m}$ are defined as follows:

$$f_{l,m} = \sum_{0 \leq j,k < b} e^{(lj+mk)(2\pi i/b)} v_{j,k}, \quad g_{l,m} = \sum_{0 \leq j,k < b} e^{(lj+mk)(2\pi i/b)} w_{j,k}.$$

It is not difficult to check that

$$\begin{aligned} T_x(f_{l,m}) &= e^{l(2\pi i/b)} f_{l,m} & T_y(f_{l,m}) &= e^{m(2\pi i/b)} f_{l,m}, \\ T_x(g_{l,m}) &= e^{l(2\pi i/b)} g_{l,m}, & \text{and } T_y(g_{l,m}) &= e^{m(2\pi i/b)} g_{l,m}. \end{aligned}$$

Since T_x and T_y no longer send vertices to their neighbors, we cannot directly compute the eigenvalues of A_1 . However, they do map a vertex v to a vertex v' whose graph distance satisfies $d(v, v') = 2$. By considering all vertices at distance two from a given vertex, we see that the graph adjacency matrix A_2 for distance two is

$$A_2 = T_x + T_x T_y + T_y + T_x^{-1} + (T_x T_y)^{-1} + T_y^{-1}.$$

We know all of the eigenvalues and eigenvectors of A_2 , the computation is identical to that of Theorem 1. A simple graph theoretical calculation verifies that $A_1^2 = 6I + A_2$, leading us to conclude:

Theorem 3. *The multiset of eigenvalues of the adjacency matrix of $\{6, 3\}_{(b,0)}$ is*

$$\left\{ \pm \sqrt{8 \cos\left(\frac{\pi l}{b}\right) \cos\left(\frac{\pi m}{b}\right) \cos\left(\frac{\pi(l+m)}{b}\right) + 4} \mid 0 \leq l, m < b \right\}.$$

The generators ρ_i of the automorphism group of $\{6, 3\}_{(b,0)}$ are indicated in Fig. 7. They permute the eigenvectors $f_{l,m}$ and $g_{l,m}$ as follows:

$$\begin{aligned} \rho_0(f_{l,m}) &= g_{-l-m,m}, & \rho_0(g_{l,m}) &= f_{-l-m,m}, \\ \rho_1(f_{l,m}) &= f_{-l,l+m}, & \rho_1(g_{l,m}) &= e^{-(2\pi i l/b)} g_{-l,l+m}, \\ \rho_2(f_{l,m}) &= f_{l+m,-m}, & \rho_2(g_{l,m}) &= g_{l+m,-m}. \end{aligned}$$

Proceeding as in Section 3, we see that the following vectors form a basis for a representation $V_{l,m}$ of $\mathcal{A}(\{6, 3\}_{(b,0)})$:

$$\begin{Bmatrix} f_{l,m}, & g_{-l,-m}, & f_{l+m,-m}, & g_{-l-m,m}, \\ f_{-l,l+m}, & g_{l,-l-m}, & f_{m,-l-m}, & g_{-m,l+m}, \\ f_{-l-m,l}, & g_{l+m,-l}, & f_{-m,-l}, & g_{m,l}. \end{Bmatrix}$$

Fig. 7. Generators of $\mathcal{A}(\{6, 3\}_{(3,0)})$.

In general, the Wythoff spaces of these representations are two-dimensional. Note that if we replace the $g_{p,q}$ with $f_{p,q}$ we get exactly the basis we had for the representation spaces $V_{l,m}$ of $\mathcal{A}(\{3, 6\}_{(b,0)})$. This will help us to enumerate these representations.

It is no longer true that the different representations $V_{l,m}$ are inequivalent; in some cases the spaces $V_{l,m}$ and $V_{-l,-m}$ are distinct vector spaces that correspond to equivalent representations. The equivalence is given by defining $T: f_{l,m} \mapsto g_{l,m}; g_{-l,-m} \mapsto f_{-l,-m}$ in the following commutative diagram:

$$\begin{array}{ccc}
 V_{l,m} & \xrightarrow{T} & V_{l',m'} \\
 g \downarrow & & g \downarrow \\
 V_{l,m} & \xrightarrow{T} & V_{l',m'}
 \end{array} \tag{4}$$

In general, suppose $V_{l,m}$ and $V_{l',m'}$ are distinct but equivalent representation spaces of $\mathcal{A}(\{6, 3\}_{(b,0)})$. Then there exists a nonsingular linear transformation $T: V_{l,m} \rightarrow V_{l',m'}$ such that (4) commutes for all $g \in \mathcal{A}(\{6, 3\}_{(b,0)})$. In particular, the diagram holds when g is replaced by T_x or T_y . So,

$$T_x T(f_{l,m}) = T T_x(f_{l,m}) = e^{(2\pi i/b)l} T(f_{l,m});$$

similarly, $T_y T(f_{l,m}) = e^{(2\pi i/b)m} T(f_{l,m})$. Hence, $V_{l',m'}$ contains a vector $T(f_{l,m})$ which is a simultaneous eigenvector of T_x and T_y on $V_{l',m'}$ with eigenvalues $e^{(2\pi i/b)l}$ and $e^{(2\pi i/b)m}$, respectively. However, this implies that $T(f_{l,m})$ must be a multiple of either $f_{l,m}$ or $g_{l,m}$. Either $V_{l,m} = V_{l',m'}$ or $T(f_{l,m}) = g_{l,m}$, implying $V_{l',m'} = V_{-l,-m}$. In other words, when $V_{l,m} = V_{-l,-m}$ the associated representation occurs with multiplicity one in the simplex representation of $\mathcal{A}(\{3, 6\}_{(b,0)})$.

