Cardinalities of Primitive Fixing Systems for Convex Bodies

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Abstract. We establish the exact upper bound of cardinalities of primitive fixing systems for any compact, convex body M , depending on $\dim M$ and $\text{md } M$. In a sense, this is the last touch to the theory of fixing systems for compact, convex bodies.

Introduction

Fejes Tóth [10] formulated a problem on *primitive fixing systems* for convex bodies. Namely, let $M \subset R^n$ be a compact, convex body. A set $F \subset \text{bd } M$ is a *fixing system* for M [10] if for any nonzero vector v there is a real number $\lambda > 0$ such that the translate $\lambda v + \text{int } M$ of the interior of M has a point in common with F . A fixing system $F \subset \text{bd } M$ is *primitive* if no proper subset $F' \subset F$ is a fixing system for M . The minimal cardinality of fixing systems for M is denoted here by $\varrho_{\min}(M)$, and the maximal cardinality of primitive fixing systems for M is denoted by $\varrho_{\max}(M)$ (if there are primitive fixing systems for M with any large integer cardinality, we write $\varrho_{\max}(M) = \infty$). We remark that every fixing system of minimal cardinality is primitive.

Fejes Tóth proved [10] that for two-dimensional compact, convex figures the maximal cardinality of primitive fixing systems is equal to 6:

$$3 \leq \varrho_{\min}(M) \leq \varrho_{\max}(M) \leq 6, \quad (1)$$

the upper bound being attained for any hexagon with pairwise parallel opposite sides.

Danzer [9] conjectured that for three-dimensional compact, convex bodies the maximal cardinality of primitive fixing systems is equal to 14 (being attained for affine images of the rhombic dodecahedron). A counterexample was given by Bollobás [1], who constructed three-dimensional convex revolution bodies with primitive fixing systems of any large integer cardinalities. Simpler examples are given in [5] and [12].

For the cardinalities of *minimal* fixing systems the exact lower and upper bounds were found by Grünbaum [11]. Namely, for any compact, convex body $M \subset R^n$ the following

inequalities hold:

$$n + 1 \leq \varrho_{\min}(M) \leq 2n. \quad (2)$$

These estimates are exact, i.e., there exists a body (for example, a ball) for which the lower estimate is attained, and there exists a body (namely, a parallelotope) for which the upper bound is attained.

In [4] and [8] some more detailed estimates are given. To formulate these results, we recall the definition of the maximin dependence functional $\text{md } M$, considered on the family of all compact, convex bodies in R^n and introduced in [2]. Namely, $m + 1$ vectors $a_0, a_1, \dots, a_m \in R^n$ are said to be *minimally dependent* if they are positively dependent (i.e., $\lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_m a_m = 0$ with some positive $\lambda_0, \lambda_1, \dots, \lambda_m$) and also every m of them are linearly independent. Furthermore, for a compact, convex body $M \subset R^n$ denote by $H(M)$ the set of all unit outward normals of M at its regular boundary points (a boundary point is regular if there is only one support hyperplane for M through the point). By $\text{md } M$ [2] we denote the *maximal* integer m such that there are minimally dependent vectors $a_0, a_1, \dots, a_m \in H(M)$. It is proved in [2] and [3] that $\text{md } M$ is equal to the *Helly dimension* (denoted by $\text{him } M$) of the body M , i.e., $\text{md } M$ is the least of the integers m with the following property: if $M_1, \dots, M_q$ are arbitrary translates of M and every $m + 1$ of the translates has a point in common, then $M_1 \cap \dots \cap M_q \neq \emptyset$.

The equality $\text{him } M = 1$ holds if and only if M is a parallelotope [2], [13]. Furthermore, in [3] a complete list of compact, convex bodies with $\text{him } M = 2$ is given. Namely, $\text{him } M = 2$ if and only if M is a *stack*, an *outcut*, a *stack-outcut*, or a four-dimensional *particular polytope*. The definition of the stack (through its polar body) is given in the beginning of the proof of Lemma 2 and the definition of the four-dimensional particular polytope (again through its polar body) is given in the beginning of the proof of Lemma 3. As to the outcuts and stack-outcuts, they are defined analogously to the stacks (we refer the reader to [3] for detailed definitions).

