Finding Optimal Shadows of Polytopes*

T. Burger¹ and P. Gritzmann²

¹Bommershöfer Weg 35, D-40670 Meerbusch, Germany
thomas_burger@westlb.de

²Zentrum Mathematik, Technische Universität München,
D-80290 München, Germany
gritzman@mathematik.tu-muenchen.de

Abstract. This paper deals with the problem of projecting polytopes in finite-dimensional Euclidean spaces on subspaces of given dimension so as to maximize or minimize the volume of the projection.

As to the computational complexity of the underlying decision problems we show that maximizing the volume of the orthogonal projection on hyperplanes is already NP -hard for simplices. For minimization, the problem is easy for simplices but NP -hard for bipyramids over parallelotopes. Similar results are given for projections on lower-dimensional subspaces. Several other related NP -hardness results are also proved including one for inradius computation of zonotopes and another for a location problem.

On the positive side, we present various polynomial-time approximation algorithms. In particular, we give a randomized algorithm for maximizing orthogonal projections of $\mathcal{H}$ -polytopes in $\mathbb{R}^n$ on hyperplanes with an error bound of essentially $O(\sqrt{n/\log n})$.

Introduction

Projection is one of the most basic operations in computational convexity. The problem of maximizing or minimizing the volumes of orthogonal projections of polytopes on subspaces generalizes the problem of computing their width or diameter, functionals with a wide range of applications [GK2]. Maximal or minimal projections on hyperplanes have received some attention since they are intimately related to various illumination problems, see, e.g., [Ma].

In this paper we present various tractability and intractability results for the *shadow problems* of maximizing or minimizing orthogonal projections of polytopes on sub-

* The research of both authors was supported by the Deutsche Forschungsgemeinschaft. The research of the second author was supported by a Max Planck Research Award.

spaces. As to the computational complexity of the underlying decision problems we show that maximizing the volume of the orthogonal projection on hyperplanes is already $\mathbf{NP}$ -hard for simplices. For minimization, the problem is easy for simplices but $\mathbf{NP}$ -hard for bipyramids over parallelotopes. Since volume computation is easy for these classes of polytopes and their orthogonal projections on hyperplanes, the problems of maximizing and minimizing projection volumes are not just hard because the volume is involved, but rather because of their combinatorial nature. Similar results are shown for projections on subspaces of intermediate dimensions.

We give several other related $\mathbf{NP}$ -hardness results including one for inradius computation of zonotopes and another for a location problem, the so-called *minsum hyperplane problem*.

On the positive side, we present various polynomial-time approximation algorithms including a randomized approximative algorithm for maximizing orthogonal projections on hyperplanes of polytopes that are given in terms of linear inequalities. It is based on structural results for projections that allow us to utilize the randomized volume algorithms of [DFK] and [KLS].

1. Preliminaries and Main Results

Our general setting is Euclidean n -space $\mathbb{R}^n$ for $n \in \mathbb{N}$, equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|_2$. The *standard basis* of $\mathbb{R}^n$ is denoted by $e_1, \dots, e_n$. An affine subspace is called a *flat*; an $(n-1)$ -dimensional flat is a *hyperplane*. A hyperplane divides $\mathbb{R}^n$ into two regions whose closures are called (closed) *half-spaces*. We write hyperplanes and half-spaces in the form

$$H_{x,\alpha} = \{y \in \mathbb{R}^n: \langle x, y \rangle = \alpha\}, \quad H_{x,\alpha}^- = \{y \in \mathbb{R}^n: \langle x, y \rangle \leq \alpha\},$$

respectively, where $x \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in \mathbb{R}$.

A *convex body* is a nonempty, compact, convex subset of $\mathbb{R}^n$; it is called *proper* if its interior is nonempty. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. The n -dimensional (Euclidean) unit ball $\{x \in \mathbb{R}^n: \|x\|_2 \leq 1\}$ is denoted by $\mathbb{B}^n$, its volume by κ_n , its boundary by $\mathbb{S}^{n-1}$. A *polytope* is a convex body which is the intersection of finitely many half-spaces or, equivalently, the convex hull of finitely many points. See [Gr] for a comprehensive study of polytopes and see [Zi] for an update.

We employ the standard *binary Turing machine* model of computation. The *size* of the input is the length of the binary encoding needed to present the input data to a Turing machine and the *time complexity* of an algorithm is also defined in terms of the operations of a Turing machine (see [GJ] or [GLS] for further details). This model includes only polytopes that allow a finite presentation over the rationals and, in fact, each computation begins with some such presentation.

A $\mathcal{V}$ -*polytope* is a string $(n, m, v_1, \dots, v_m)$ consisting of positive integers n and m , and an m -tuple $v_1, \dots, v_m$ of points of $\mathbb{Q}^n$. It represents the polytope

$$P = \text{conv}\{v_1, \dots, v_m\}$$

in $\mathbb{R}^n$.

An $\mathcal{H}$ -polytope is a string $(n, m, w_1, \dots, w_m, \beta_1, \dots, \beta_m)$ consisting of positive integers n and m , an m -tuple $w_1, \dots, w_m$ of nonzero vectors of $\mathbb{Q}^n$, and rationals $\beta_1, \dots, \beta_m$ such that set

$$P = \bigcap_{i=1}^m H_{w_i, \beta_i}^-,$$

which it presents is a polytope. In the following we do not formally distinguish a $\mathcal{V}$ - or an $\mathcal{H}$ -polytope from the geometric object it represents.

It is well known that each $\mathcal{V}$ -polytope P admits an $\mathcal{H}$ -presentation and vice versa. However, since P may have many more vertices than facets (or vice versa), it may happen that the minimum size for one sort of presentation is much larger than the minimum size for the other. That is why the complexity results for $\mathcal{V}$ -polytopes and $\mathcal{H}$ -polytopes differ substantially.

Zonotopes are special polytopes that are Minkowski-sums of finitely many line segments. Since they play a special role in our study we specify a suitable presentation as follows. An $\mathcal{S}$ -zonotope is a string $(n, m, c, z_1, \dots, z_m)$, where $n, m \in \mathbb{N}$ and $c, z_1, \dots, z_m \in \mathbb{Q}^n$. The string represents the geometric object

$$Z = c + \sum_{i=1}^m [-1, 1]z_i.$$

There are two natural representations of a linear subspace U of $\mathbb{R}^n$ of dimension k . It can be viewed either as the linear hull of k linearly independent vectors in $\mathbb{R}^n$ or as the orthogonal complement of the linear hull of $n - k$ linearly independent vectors of $\mathbb{R}^n$. Since each representation can easily be transformed into the other by solving a system of linear equations which can be done in polynomial time if U is generated by rational vectors, either representation is equally appropriate.

For a k -dimensional linear subspace $U \subset \mathbb{R}^n$ and a convex body $K \in \mathcal{K}^n$ we denote the *orthogonal projection* of K on U by $\Pi(K; U)$. The k -dimensional volume of the orthogonal projection of K on U is denoted by $\sigma(K, U)$. It is often called the *k-dimensional outer quermass*; we however use the term *k-shadow volume*. For the maximum and minimum k -shadow volume of a convex body K we write $Q_k(K)$ and $q_k(K)$, respectively.

Note that the restriction to orthogonal projections comes naturally, since on one hand the volume of nonorthogonal projections of a polytope can be made arbitrarily large and on the other hand it is easy to see that minimum projections are necessarily orthogonal.

We assume that the dimensions of the subspaces which are considered for the projections depend only on the dimension n of the original space $\mathbb{R}^n$. We use a function $\theta: \mathbb{N} \rightarrow \mathbb{N}$ with $\theta(n) \leq n - 1$ to specify the dimension of the subspaces for each $n \in \mathbb{N}$. The function θ will not be considered as part of the input. Hence its computation does not affect the running time of the algorithms. In the following we mainly focus on the decision

problems related to maximal and minimal projections: The first of these is as follows:

$\mathcal{V}$ -MAX- $\theta(n)$ -PROJ

Instance: A positive integer n , a $\mathcal{V}$ -polytope $P \subset \mathbb{R}^n$, a positive rational λ .

Question: Is there a $\theta(n)$ -dimensional subspace U of $\mathbb{R}^n$ such that the $\theta(n)$ -dimensional volume $\sigma(P, U)$ of the orthogonal projection of P on U is greater than or equal to λ ?

The problems $\mathcal{H}$ -MAX- $\theta(n)$ -PROJ, $\mathcal{V}$ -MIN- $\theta(n)$ -PROJ, and $\mathcal{H}$ -MIN- $\theta(n)$ -PROJ are defined accordingly. We also speak of MAX- $\theta(n)$ -PROJ for $\mathcal{H}$ -polytopes, etc., if this seems more appropriate. OPTMAX- $\theta(n)$ -PROJ, OPTMIN- $\theta(n)$ -PROJ, etc., denote the corresponding optimization problems.