Note that if $V_{l,m} = V_{-l,-m}$, there is some k for which $V_{k,k} = V_{l,m}$. The representation spaces $V_{l,m}$ fall into the following three categories:

1. $V_{l,m} \neq V_{-l,-m}$ has Wythoff dimension two and multiplicity two in the simplex representation,
2. $V_{b/3,b/3}$ has Wythoff dimension one and multiplicity one, and
3. $V_{l,l}$ has Wythoff dimension two and multiplicity one.

When $V_{l,m} \neq V_{-l,-m}$, the basis

$$\left\{ \begin{array}{ll} f_{l,m} + e^{-2\pi i m/b} g_{-l,-m}, & if_{l,m} - ie^{-2\pi i m/b} g_{-l,-m}, \\ f_{l+m,-m} + e^{-2\pi i m/b} g_{-l-m,m}, & if_{l+m,-m} - ie^{-2\pi i m/b} g_{-l-m,m}, \\ f_{-l,l+m} + e^{2\pi i (l-m)/b} g_{l,-l-m}, & if_{-l,l+m} - ie^{2\pi i (l-m)/b} g_{l,-l-m}, \\ f_{m,-l-m} + e^{2\pi i (l-m)/b} g_{-m,l+m}, & if_{m,-l-m} - ie^{2\pi i (l-m)/b} g_{-m,l+m}, \\ f_{-l-m,l} + e^{2\pi i l/b} g_{l+m,-l}, & if_{-l-m,l} - ie^{2\pi i l/b} g_{l+m,-l}, \\ f_{-m,-l} + e^{2\pi i l/b} g_{m,l}, & if_{-m,-l} - ie^{2\pi i l/b} g_{m,l} \end{array} \right\}$$

provides a representation of $\mathcal{A}(\{6, 3\}_{(b,0)})$ in which the ρ_i act with coefficients in $\mathbb{R}$. When $l = 0, m = 0$, or $l + m = 0$, this representation has dimension six. Otherwise, its dimension is 12. In either case, the Wythoff space is two dimensional; its basis is obtained by summing the elements of each of the two orbits of the basis vectors under the action of $\langle \rho_1, \rho_2 \rangle$.

The space $V_{b/3,b/3} = V_{-b/3,-b/3}$ corresponds to an irreducible four-dimensional realization of $\mathcal{A}(\{6, 3\}_{(b,0)})$. Its Wythoff space is one-dimensional and is spanned by $f_{b/3,b/3} + f_{-b/3,-b/3}$. The ρ_i act with real coefficients on the elements of the basis

$$\left\{ \begin{array}{ll} f_{b/3,b/3} + f_{-b/3,-b/3}, & if_{b/3,b/3} - if_{-b/3,-b/3}, \\ g_{b/3,b/3} + g_{-b/3,-b/3}, & ig_{b/3,b/3} - ig_{-b/3,-b/3}. \end{array} \right\}$$

The remaining representations $V_{l,l}$ have Wythoff dimension two, but are not irreducible as representations of $\mathcal{A}(\{6, 3\}_{(b,0)})$. The representation decomposes: $V_{l,l} = W_{l,l} \oplus W'_{l,l}$, where $W_{l,l}$ and $W'_{l,l}$ are inequivalent irreducible representation spaces with Wythoff dimension one. Except for the special cases $V_{0,0}$ and $V_{b/2,b/2}$, the ρ_i act on the following basis of $W_{l,l}$ with coefficients in $\mathbb{R}$, where $\zeta = e^{-2\pi i l/b}$:

$$\left\{ \begin{array}{ll} f_{l,l} + \zeta g_{-l,-l} + f_{-l,-l} + \bar{\zeta} g_{l,l}, & if_{l,l} - i\zeta g_{-l,-l} - if_{-l,-l} + i\bar{\zeta} g_{l,l}, \\ f_{2l,-l} + \zeta g_{-2l,l} + f_{-2l,l} + \zeta g_{2l,-l}, & if_{2l,-l} - i\zeta g_{-2l,l} - if_{-2l,l} + i\zeta g_{2l,-l}, \\ f_{-l,2l} + g_{l,-2l} + f_{l,-2l} + g_{-l,2l}, & if_{-l,2l} - ig_{l,-2l} - if_{l,-2l} + ig_{-l,2l}. \end{array} \right\}$$

The basis for $W'_{l,l}$ is similar. $V_{0,0}$ and $V_{b/2,b/2}$ are easily decomposed into pairs of one- and three-dimensional realizations over $\mathbb{R}$, respectively.