Note that in order to solve the stated problems it is strictly speaking not necessary to compute the volume of the optimal projections. All that is required is to determine whether there exists an appropriate subspace. It is clear, however, that the volume of an optimal projection of a given polytope can be approximated by means of any algorithm for the corresponding decision problem within a binary search framework. (This would be different, however, for optimization versions of the problems that ask for an optimal subspace.) It is, hence, not very surprising that for many (but not for all!) choices of functions θ the problems MAX- $\theta(n)$ -PROJ and MIN- $\theta(n)$ -PROJ are no easier than volume computation, a problem which is known to be $\#\mathbb{P}$ -hard in general [DF], [Kh]. Hence on the side of intractability results our main interest here is in simple classes of polytopes for which the problem of computing the volume is easy.

In Section 2 we analyze the computational complexity of the shadow problems. Our main results show that these problems are not just hard because the volume is involved. For example, MAX- $(n-1)$ -PROJ is already $\mathbb{NP}$ -complete for simplices. Note that the volume of a rational $\mathcal{V}$ -presented simplex is just the $n!$ th fraction of a corresponding determinant, and hence its volume can be computed in polynomial time. Also, the volume of a projection of a simplex on any rational hyperplane can be computed in polynomial time. This is part of the reason why, when restricted to the class of simplices, MAX- $(n-1)$ -PROJ is in $\mathbb{NP}$.

Table 1 lists the complexity results for shadow maximization obtained in this paper.

Since k is an arbitrary fixed positive integer constant the results are nearly complete; the only major case which remains open is that of $\mathcal{V}$ -MAX- $\theta(n)$ -PROJ for functions θ with $2 \leq \theta(n)$ that are in $o(n^{1/k})$ for any $k \in \mathbb{N}$.

Table 1. The computational complexity of shadow maximization.

MAX- $(n-1)$ -PROJ		is $\mathbb{NP}$ -complete	for simplices
MAX- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\mathbb{NP}$ -hard	for simplices
MAX- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\#\mathbb{P}$ -hard	for $\mathcal{V}$ -polytopes
MAX- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\#\mathbb{P}$ -hard	for $\mathcal{H}$ -polytopes
MAX- $\theta(n)$ -PROJ	with $\theta(n) = O(n - n^{1/k})$	is $\mathbb{NP}$ -hard	for $\mathcal{H}$ -bipyramids
MAX-1-PROJ		is $\mathbb{NP}$ -complete	for $\mathcal{H}$ -parallelotopes
MAX-1-PROJ		is in $\mathbb{P}$	for $\mathcal{V}$ -polytopes

Table 2. The computational complexity of shadow minimization.

MIN- $(n - 1)$ -PROJ		is in $\mathbb{P}$	for simplices and parallelotopes
MIN- $(n - 1)$ -PROJ		is $\mathbb{NP}$ -complete	for $\mathcal{H}$ -bipyramids
MIN- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\mathbb{NP}$ -hard	for $\mathcal{H}$ -prisms
MIN- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\#\mathbb{P}$ -hard	for $\mathcal{V}$ -polytopes
MIN- $\theta(n)$ -PROJ	with $\theta(n) = \Omega(n^{1/k})$	is $\#\mathbb{P}$ -hard	for $\mathcal{H}$ -polytopes
MIN- $\theta(n)$ -PROJ	with $\theta(n) = O(n - n^{1/k})$	is $\mathbb{NP}$ -hard	for $\mathcal{H}$ -polytopes
MIN- $\theta(n)$ -PROJ	with $\theta(n) = O(\log n)$	is $\mathbb{NP}$ -hard	for $\mathcal{V}$ -polytopes
MIN-1-PROJ		is $\mathbb{NP}$ -complete	for simplices

MIN- $(n - 1)$ -PROJ can be solved in polynomial time for simplices. However, in general MIN- $(n - 1)$ -PROJ is $\mathbb{NP}$ -complete for $\mathcal{H}$ -presented bipyramids over parallelotopes. The proof uses the fact that the problem of computing the inradius of certain zonotopes is $\mathbb{NP}$ -hard, a result that is also proved here.

Table 2 gives an overview of our complexity results for shadow minimization; k is again an arbitrary fixed positive integer.

As an additional consequence of the complexity result for inradius computations of zonotopes we show that the following location problem is $\mathbb{NP}$ -complete:

MINSUM

Instance: Positive integers n, m , positive rationals $\omega_1, \dots, \omega_m$, rational vectors

$v_1, \dots, v_m \in \mathbb{R}^n$, a positive rational λ .

Question: Does there exist a nonzero vector $y \in \mathbb{Q}^n$ and a real number $\gamma \in \mathbb{Q}$ such that the square of $\sum_{i=1}^m \omega_i \text{dist}(v_i, H_{y,\gamma})$ is less than or equal to λ ?

For a survey on MINSUM, some of its applications, and for additional references see [KM], [MS2], and [MS3].

In Section 3 we present (deterministic and randomized) approximation algorithms for all shadow problems. In particular we give a randomized algorithm which solves $\mathcal{H}$ -OPTMAX- $(n - 1)$ -PROJ with probability $1 - \beta \in [0, 1)$ up to a relative error of $O(\varepsilon\sqrt{m} + \sqrt{n/\ln n} + \varepsilon\sqrt{mn/\ln n})$ in running time that is polynomial in $n, m, 1/\varepsilon$, and $\ln(1/\beta)$.

2. Computational Complexity

All information concerning the volumes of projections on hyperplanes is encoded in the *projection polytope*. Let P be a proper polytope in $\mathbb{R}^n$ with facets $F_1, \dots, F_m$ with $(n - 1)$ -dimensional volume $\mu_i = \text{vol}_{n-1}(F_i)$ and corresponding outer normal vectors $u_1, \dots, u_m \in \mathbb{S}^{n-1}$, respectively. For abbreviation, we write $\tilde{u}_i = \mu_i u_i$ and call the vectors $\tilde{u}_i$ the *scaled (outer facet) normal vectors* of P . It is elementary to see (see [BF1] or [Sc]) that

$$\sum_{i=1}^m \tilde{u}_i = 0$$

and that, for each $u \in \mathbb{S}^{n-1}$, the shadow volume in direction u equals

$$\sigma(P, u^\perp) = \frac{1}{2} \sum_{i=1}^m |\langle u, \tilde{u}_i \rangle|. \quad (1)$$

Note that the right-hand side of (1) is the support function of the zonotope

$$Z_P = \frac{1}{2} \sum_{i=1}^m [-\tilde{u}_i, \tilde{u}_i], \quad (2)$$

called the *projection polytope* of P . Its importance in our context comes from the fact that

$$q_{n-1}(P) = r_n(Z_P) \quad \text{and} \quad Q_{n-1}(P) = R_n(Z_P),$$

where r_n and R_n denote the (Euclidean) inradius and circumradius, respectively. We also obtain a description of the optimum directions. Each maximizing direction is parallel to a vertex of Z_P ; minimizing directions are orthogonal to a facet of Z_P ; see [MW], [Ma], and [BGK].

In order to be able to deal with projections on lower-dimensional subspaces we begin with a few geometric transformations that can be used for extending some hardness results.

We only consider transformations from results for maximum projections on hyperplanes in detail here. Other transformations (from maximum projection on lines and from minimum projection on hyperplanes and lines) can be treated similarly, see [Bu].

A central tool employed in this section is the mixed volume $V: (\mathcal{K}^n)^n \rightarrow \mathbb{R}$ which generalizes the ordinary volume. For a definition and elementary properties of the mixed volume we refer to [Sc]. Of particular importance to us is the following connection between mixed volume and volume of projection, which is a consequence of Theorem 19.4 in [BZ], see also p. 294 of [Sc].

Proposition 2.1. *Let $K \in \mathcal{K}^n$ and $k \in \mathbb{N}$ with $k < n$. Further, let $u_1, \dots, u_{n-k} \in \mathbb{S}^{n-1}$ be orthogonal and $U = \{u_1, \dots, u_{n-k}\}^\perp$. Then*

$$\sigma(K, U) = \frac{n!}{k!} V(K[k], [0, 1]u_1, \dots, [0, 1]u_{n-k}),$$

where $K[k]$ stands for the k -tuple $(K, \dots, K)$.

Lemma 2.2. *Let $K \subset \mathbb{R}^n \times \{0\}$ be a convex body, let $q \in \mathbb{R}^n$, and let $\rho > 0$ such that $q + \rho\mathbb{B}^n \subset K$. Further, let $0 < |\eta| \leq \sqrt{3}\rho$, let $p = q + \eta e_{n+1}$, and set $K_\eta = \text{conv}(K \cup \{p\})$. Then*

$$Q_{n-l}(K_\eta) = Q_{n-l}(K),$$

for all $l = 0, \dots, n-1$.

Proof. Since the reverse inequality is obvious we have to show that

$$\begin{aligned} & \max\{\sigma(K_\eta, \{z_1, \dots, z_{l+1}\}^\perp) : z_1, \dots, z_{l+1} \in \mathbb{S}^n \text{ orthonormal}\} \\ & \leq \max\{\sigma(K, \{z_1, \dots, z_l\}^\perp) : z_1, \dots, z_l \in \mathbb{S}^{n-1} \times \{0\} \text{ orthonormal}\}. \end{aligned}$$

Let orthonormal vectors $z_1, \dots, z_{l+1} \in \mathbb{S}^n$ be given. Without loss of generality we can assume that $z_1, \dots, z_l \perp e_{n+1}$ and $\langle z_{l+1}, e_{n+1} \rangle \geq 0$. By Proposition 2.1, it suffices to show that

$$(n+1)V(K_\eta[n-l], [0, 1]z_1, \dots, [0, 1]z_{l+1}) \leq V(K[n-l], [0, 1]z_1, \dots, [0, 1]z_l).$$

Of course, equality holds for $z_{l+1} = e_{n+1}$.

Since $z_1, \dots, z_l \perp e_{n+1}$, $\Pi(K_\eta; \{z_1, \dots, z_l\}^\perp)$ is an $(n-l+1)$ -dimensional pyramid of base $\Pi(K; \{z_1, \dots, z_l\}^\perp)$ and height η . Therefore,

$$\begin{aligned} V(K_\eta[n-l+1], [0, 1]z_1, \dots, [0, 1]z_l) &= \frac{(n-l+1)!}{(n+1)!} \sigma(K_\eta, \{z_1, \dots, z_l\}^\perp) \\ &= \frac{(n-l+1)!}{(n+1)!} \frac{\eta}{n-l+1} \sigma(K, \{z_1, \dots, z_l\}^\perp) \\ &= \frac{\eta}{n+1} V(K[n-l], [0, 1]z_1, \dots, [0, 1]z_l). \end{aligned}$$

It thus remains to show that

$$\eta V(K_\eta[n-l], [0, 1]z_1, \dots, [0, 1]z_{l+1}) \leq V(K_\eta[n-l+1], [0, 1]z_1, \dots, [0, 1]z_l). \quad (3)$$

We write $z_{l+1} = \sqrt{1-\zeta^2}\hat{z} + \zeta e_{n+1}$ for some $\hat{z} \in \mathbb{S}^{n-1} \times \{0\}$ and $\zeta \in [0, 1]$ and consider two alternatives.

Suppose first that $\zeta \geq \sqrt{3}/2$. Then $\zeta/\sqrt{1-\zeta^2} \geq \sqrt{3} \geq \eta/\rho$ and thus $p - [0, \eta/\zeta]z_{l+1} \subset K_\eta$, by definition of K_η . Therefore,

$$\begin{aligned} V(K_\eta[n-l], [0, 1]z_1, \dots, [0, 1]z_{l+1}) &= \frac{\zeta}{\eta} V\left(K_\eta[n-l], p - [0, 1]\frac{\eta}{\zeta}z_{l+1}, [0, 1]z_1, \dots, [0, 1]z_l\right) \\ &\leq \frac{1}{\eta} V(K_\eta[n-l+1], [0, 1]z_1, \dots, [0, 1]z_l), \end{aligned}$$

which implies (3).

Now let $\zeta \leq \sqrt{3}/2$. Elementary analytic geometry shows that

$$q - \rho\hat{z} + \frac{2\eta}{\zeta + (\eta/\rho)\sqrt{1-\zeta^2}}[0, 1]z_{l+1} \subset K_\eta.$$

Therefore,

$$\begin{aligned} V(K_\eta[n-l], [0, 1]z_1, \dots, [0, 1]z_{l+1}) &\leq \frac{\zeta + (\eta/\rho)\sqrt{1-\zeta^2}}{2\eta} V(K_\eta[n-l+1], [0, 1]z_1, \dots, [0, 1]z_l). \end{aligned}$$

It is easy to check that the numerator of the fraction on the right-hand side assumes its maximum for $\zeta = 1/\sqrt{1+\eta^2/\rho^2}$ and is, therefore, not greater than $\sqrt{1+\eta^2/\rho^2}$. The latter is at most 2 because $\eta/\rho \leq \sqrt{3}$. So (3) follows again. $\square$

Now we generalize the previous lemma by continuing the process of constructing pyramids. We also formulate it in a computational manner so that we can apply it directly later.

Lemma 2.3. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$, let $P \subset \mathbb{R}^n$ be a proper $\mathcal{W}$ -polytope of size L , and let k be a positive integer. There is a proper k -fold pyramid $P_k \subset \mathbb{R}^{n+k}$ over P such that*

$$Q_l(P) = Q_l(P_k)$$

for $l = 1, \dots, n$. A $\mathcal{W}$ -presentation of P_k can be computed in time polynomial in k and L .

Proof. By use of the ellipsoid algorithm, we can compute, in time polynomial in L , a rational point q_0 and a rational number $\rho_0 \geq 2^{-4L}$ such that $q_0 + \rho_0 \mathbb{B}^n \subset P$. We embed P in $\mathbb{R}^n \times \{0\}^k$ and call it P_0 . For $i = 1, \dots, k$, we define inductively

$$\begin{aligned} \eta_i &= \rho_{i-1}, & \rho_i &= \eta_i/3, & q_i &= q_{i-1} + \rho_i e_{n+i}, \\ p_i &= q_{i-1} + \eta_i e_{n+i}, & P_i &= \text{conv}(P_{i-1} \cup \{p_i\}). \end{aligned}$$

Then $q_i = q_0 + \sum_{j=1}^i \rho_j e_{n+j}$ and $\rho_i = 3^{-i} \rho_0$, for $i = 0, \dots, k$. By elementary geometry,

$$q_i + \rho_i (\mathbb{B}^{n+i} \times \{0\}^{k-i}) \subset P_i \subset \mathbb{R}^{n+i} \times \{0\}^{k-i}, \quad i = 0, \dots, k.$$

Applying Lemma 2.2 inductively in $\mathbb{R}^n, \mathbb{R}^{n+1}, \dots, \mathbb{R}^{n+k-1}$ yields

$$Q_l(P_0) = Q_l(P_1) = \dots = Q_l(P_k).$$

It remains to show that P_k can be presented the same way as P in polynomial time. This is obvious for $\mathcal{V}$ -polytopes. For $\mathcal{H}$ -polytopes note that if

$$P = H_{e_{n+1}, 0} \cap \bigcap_{i=1}^m H_{w_i, \beta_i}^- \subset \mathbb{R}^n \times \{0\},$$

where $w_1, \dots, w_m \in \mathbb{R}^n \setminus \{0\}$, $\beta_1, \dots, \beta_m \in \mathbb{R}$, and $p = q + \eta e_{n+1}$ for some $q \in \mathbb{R}^n \times \{0\}$ and some positive $\eta \in \mathbb{R}$, then

$$\text{conv}(P \cup \{p\}) = H_{-e_{n+1}, 0}^- \cap \bigcap_{i=1}^m H_{w'_i, \beta_i}^-,$$

where $w'_i = w_i + ((\beta_i - \langle w_i, q \rangle)/\eta)e_{n+1}$ for $i = 1, \dots, m$. Thus an $\mathcal{H}$ -presentation of P_k can be computed in polynomial time. $\square$

The following lemmas can be proved in a similar way, see [Bu] for details.

Lemma 2.4. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$ and let $P \subset \mathbb{R}^n$ be a symmetric proper $\mathcal{W}$ -polytope of size L . Let k be a positive integer. There is a constant γ , depending only on k and P , and a k -fold pyramid $P_k \subset \mathbb{R}^{n+k}$ over P such that*

$$Q_{k+1}(P_k) = \gamma Q_1(P).$$

The constant γ and a $\mathcal{W}$ -presentation of P_k can be computed in time polynomial in k and L .

Lemma 2.5. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$, let $P \subset \mathbb{R}^n$ be a proper $\mathcal{W}$ -polytope of size L , and let k be a positive integer, which is in $O(\log L)$ if $\mathcal{W} = \mathcal{V}$. There is a k -fold prism $P_k \subset \mathbb{R}^{n+k}$ over P such that, for all integers $l = 1, \dots, n$,*

$$q_l(P_k) = q_l(P).$$

A $\mathcal{W}$ -presentation of P_k can be computed in time polynomial in k and L .

Lemma 2.6. *Let $P \subset \mathbb{R}^n$ and let $k \in \mathbb{N}$ as in Lemma 2.5. There is a constant γ , depending only on k and P , and a k -fold prism $P_k \subset \mathbb{R}^{n+k}$ over P such that*

$$q_{k+1}(P_k) = \gamma q_1(P).$$

The constant γ and a $\mathcal{W}$ -presentation of P_k can be computed in time polynomial in k and L .