Recall the surjection from bases of the $V_{l,m}$ to bases of the irreducible representations of $\mathcal{A}(\{3, 6\}_{(b,0)})$ given by replacing the $g_{l,m}$ with $f_{l,m}$. We use this to enumerate the irreducible representations of $\mathcal{A}(\{6, 3\}_{(b,0)})$; our conclusions are presented in Table 3.

The two-dimensional realization of $\{3, 6\}_{(b,0)}$ corresponds to $W_{b/3,b/3}$. The one- and three-dimensional realizations and the six-dimensional realizations $R_{l,l}$ correspond to the inequivalent pairs of one-, three-, and six-dimensional irreducible representations of $\mathcal{A}(\{6, 3\}_{(b,0)})$ described in category 3 above. The remaining realizations of $\{3, 6\}_{(b,0)}$ correspond to equivalent pairs $V_{l,m} \cong V_{-l,-m}$ of six- or twelve-dimensional irreducible representations with Wythoff dimension two.

The real representations arising from the $V_{l,l}$ (categories 2 and 3) have one-dimensional Wythoff spaces and are clearly irreducible. The fact that $\sum w_G d_G = 2b^2$ confirms that those of category 1 are irreducible as well. Table 3 tallies the irreducible components of the simplex representation of $\mathcal{A}(\{6, 3\}_{(b,0)})$.

Table 3. Number and dimension of irreducible representations of $\mathcal{A}(\{6, 3\}_{(b,0)})$, counted without multiplicity.

$b \bmod 6$	Dimension over $\mathbb{R}$						Total
	$w_G = 1$				$w_G = 2$		
	1	3	4	6	6	12	
0	2	2	1	$b - 4$	$(b - 2)/2$	$(b^2 - 6b + 12)/12$	$(b^2 + 12b + 6)/12$
1	2	0	0	$b - 1$	$(b - 1)/2$	$(b^2 - 6b + 5)/12$	$(b^2 + 12b + 11)/12$
2	2	2	0	$b - 2$	$(b - 2)/2$	$(b^2 - 6b + 8)/12$	$(b^2 + 12b + 20)/12$
3	2	0	1	$b - 3$	$(b - 1)/2$	$(b^2 - 6b + 9)/12$	$(b^2 + 12b + 3)/12$
4	2	2	0	$b - 2$	$(b - 2)/2$	$(b^2 - 6b + 8)/12$	$(b^2 + 12b + 20)/12$
5	2	0	0	$b - 1$	$(b - 1)/2$	$(b^2 - 6b + 5)/12$	$(b^2 + 12b + 11)/12$

When the Wythoff space of a representation is two-dimensional, there are uncountably many different realizations associated with the same representation [9], so it is impossible to enumerate the irreducible realizations of $\{6, 3\}_{(b,0)}$. However, by taking linear combinations of realizations generated by applying Wythoff's construction to the representations described above, we can construct any realization of $\{6, 3\}_{(b,0)}$.

6. Realizations of $\{6, 3\}_{(b,b)}$

The regular abstract polyhedron $\{6, 3\}_{(b,b)}$ is the dual of $\{3, 6\}_{(b,b)}$, with $6b^2$ vertices. The techniques used to determine its irreducible realizations are a combination of those presented in the previous sections. In particular, some representations of $\mathcal{A}(\{3, 6\}_{(b,b)})$ have Wythoff spaces of dimension two. The distinct irreducible representations of $\mathcal{A}(\{3, 6\}_{(b,b)})$ with nontrivial Wythoff spaces are enumerated in Table 4.

7. Remarks

Remark 1. The techniques used in this paper rely on the fact that the automorphism groups of the regular abstract polyhedra studied have large commutative subgroups. This will naturally be true for other abstract polytopes defined by taking quotients of a Euclidean tiling. The toroidal quotients of Euclidean regular tessellations which are themselves regular are completely enumerated in [12]. We do not know any other abstract polytopes which have this property.

Table 4. Number and dimension of irreducible representations of $\mathcal{A}(\{6, 3\}_{(b,b)})$, counted without multiplicity.

b	Dimension over $\mathbb{R}$						Total
	$w_G = 1$				$w_G = 2$		
	1	3	4	6	6	12	
Even	2	2	1	$3b - 4$	$(b - 2)/2$	$(3b^2 - 12b + 12)/12$	$(3b^2 + 30b + 12)/12$
Odd	2	0	1	$3b - 3$	$(b - 1)/2$	$(3b^2 - 12b + 9)/12$	$(3b^2 + 30b + 3)/12$

Remark 2. It may be possible to extend these results to the nonregular *chiral* abstract polyhedra of types $\{3, 6\}_{(b,c)}$ and $\{6, 3\}_{(b,c)}$.

Remark 3. In studying the action of the automorphism group of a regular abstract polyhedron on the representation spaces discussed here, it is convenient to reduce the space to a sum of two-dimensional subspaces. The generators ρ_i can then be described in terms of reflections in these subspaces and a small number of permutations interchanging the subspaces.

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