We are now ready to deduce results concerning the computational complexity of shadow optimization stated in Section 1. We begin with shadow maximization.

It is easy to check that the projection polytope of a proper simplex $S = \text{conv}\{0, v_1, \dots, v_n\} \subset \mathbb{R}^n$ is the parallelotope

$$Z_S = \frac{n}{2} \text{vol}_n(S) \sum_{i=0}^n [-1, 1] w_i,$$

where $w_1, \dots, w_n$ is the dual basis of $v_1, \dots, v_n$ and $w_0 = -\sum_{i=1}^n w_i$. In particular, the vectors $w_0, \dots, w_n \in \mathbb{R}^n$ can be computed from $v_1, \dots, v_n$ in polynomial time, and vice versa. Thus for proper simplices MAX- $(n-1)$ -PROJ is equivalent to the problem $[-1, 1]$ -PARMAX₂ of bounding the square of the Euclidean norm of $(\mathcal{H}$ - or $\mathcal{S}$ -) parallelotopes from below. The latter problem is $\mathbb{NP}$ -complete [BGKvL], so the former is $\mathbb{NP}$ -hard. Since projection volumes of simplices can easily be computed using the projection polytope, the problem is also in $\mathbb{NP}$. Thus we have proved the first part of the following theorem.

Theorem 2.7. *MAX- $(n-1)$ -PROJ is $\mathbb{NP}$ -complete for $(\mathcal{V}$ - or $\mathcal{H}$ -) simplices. Further, the problem of deciding whether the maximum $(n-1)$ -shadow volume of a given $(\mathcal{V}$ - or $\mathcal{H}$ -) simplex is attained for more than one hyperplane is $\mathbb{NP}$ -hard.*

The latter statement follows from the fact that UNIDIAM, the uniqueness problem associated with $[-1, 1]$ -PARMAX₂, is $\mathbb{NP}$ -hard [GK1].

With the aid of Lemma 2.3 we can extend this $\mathbb{NP}$ -completeness result to MAX- $\theta(n)$ -PROJ for a large class of functions $\theta: \mathbb{N} \rightarrow \mathbb{N}$. We need another polynomially bounded function $\theta': \mathbb{N} \rightarrow \mathbb{N}$ corresponding to θ such that, for all $n \in \mathbb{N}$,

$$\theta(\theta'(n) + n) = n - 1.$$

We embed a given proper simplex S in $\mathbb{R}^n \times \{0\}^{\theta'(n)} \subset \mathbb{R}^{n+\theta'(n)}$, and then construct an $(n + \theta'(n))$ -simplex S' over S as described in Lemma 2.3. Since $\theta(\theta'(n) + n) = n - 1$ we have reduced MAX- $(n-1)$ -PROJ to the problem of finding the maximum volume of the $\theta(n)$ -dimensional projections of S' . That proves the following theorem.

Theorem 2.8. *Let polynomial time functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for each $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = n - 1$. Then the problem $\text{MAX-}\theta(n)\text{-PROJ}$ is $\mathbb{NP}$ -hard, even for ($\mathcal{V}$ - or $\mathcal{H}$ -) simplices.*

It is easy to see that $\theta'(n)$ is bounded by a polynomial in n if, and only if, $\theta(n) \geq \alpha \sqrt[k]{n}$ for some positive $\alpha, \beta \in \mathbb{R}$. In this case the reduction can be accomplished in polynomial time. Examples of pairs of functions θ, θ' are listed in the following table where k is a fixed positive integer:

$\theta(n)$	$n - k$	$\lceil n/k \rceil$	$\lceil \sqrt[k]{n} \rceil$
$\theta'(n)$	$k - 1$	$(k - 1)n - k$	$(n - 1)^k - n$

In Theorem 2.8 the relation between θ and θ' implies a lower bound for $\theta(n)$. In the following we want to deduce a similar result for slowly increasing functions θ . The problem MAX-1-PROJ is equivalent to the problem of bounding the diameter of a given polytope from below. That problem is $\mathbb{NP}$ -complete for centered $\mathcal{H}$ -parallelotopes [BGKvL, Theorem 15]. We can extend this result using k -fold bipyramids over parallelotopes. Lemma 2.4 shows that we need to have a polynomially bounded function $\theta': \mathbb{N} \rightarrow \mathbb{N}$ corresponding to θ , construct a bipyramid P' in $\mathbb{R}^{n+\theta'(n)}$ and then project on $(\theta'(n) + 1)$ -dimensional subspaces. Thus the auxiliary function θ' is related to θ by the condition

$$\theta(\theta'(n) + n) = \theta'(n) + 1, \quad \text{for all } n \in \mathbb{N}.$$

This leads to the following theorem for $\mathcal{H}$ -polytopes.

Theorem 2.9. *Let polynomial time functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for all $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = \theta'(n) + 1$. Then the problem $\text{MAX-}\theta(n)\text{-PROJ}$ is $\mathbb{NP}$ -hard for symmetric $\mathcal{H}$ -polytopes that are k -fold bipyramids over parallelotopes.*

It is easy to check that if $\theta(n) \leq n - \alpha \sqrt[k]{n}$ for some positive $\alpha, \beta \in \mathbb{R}$, then $\theta'(n) \leq (n/\alpha)^\beta - n$. Examples of pairs θ, θ' are given in the following table where k is again a fixed positive integer:

$\theta(n)$	k	$\lceil n/(k + 1) \rceil$	$\lceil n + 1 - \sqrt[k]{n} \rceil$
$\theta'(n)$	$k - 1$	$\lceil (n - k - 1)/k \rceil$	$n^k - n$

The problem of computing the diameter is easy for $\mathcal{V}$ -polytopes. Just determine the distance for every pair of vertices. The natural generalization of this approach to two-dimensional subspaces fails. More precisely, among the optimal two-dimensional subspaces there need not be one parallel to the affine hull of a triple of vertices; the regular simplex provides a simple example. The computational complexity of the problem of maximizing the $\theta(n)$ -shadow volume remains open for $\mathcal{V}$ -polytopes for functions θ greater 1 that are in $o(n^{1/k})$ for any fixed positive integer k .

Of course, the above lifting techniques allow us also to extend the results of [DF] and [Kh].

Theorem 2.10. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$ and let polynomial time functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for each $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = n - 1$. Then the problem $\mathcal{W}$ -MAX- $\theta(n)$ -PROJ is $\#\mathbb{P}$ -hard.*

We now turn to shadow minimization. First we consider the projection on hyperplanes. For simplices, the minimum problem is easy to solve, since the normal vector of an optimal hyperplane is orthogonal to at least $n - 1$ outer normals. Thus there are only $\binom{n+1}{n-1} = O(n^2)$ directions to consider. It is even easier for parallelotopes since in that case the number of alternatives reduces to n . Hence MIN- $(n - 1)$ -PROJ is in $\mathbb{P}$ when restricted to $\mathcal{V}$ - or $\mathcal{H}$ -simplices or $\mathcal{H}$ -parallelotopes.

For an arbitrary polytope with m facets, this characterization does not suffice, since there are $\binom{m}{n-1}$ directions to consider, which is proportional to m^{n-1} . It turns out that the minimum problem becomes $\mathbb{NP}$ -hard for $\mathcal{H}$ -presented bipyramids over parallelotopes. For the proof, we proceed in two steps.

Since the minimum shadow of a polytope P equals the inradius of the corresponding projection polytope Z_P , the first (and major) step is to prove $\mathbb{NP}$ -hardness for the problem of computing the inradius of a zonotope. Since we want to prove $\mathbb{NP}$ -hardness of MIN- $(n - 1)$ -PROJ for a class of polytopes for which volume computation is easy, we prove $\mathbb{NP}$ -completeness for the following problem INRADCROSSZON. The name INRADCROSSZON indicates that the convex hull of the vectors $v'_1, \dots, v'_{2n}$ is a crosspolytope.

INRADCROSSZON

Instance: A positive integer n , n linearly independent rational vectors $v_1, \dots, v_n \in \mathbb{R}^n \times \{0\}$, positive rationals α and λ .

Question: Let $v'_i = v_i + \alpha e_{n+1}$ and $v'_{n+i} = -v_i + \alpha e_{n+1}$ for $i = 1, \dots, n$. Is the square of the inradius of the zonotope $\sum_{i=1}^{2n} [0, 1]v'_i$ less than or equal to λ ?

Theorem 2.11. *The problem INRADCROSSZON is $\mathbb{NP}$ -complete.*

Proof. The problem is in $\mathbb{NP}$ since for a zonotope Z (being a symmetric polytope) the inradius equals half the width, and the width is attained for a direction which is orthogonal to a facet of Z and, therefore, orthogonal to $n - 1$ of the vectors generating Z .

Using the fact that $[-1, 1]$ -PARMAX₂ is $\mathbb{NP}$ -complete, we proceed as follows. We consider an arbitrary proper parallelotope P in $\mathbb{R}^n$ centered at the origin. We embed P 's polar P° in a hyperplane $H \subset \mathbb{R}^{n+1}$ parallel to $\mathbb{R}^n \times \{0\}$ at a sufficiently small positive height α and consider the zonotope Z which is generated by the segments joining 0 and the vertices of the embedded P° . Then we show that (since the height is sufficiently small) the insphere of Z is uniquely determined by the minimum of the distances of the center of Z to the facets of Z which contain the origin. Thus the inradius problem for Z is related to the norm maximization problem for P by polarity.

We now give the details of the proof. Consider an instance $(n, z_1, \dots, z_n, \lambda)$ of $[-1, 1]$ -PARMAX₂. Of course, $z_1, \dots, z_n$ are a basis of $\mathbb{R}^n$. Let $v_1, \dots, v_n$ be the corresponding dual basis, and put $v_{n+i} = -v_i$ for $i = 1, \dots, n$. Clearly, $v_1, \dots, v_{2n}$ can be

Fig. 1. (a) The zonotope Z containing $P^\circ \times [\alpha, (2n-1)\alpha]$ (with dotted boundary). (b) The ball B inscribed Z (two-dimensional section).

computed in polynomial time. Then

$$P = \sum_{i=1}^n [-1, 1] z_i = \bigcap_{i=1}^{2n} H_{v_i, 1}^-.$$

Since $P^\circ = \text{conv}\{v_1, \dots, v_{2n}\}$ and P is centered, $\max_{x \in P} \|x\|_2^2 = R_n^2(P) \geq \lambda$ if, and only if, $r_n^2(P^\circ) \leq 1/\lambda$ for every positive λ .

Let L be the size of the above $\mathcal{H}$ -presentation of P and let $\alpha = 2^{-4L}/n$. Embed $v_1, \dots, v_{2n}$ in $\mathbb{R}^n \times \{0\}$ and define

$$Z = \sum_{i=1}^{2n} [0, 1] v'_i, \quad \text{where } v'_i = v_i + \alpha e_{n+1}, \quad i = 1, \dots, 2n.$$

We show that

$$P^\circ \times [\alpha, (2n-1)\alpha] \subset Z,$$

see Fig. 1(a). In fact, let $x \in P^\circ \times [\alpha, (2n-1)\alpha]$, say $x = \sum_{i=1}^{2n} \lambda_i v_i + \lambda_0 \alpha e_{n+1}$ for some $\lambda_0 \in [1, 2n-1]$ and $\lambda_1, \dots, \lambda_{2n} \in [0, 1]$ with $\sum_{i=1}^{2n} \lambda_i = 1$. Without loss of generality we can assume that $\lambda_i = 0$ for $i = n+1, \dots, 2n$. Let $\beta = (\lambda_0 - 1)/(2n-2)$,

$$\lambda'_i = \lambda_i + \beta(1 - \lambda_i) \quad \text{and} \quad \lambda'_{n+i} = \beta(1 - \lambda_i), \quad i = 1, \dots, n.$$

Then it is easy to check that $\beta \in [0, 1]$, $\lambda'_1, \dots, \lambda'_{2n} \in [0, 1]$, $\sum_{i=1}^{2n} \lambda'_i = \lambda_0$, and $\sum_{i=1}^{2n} \lambda'_i v_i = \sum_{i=1}^{2n} \lambda_i v_i$. Therefore,

$$x = \sum_{i=1}^{2n} \lambda'_i v_i + \alpha \left(\sum_{i=1}^{2n} \lambda'_i \right) e_{n+1} = \sum_{i=1}^{2n} \lambda'_i v'_i \in Z,$$

which shows the asserted inclusion.

The zonotope Z has center $c = n\alpha e_{n+1}$. Let B be a maximum inscribed ball B centered at c , and let q be a point where B touches the boundary of Z . Since $P^\circ \times [\alpha, (2n-1)\alpha] \subset$

Z and $r_{n+1}(Z) < \|c\|_2 = n\alpha = 2^{-4L} \leq r_n(P^\circ)$ we can assume that $0 < \langle q, e_{n+1} \rangle < \alpha$. Let p be the (unique) point on the line through 0 and q such that $\langle p, e_{n+1} \rangle = \alpha$. By the similarity of the triangles $\text{conv}\{0, p, \alpha e_{n+1}\}$ and $\text{conv}\{0, q, c\}$ it follows that

$$\frac{r_{n+1}(Z)}{n\alpha} = \frac{\|c - q\|_2}{n\alpha} = \frac{\|p - \alpha e_{n+1}\|_2}{\|p\|_2} = \frac{r_n(P^\circ)}{\sqrt{\alpha^2 + r_n^2(P^\circ)}}, \quad (4)$$

see Fig. 1(b). Therefore, $r_n^2(P^\circ) \leq 1/\lambda$ if, and only if, $r_{n+1}^2(Z) \leq n^2\alpha^2/(\lambda\alpha^2 + 1)$. Thus we have reduced the given instance of $[-1, 1]$ -PARMAX₂ to the instance $(n, v_1, \dots, v_n, \alpha, n^2\alpha^2/(\lambda\alpha^2 + 1))$ of INRADCROSSZON. Clearly, the reduction runs in polynomial time. $\square$

We now show that zonotopes of the form appearing in INRADCROSSZON occur as projection polytopes of certain $\mathcal{H}$ -polytopes. Using this fact we can, in a second step, reduce INRADCROSSZON to MIN- $(n-1)$ -PROJ and establish NP-hardness of MIN- $(n-1)$ -PROJ for bipyramids over parallelotopes.

Lemma 2.12. *Let $w_1, \dots, w_n \in \mathbb{R}^n \times \{0\}$ be linearly independent vectors. Define the parallelotope $P = \bigcap_{i=1}^n H_{\pm w_i, 1}^- \cap H_{e_{n+1}, 0}$ and the bipyramid $P' = \text{conv}(P \cup \{\pm(1/\alpha)e_{n+1}\})$ for some positive $\alpha \in \mathbb{R}$. Then*

$$Z_{P'} = \frac{\text{vol}_n(P)}{\alpha n} \sum_{i=1}^{2n} [-1, 1]w'_i, \quad (5)$$

where $w'_i = w_i + \alpha e_{n+1}$ and $w'_{n+i} = w_i - \alpha e_{n+1}$, $i = 1, \dots, n$.

Proof. It is easy to check that $(n+1, 4n; \pm w'_1, \dots, \pm w'_{2n}, 1, \dots, 1)$ is an $\mathcal{H}$ -presentation of P' . Let $F = P \cap H_{w, 1}$ be a facet of P and let $F' = P' \cap H_{w', 1}$ be one of the two corresponding facets of P' . Of course, $w \in \{\pm w_1, \dots, \pm w_n\}$ and $w' \in \{\pm w'_1, \dots, \pm w'_{2n}\}$. Then

$$\text{vol}_n(F') = \frac{1}{n} \sqrt{\frac{1}{\|w\|_2^2} + \frac{1}{\alpha^2}} \text{vol}_{n-1}(F) = \frac{1}{\alpha n} \frac{\|w'\|_2}{\|w\|_2} \text{vol}_{n-1}(F),$$

which implies, for the scaled normal $\tilde{w}'$,

$$\tilde{w}' = \text{vol}_n(F') \frac{w'}{\|w'\|_2} = \frac{1}{\alpha n} \frac{\text{vol}_{n-1}(F)}{\|w\|_2} w' = \frac{\text{vol}_n(P)}{2\alpha n} w'.$$

By (2), the assertion (5) follows. $\square$

We now reduce INRADCROSSZON to MIN- $(n-1)$ -PROJ.

Theorem 2.13. *$\mathcal{H}$ -MIN- $(n-1)$ -PROJ is NP-complete for the class of $\mathcal{H}$ -presented bipyramids over parallelotopes. For that class it is also NP-hard to decide whether there is more than one direction minimizing the shadow volume.*

Proof. Let an instance $(n, v_1, \dots, v_n, \alpha, \lambda)$ of INRADCROSSZON be given and define the zonotope $Z = \sum_{i=1}^{2n} [0, 1]v'_i \subset \mathbb{R}^{n+1}$, where $v'_i = v_i + \alpha e_{n+1}$ and $v'_{n+i} = -v_i + \alpha e_{n+1}$ for $i = 1, \dots, n$. Let

$$Z' = Z - n\alpha e_{n+1} = Z - \frac{1}{2} \sum_{i=1}^{2n} v'_i = \frac{1}{2} \sum_{i=1}^{2n} [-1, 1]v'_i,$$

$$P = \bigcap_{i=1}^n H_{\pm v_i, 1}^- \cap H_{e_{n+1}, 0}, \quad \text{and} \quad P' = \text{conv} \left(P \cup \left\{ \pm \frac{1}{\alpha} e_{n+1} \right\} \right).$$

By Lemma 2.12,

$$Z_{P'} = \frac{\text{vol}_n(P)}{\alpha n} \sum_{i=1}^{2n} [-1, 1]v'_i = \frac{2 \text{vol}_n(P)}{\alpha n} Z'. \quad (6)$$

Therefore, $r_{n+1}(Z) = r_{n+1}(Z') \leq \lambda$ if, and only if,

$$\min_{u \in \mathbb{S}^n} \sigma(P', u^\perp) \leq \frac{2 \text{vol}_n(P)}{n\alpha} \lambda.$$

Clearly, the fraction $(2 \text{vol}_n(P))/(n\alpha)$ and an $\mathcal{H}$ -presentation of P' can be computed in time bounded by a polynomial in the size of the given instance. Thus we have polynomially reduced INRADCROSSZON to $\mathcal{H}$ -MIN- $(n-1)$ -PROJ.

The explicit expression (6) for $Z_{P'}$ shows that any optimum direction is rational with size bounded polynomially by the input size. Furthermore, for a given rational vector $v \in \mathbb{R}^n \setminus \{0\}$, one can check in polynomial time if $\sigma(P', v^\perp) \leq \lambda$. This implies that the restricted problem is in $\mathbb{NP}$.

The previous construction allows us to deduce the second assertion directly from the $\mathbb{NP}$ -hardness of UNIDIAM [GK1]. $\square$

Using Lemma 2.5 we can again extend the result of Theorem 2.13 to projections on lower-dimensional subspaces.

Theorem 2.14. *Let polynomially bounded functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for all $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = n - 1$ and $\theta(n) \geq \alpha \sqrt[n]{n}$ for some positive $\alpha, \beta \in \mathbb{R}$. Then the problem MIN- $\theta(n)$ -PROJ is $\mathbb{NP}$ -hard for $\mathcal{H}$ -presented prisms over bipyramids.*

We now turn to the $\mathbb{NP}$ -hardness of MIN- $\theta(n)$ -PROJ for functions θ that are only slowly increasing. Here we start from the fact, proved in [GK2], that the problem of computing the width of a simplex is $\mathbb{NP}$ -hard. Using Lemma 2.6 we obtain $\mathbb{NP}$ -hardness of MIN- $\theta(n)$ -PROJ for certain functions $\theta: \mathbb{N} \rightarrow \mathbb{N}$. Again a second function θ' is needed, related to θ by the equation $\theta(\theta'(n) + n) = \theta'(n) + 1$. Further, in the case of $\mathcal{V}$ -polytopes the condition $\theta'(n) = O(\log n)$ must hold. We show that this condition is satisfied if $\theta(n) \leq \alpha \log n$ for all n , where α is some positive real number. So suppose the latter is the case. Then

$$1 + \theta'(n) = \theta(\theta'(n) + n) \leq \alpha \log(\theta'(n) + n), \quad (7)$$

for all $n \in \mathbb{N}$. The assumption $\theta'(n) > n$ for infinitely many $n \in \mathbb{N}$ implies that $n < \theta'(n) < \alpha \log(2\theta'(n))$ for infinitely many n which is impossible. Therefore, $\theta'(n) =$

$O(n)$ which together with (7) implies that $\theta'(n) = O(\log n)$. Altogether we have the following theorem.

Theorem 2.15. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$, and let functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for all $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = \theta'(n) + 1$ and*

$$\theta(n) \leq \begin{cases} \alpha \log n, & \text{if } \mathcal{W} = \mathcal{V}, \\ n - \alpha \sqrt[\beta]{n}, & \text{if } \mathcal{W} = \mathcal{H}, \end{cases}$$

for some positive $\alpha, \beta \in \mathbb{R}$. Then $\mathcal{W}$ -MIN- $\theta(n)$ -PROJ is $\mathbb{NP}$ -hard for k -fold prisms over simplices.

For general polytopes Lemma 2.5 allows us again to extend the results of [DF] and [Kh] to show that MIN- $\theta(n)$ -PROJ is $\#\mathbb{P}$ -hard.

Theorem 2.16. *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{H}\}$, and let functions $\theta, \theta': \mathbb{N} \rightarrow \mathbb{N}$ be given such that, for all $n \in \mathbb{N}$, $\theta(\theta'(n) + n) = n$ and*

$$\theta(n) \geq \begin{cases} n - \alpha \log n, & \text{if } \mathcal{W} = \mathcal{V}, \\ \alpha \sqrt[\beta]{n}, & \text{if } \mathcal{W} = \mathcal{H}, \end{cases}$$

for some positive $\alpha, \beta \in \mathbb{R}$. Then the problem $\mathcal{W}$ -MIN- $\theta(n)$ -PROJ is $\#\mathbb{P}$ -hard.

We close this section by determining the computational complexity of the location problem MINSUM introduced in Section 1. The connection between MINSUM and zonotopes was established in [KM], where it was also proved that the problem is in $\mathbb{P}$ when the dimension is fixed. Here we show its intractability when the dimension is part of the input.

Theorem 2.17. *MINSUM is $\mathbb{NP}$ -complete.*

Proof. Let $(n, m, 0, z_1, \dots, z_m)$ be an $\mathcal{S}$ -zonotope. For $i = 1, \dots, m$ let $v_i = z_i$, $v_{m+i} = -z_i$, and set $\omega_1 = \dots = \omega_{2m} = 1$, defining an instance of MINSUM. Then among the optimal hyperplanes for this instance there is one, say $H_{y,0}$, through the origin. Then

$$\sum_{i=1}^{2m} \omega_i \operatorname{dist}(v_i, H_{y,0}) = \frac{2}{\|y\|_2} \sum_{i=1}^m |\langle v_i, y \rangle|.$$

Hence $u = y/\|y\|_2$ is a direction for which the breadth of the zonotope $Z = \sum_{i=1}^m [-1, 1]z_i$ is minimal. Further, u must be orthogonal to a facet of Z whose distance from the origin is precisely $r_n(Z)$. The $\mathbb{NP}$ -hardness of MINSUM follows now from Theorem 2.11.

To see that MINSUM is in $\mathbb{NP}$ let an arbitrary instance $(n, m, v_1, \dots, v_m, \omega_1, \dots, \omega_m, \lambda)$ of MINSUM be given. For any $y \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$, we have

$$\sum_{i=1}^m \omega_i \operatorname{dist}(v_i, H_{y,\gamma}) = \frac{1}{\|y\|_2} \sum_{i=1}^m \omega_i |\gamma - \langle v_i, y \rangle|,$$

a function which is piecewise linear in γ . Therefore, for a given y , the minimum is attained at a vertex of its graph, whence $|\gamma - \langle v_i, y \rangle| = 0$ for at least one index i_0 , say. Let $u = y / \|y\|_2$ and

$$Z_{i_0} = \sum_{\substack{i=1 \\ i \neq i_0}}^m [-\omega_i, \omega_i](v_i - v_{i_0}).$$

Then we have

$$\frac{1}{\|y\|_2} \sum_{i=1}^m \omega_i |\gamma - \langle v_i, y \rangle| = \sum_{\substack{i=1 \\ i \neq i_0}}^m |\omega_i \langle v_{i_0} - v_i, u \rangle| = \frac{1}{2} b(Z_{i_0}, u),$$

where b denotes the breadth function. Finally note again that a direction u minimizing $b(Z_{i_0}, u)$ is orthogonal to a facet of Z_{i_0} . Thus u is orthogonal to at least $n - 1$ of the vectors $v_i - v_{i_0}$, implying that MINSUM is in $\mathbb{NP}$. $\square$

3. Polynomial-Time Approximations

It is easy to devise approximative algorithms based on polynomial-time approximations of the Löwner–John ellipsoid. In fact, using Theorem 4.6.1 of [GLS] it can be shown that OPTMAX- $\theta(n)$ -PROJ can be solved approximately in polynomial time with relative error $O(n^{3\theta(n)/2})$. Using [KT] this can be improved for $\mathcal{H}$ -polytopes essentially to $O(n^{\theta(n)})$. The same error bound can also be obtained for $\mathcal{V}$ -polytopes by use of the randomized algorithm given in [KLS].

While these error bounds seem rather weak one cannot hope for substantially better bounds when deterministic algorithms are used. In fact, using [BF2] and [BF3] in conjunction with Lemmas 2.2 and 2.5 it follows that a deterministic algorithm that outputs, for a convex body $K \subset \mathbb{R}^n$ given by a strong optimization oracle, estimates $\underline{Q}_k(K)$ and $\overline{Q}_k(K)$, $k \in \{1, \dots, n - 1\}$, such that

$$\underline{Q}_k(K) \leq Q_k(K) \leq \overline{Q}_k(K),$$

that are based on s calls to the oracle must make an error

$$\sup_{K \in \mathcal{K}^n} \frac{\overline{Q}_k(K)}{\underline{Q}_k(K)} \geq \left(\frac{k}{2e \ln s} \right)^{k/2}.$$

Of course, a similar result holds for q_k .

In the following we show, however, that the positive results can be improved for $\mathcal{H}$ -MAX- $(n - 1)$ -PROJ. We utilize again the fact that all relevant information about the volumes of projections on hyperplanes is encoded in the projection polytope.

As was pointed out before, each maximizing direction is parallel to a vertex of Z_P while minimizing directions are orthogonal to a facet of Z_P . Hence there are finite algorithms for finding optimum $(n - 1)$ -shadows for polytopes when the m scaled normals are known. The time complexity of the algorithms is $O(m^n)$ for both the minimum and the maximum problem [MS1]. In fixed dimension, the problem of computing the scaled normals of a proper ($\mathcal{W}$ - or $\mathcal{H}$ -) polytope is also in $\mathbb{P}$; see [GK3] for a survey on volume computation.

In trying to extend this result to the case of the dimension being part of the input, we encounter major obstacles. The computation of the scaled normals of a proper polytope is $\#\mathbb{P}$ -hard, and even if they are available, the problems of computing the circumradius and the inradius of a zonotope are both still $\mathbb{NP}$ -hard, see [BGKvL] and Section 2. Further, note that the binary size of the volume of $\mathcal{H}$ -polytopes is, in general, not bounded by a polynomial in their size [La]. By Lemmas 2.2 and 2.5 the same is true in general for maximum and minimum shadows of $\mathcal{H}$ -polytopes.

In order to devise approximative algorithms suppose first that we have access to an oracle $\mathcal{O}_\rho$, which, for a given ($\mathcal{V}$ - or $\mathcal{H}$ -) polytope in $\mathbb{R}^n$, returns its n -dimensional volume with a positive relative error of ρ . Later we examine how such oracles can be obtained.

When an approximative volume oracle is available and the facet normals of the given polytope are known, vectors close to the scaled normals can be used to present an approximation of the projection polytope. The next lemma gives an estimate of how the initial error affects accuracy of the corresponding approximation of the circumradius of the projection polytope.

Lemma 3.1. *Let $m \in \mathbb{N}$, let $z_1, \dots, z_m, z'_1, \dots, z'_m \in \mathbb{R}^n \setminus \{0\}$, and set $Z = \sum_{i=1}^m [-1, 1]z_i$, $Z' = \sum_{i=1}^m [-1, 1]z'_i$. Further, let $\rho > 0$ and suppose that, for $i = 1, \dots, m$,*

$$\|z_i - z'_i\|_2 \leq \rho \min\{\|z_i\|_2, \|z'_i\|_2\}.$$

Then

$$Z \subset Z' + \rho\sqrt{m}R_n(Z)\mathbb{B}^n \quad \text{and} \quad Z' \subset Z + \rho\sqrt{m}R_n(Z)\mathbb{B}^n.$$

In particular, $R_n(Z')$ approximates $R_n(Z)$ with a relative error of $\sqrt{m}\rho$.

Proof. For all $x \in \mathbb{R}^n$, we have the following estimates for the support function:

$$\begin{aligned} h(Z, x) &= \sum_{i=1}^m |\langle z_i, x \rangle| \leq \sum_{i=1}^m |\langle z'_i, x \rangle| + \sum_{i=1}^m |\langle z_i - z'_i, x \rangle| \\ &\leq h(Z', x) + \sum_{i=1}^m \|z_i - z'_i\|_2 \|x\|_2 \leq h(Z', x) + \rho \|x\|_2 \sum_{i=1}^m \|z_i\|_2 \\ &\leq h(Z', x) + \rho\sqrt{m} \left(\sum_{i=1}^m \|z_i\|_2^2 \right)^{1/2} \|x\|_2. \end{aligned}$$

Since $R_n^2(\sum_{i=1}^m [-1, 1]z_i) \geq R_n^2(\sum_{i=1}^{m-1} [-1, 1]z_i) + \|z_m\|_2^2$, induction on m shows that $R_n^2(Z) \geq \sum_{i=1}^m \|z_i\|_2^2$, and therefore

$$h(Z, x) \leq h(Z', x) + \rho\sqrt{m}R_n(Z)\|x\|_2 = h(Z' + \rho\sqrt{m}R_n(Z)\mathbb{B}^n, x),$$

for all $x \in \mathbb{R}^n$. Hence $Z \subset Z' + \rho\sqrt{m}R_n(Z)\mathbb{B}^n$. The other inclusion follows by reversing the roles of Z and Z' . $\square$

Since circumradius computation is $\mathbb{NP}$ -hard we need to resort to a second approximation now. Clearly, one can approximate the Euclidean circumradius of a zonotope

$Z = \sum_{i=1}^m [-1, 1]z_i \subset \mathbb{R}^n$ with relative error of $\sqrt{n}$ by using the maximum norm instead, i.e., by computing

$$\max_{x \in Z} \|x\|_\infty = \max_{i=1, \dots, n} \max_{x \in Z} |\langle e_i, x \rangle| = \max_{i=1, \dots, n} \sum_{j=1}^m |\langle e_i, z_j \rangle|.$$

The error can be improved to $O(\sqrt{n/\ln n})$ (and this is actually best possible in the oracle model of the algorithmic theory of convex bodies even for randomized algorithms); see [BGK⁺1] and [BGK⁺2].

It is now easy to combine the two steps—computation of the scaled normals and determination of the circumradius of the projection polytope—to obtain an approximative algorithm for $(n-1)$ -shadow volume maximization.

Theorem 3.2. *Suppose that an oracle $\mathcal{O}_\rho$ is available which computes the volume for $\mathcal{H}$ -polytopes up to a positive relative error of ρ . Then there is a polynomial time algorithm which solves $\mathcal{H}$ -OPTMAX- $(n-1)$ -PROJ up to a relative error of $O(\rho\sqrt{m} + \sqrt{n/\ln n} + \rho\sqrt{mn/\ln n})$, that depends only on ρ , the dimension n , and the number m of facets.*

The volume of a proper convex body $K \subset \mathbb{R}^n$ given by a strong optimization oracle can be approximated in oracle polynomial time up to a relative error of $n!$, see [BH]. Hence we obtain the following corollary.

Corollary 3.3. *There exists a polynomial time algorithm for $\mathcal{H}$ -OPTMAX- $(n-1)$ -PROJ which approximates the optimal solution by a relative error of $O((n-1)!\sqrt{mn/\ln n})$.*

The inradius of a zonotope is more sensitive to changes in the generating vectors than the circumradius. The reason is that large generating vectors can still result in a small inradius. Hence for $\mathcal{H}$ -MIN- $(n-1)$ -PROJ we can only derive a much weaker result that way.

While any deterministic polynomial time algorithm for volume computation produces an error that is at least exponential, we can bring down the approximation error by utilizing a randomized volume algorithm which approximates the volume of an $\mathcal{H}$ -polytope up to a positive relative error ε with probability $1 - \beta \in [0, 1]$, in time that is polynomial in ε , in the size of the input and in the size of β . (Note that ε does not enter the running time logarithmically.) The currently best running time is achieved by an algorithm of [KLS]

$$O(n^5 \varepsilon^{-2} (\ln n)^5 \ln(1/\varepsilon)^3 \ln(1/\beta)).$$

For the approximation of the scaled normals of some proper polytope P we need m volume computations where m is the number of facets of P . To guarantee an error probability of at most $\beta \in [0, 1]$ we have to perform each volume computation with an error probability not greater than β/m because, by Bernoulli's inequality, $1 - (1 - \beta/m)^m \leq 1 - (1 - \beta) = \beta$. With the aid of Theorem 3.2 we obtain the following result.

Theorem 3.4. *There is a randomized algorithm which solves $\mathcal{H}$ -OPTMAX- $(n - 1)$ -PROJ for $\mathcal{H}$ -polytopes in $\mathbb{R}^n$ with m facets with probability $1 - \beta$, $\beta \in (0, 1]$, up to a relative error of $O(\varepsilon\sqrt{m} + \sqrt{n/\ln n} + \varepsilon\sqrt{mn/\ln n})$ in running time polynomial in n , m , $1/\varepsilon$, and $\ln(1/\beta)$.*

In order to apply the above approach to $\mathcal{V}$ -polytopes we would need access to their facet normals. However, as their number may be exponential in the size of the input the question of efficient randomized algorithms for $\mathcal{V}$ -OPTMAX- $(n - 1)$ -PROJ remains open.

Not much is known for optimal projections on subspaces of dimension and codimension different from 1. The reason is the lack of a suitable geometric characterization of minimal and maximal projections. It is, of course, natural to try to extend the positive results for $(n - 1)$ -shadow volumes by proceeding inductively, i.e., finding a hyperplane H first such that $\sigma(P, H)$ is optimal for a given polytope, then a projection of $\Pi(P; H)$ onto an optimal hyperplane in H , etc. We close this section by showing that one has to accept additional errors when following this strategy.

Theorem 3.5. *Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n - 1$. There exist polytopes P and Q in $\mathbb{R}^n$ with the following properties. Let U, V be subspaces of dimension k such that $\sigma(P, U) = q_k(P)$ and $\sigma(Q, V) = q_k(Q)$, and set $P' = \Pi(P; U)$ and $Q' = \Pi(Q; V)$. Then*

$$q_l(P') > q_l(P) \quad \text{and} \quad q_l(Q') < q_l(Q) \quad (8)$$

for any $l = 1, \dots, k - 1$.

Proof. It suffices to consider the case $k = n - 1$. We only give a proof for the shadow minimization; the assertion for shadow maximization follows similarly (with the prism replaced by suitable double pyramids); see [Bu].

Let $S \subset \mathbb{R}^{n-1} \times \{0\}$ be a polytope, and define the prism $T = S + [0, \eta]e_n$ for some positive η . Suppose that $\sigma(T, U)$ is minimal, and let $u \in \mathbb{S}^{n-1}$ such that $U = u^\perp$. Then at least $n - 1$ linearly independent outer normal vectors of T are orthogonal to u . If one of the corresponding facets is $S \times \{0\}$, then $u \perp e_n$ and therefore $q_{n-1}(T) = \eta q_{n-2}(S)$. Otherwise $u = \pm e_n$, whence $q_{n-1}(T) = \text{vol}_{n-1}(S)$. By considering a sequence of polytopes $S_j \subset \mathbb{R}^{n-1} \times \{0\}$ converging to $\mathbb{B}^{n-1} \times \{0\}$, it follows by the continuity of q_k that, for the cylinder $K_\eta = \mathbb{B}^{n-1} + [0, \eta]e_n$

$$q_{n-1}(K_\eta) = \begin{cases} \eta \kappa_{n-2}, & \text{if } \eta < \kappa_{n-1}/\kappa_{n-2}, \\ \kappa_{n-1}, & \text{otherwise.} \end{cases}$$

Moreover, if $\eta > \kappa_{n-1}/\kappa_{n-2}$, then the optimal subspace is $U = e_n^\perp$, and the minimum projection is the $(n - 1)$ -dimensional unit ball. On the other hand, for $l = 1, \dots, n - 2$,

$$q_l(K_\eta) \leq \begin{cases} \eta \kappa_{l-1}, & \text{if } \eta < \kappa_l/\kappa_{l-1}, \\ \kappa_l, & \text{otherwise.} \end{cases}$$

Therefore, if $\eta \in (\kappa_{n-1}/\kappa_{n-2}, \kappa_l/\kappa_{l-1})$, then

$$q_l(K_\eta) \leq \eta \kappa_{l-1} < \kappa_l = q_l(\mathbb{B}^{n-1}) = q_l(\Pi(K_\eta; U)).$$

An appropriate η always exists, since the sequence $(\kappa_n/\kappa_{n-1})_{n \in \mathbb{N}}$ is strictly monotonously decreasing. Thus $K = K_\eta$ satisfies (8) for η as described. Setting $P = S_j + [0, \eta]e_n$ for sufficiently large j yields a polytope that satisfies the assertion. $\square$

References

- [BF1] T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, vol. 3, Erg. d. Math. u. ihrer Grenzgebiete, no. 1, Julius Springer, Berlin, 1934. Corrected reprint: Springer-Verlag, Berlin, 1974. Translation: *Theory of Convex Bodies*, BCS Associates, Moscow, ID, 1987.
- [BF2] I. Bárány and Z. Füredi, Computing the volume is difficult, *Proc. 18th Symp. Theory Comput.*, Assoc. Comput. Mach., New York, 1986, pp. 442–447.
- [BF3] I. Bárány and Z. Füredi, Computing the volume is difficult, *Discrete Comput. Geom.* **2** (1987), 319–326.
- [BGK] T. Burger, P. Gritzmann, and V. Klee, Polytope projection and projection polytopes, *Amer. Math. Monthly* **103** (1996), 742–755.
- [BGK⁺1] A. Brieden, P. Gritzmann, R. Kannan, V. Klee, L. Lovász, and M. Simonovits, Approximation of diameters: randomization doesn't help, *Proc. 39th IEEE Symp. Found. Comput. Sci. (FOCS '98)*, 1998, pp. 244–251.
- [BGK⁺2] A. Brieden, P. Gritzmann, R. Kannan, V. Klee, L. Lovász, and M. Simonovits, Deterministic and randomized polynomial-time approximation of radii, Preprint, 1999.
- [BGKvL] H.L. Bodlaender, P. Gritzmann, V. Klee, and J. van Leeuwen, Computational complexity of norm-maximization, *Combinatorica* **10**(2) (1990), 203–225.
- [BH] U. Betke and M. Henk, Approximating the volume of convex bodies, *Discrete Comput. Geom.* **10** (1) (1993), 15–21.
- [Bu] T. Burger, Optimal orthogonal projections, Dissertation, Universität Trier, 1997.
- [BZ] Y.D. Burago and V.A. Zalgaller, *Geometric Inequalities*, Grundle. d. math. Wissenschaften, vol. 285, Springer-Verlag, Berlin, 1988. Russian original: *Geometricheskie Neravenstva*, “Nauka”, Leningradskoe Otdelenie, Leningrad, 1980.
- [DF] M.E. Dyer and A.M. Frieze, On the complexity of computing the volume of a polyhedron, *SIAM J. Comput.* **17** (1988), 967–974.
- [DFK] M.E. Dyer, A.M. Frieze, and R. Kannan, A random polynomial-time algorithm for estimating volume of convex bodies, *Proc. 21st Symp. Theory Comput.*, Assoc. Comput. Mach., New York, 1989, pp. 375–381.
- [GJ] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, CA, 1979.
- [GK1] P. Gritzmann and V. Klee, Deciding uniqueness in norm-maximization, *Math. Programming* **57** (1992), 203–214.
- [GK2] P. Gritzmann and V. Klee, Computational complexity of inner and outer j -radii of polytopes in finite-dimensional normed spaces, *Math. Programming* **59** (1993), 163–213.
- [GK3] P. Gritzmann and V. Klee, On the complexity of some basic problems in computational convexity: II. Volume and mixed volumes, in *Polytopes: Abstract, Convex and Computational* (T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivić Weiss, eds.), NATO ASI Ser., Ser. C, Math. Phys. Sci., vol. 440, Kluwer, Dordrecht, 1994, pp. 373–466.
- [GLS] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, Berlin, 1988, 2nd edn., 1993.
- [Gr] B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, London, 1967.
- [Kh] L.G. Khachiyan, Complexity of polytope volume computation, in *New Trends in Discrete and Computational Geometry* (J. Pach, ed.), Springer-Verlag, Berlin, 1993, pp. 91–101.
- [KLS] R. Kannan, L. Lovász, and M. Simonovits, Random walks and an $O^*(n^5)$ volume algorithm for convex bodies, *Random Structures Algorithms* **11** (1997), 1–50.
- [KM] N.M. Korneenko and H. Martini, Hyperplane approximation and related topics, in *New Trends in Discrete and Computational Geometry* (J. Pach, ed.), Springer-Verlag, Berlin, 1993, pp. 135–161.

- [KT] L.G. Khachiyan and M.J. Todd, The problem of computing the volume of polytopes is enumerably hard, *Math. Programming* **61** (1993), 137–159.
- [La] J. Lawrence, Polytope volume computation, *Math. Comp.* **57**(195) (1991), 259–271.
- [Ma] H. Martini, Some results and problems around zonotopes, in *Intuitive Geometry* (K. Böröczky and G. Fejes Tóth, eds.), 48. Coll. Math. Soc. János Bolyai, Pap. Int. Conf. Siófok (Hungary), 1985, North-Holland, Amsterdam, 1985, pp. 383–418.
- [MS1] M. McKenna and R. Seidel, Finding the optimal shadow of a convex polytope, *Proc. 1st Symp. Comput. Geom.*, Assoc. Comput. Mach., New York, 1985, pp. 24–28.
- [MS2] H. Martini and A. Schöbel, Median hyperplanes in normed spaces - a survey, *Discrete Appl. Math.* **89** (1998), 181–195.
- [MS3] H. Martini and A. Schöbel, Median and center hyperplanes in minkowski spaces – a unifying approach, *Discrete Math.*, to appear.
- [MW] H. Martini and B. Weißbach, Zur besten Beleuchtung konvexer Polyeder, *Beiträge Algebra Geom.* **17** (1984), 151–168.
- [Sc] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.
- [Zi] G.M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, Berlin, 1995.

Received February 17, 1999. Online publication May 3, 2000.