Now we can formulate the results established in [4] and [8]: *For any compact, convex body $M \subset R^n$ the estimates*

$$n + \frac{n}{\text{md } M} \leq \varrho_{\min}(M) \leq 2n + 1 - \text{md } M \quad (3)$$

hold, both the estimates being exact.

For *maximal* cardinalities of primitive fixing systems of any compact, convex body $M \subset R^n$ we only have the estimate

$$\varrho_{\max}(M) \leq \infty \quad (4)$$

that also is exact. This follows from [5] where an n -dimensional, compact, convex body is constructed with $\varrho_{\max}(M) = \infty$ (see also [12]). However, the estimate (4) is not written depending on $\text{md } M$. The problem of finding the estimates for maximal cardinalities of primitive fixing systems depending on $\dim M$ and $\text{md } M$ was open, and this was the last problem from the considered circle of questions. This problem is solved in the next section.

1. Main Theorem

The following theorem contains the exact upper bound of the cardinalities of primitive fixing systems for any compact, convex body M depending on $\dim M$ and $\text{md } M$.

Theorem. *Let $M \subset R^n$ be a compact, convex body. If $\text{md } M = 1$ (i.e., M is an n -dimensional parallelotope), then $\varrho_{\max}(M) = 2n$. If $\text{md } M = 2$, then*

$$\varrho_{\max}(M) \leq \frac{9}{2}n + \alpha_n, \quad (5)$$

where

$$\alpha_n = 0 \quad \text{for } n \equiv 0 \pmod{4}, \quad \alpha_n = -\frac{5}{2} \quad \text{for } n \equiv 1 \pmod{4}, \quad (6)$$

$$\alpha_n = -3 \quad \text{for } n \equiv 2 \pmod{4}, \quad \alpha_n = -\frac{11}{2} \quad \text{for } n \equiv 3 \pmod{4}. \quad (7)$$

If $\text{md } M \geq 3$, then $\varrho_{\max}(M) \leq \infty$.

All the estimates are exact (i.e., there exists a compact, convex body $M \subset R^n$ with the prescribed $\text{md } M$ for which the indicated upper bound is attained).

Thus, together with (3), we obtain the following exact estimates:

$$\varrho_{\min}(M) = \varrho_{\max}(M) = 2n \quad \text{for } \text{md } M = 1,$$

$$\frac{3}{2}n \leq \varrho_{\min}(M) \leq \varrho_{\max}(M) \leq \frac{9}{2}n + \alpha_n \quad \text{for } \text{md } M = 2, \quad (8)$$

$$n + \frac{n}{\text{md } M} \leq \varrho_{\min}(M) \leq \varrho_{\max}(M) \leq \infty \quad \text{for } \text{md } M \geq 3.$$

We remark that for the case $\text{md } M = 2$ not every integer contained between the indicated lower and upper bounds (see (8)) is the cardinality of a primitive fixing system for M (see the remark at the end of the paper).

2. Proof of the Main Theorem

First, we describe the way to find the number $\varrho_{\max}(M)$ with the help of the polar body M^* . Let $M \subset R^n$ be a compact, convex body with the origin in its interior and let M^* be its polar body, i.e.,

$$M^* = \{y \in R^n: \langle y, x \rangle \leq 1 \text{ for all } x \in M\}.$$

Let $\Gamma_1, \dots, \Gamma_k$ be support hyperplanes of M^* and let $B_i = M^* \cap \Gamma_i, i = 1, \dots, k$. We say that $\{\Gamma_1, \dots, \Gamma_k\}$ is a *fixing cosystem* for M^* if for every open half-space $P \subset R^n$ with the boundary hyperplane $\text{bd } P$ through the origin, at least one of the sets $B_1, \dots, B_k$ is completely contained in P . The fixing cosystem $\{\Gamma_1, \dots, \Gamma_k\}$ is *primitive* if no proper subset of $\{\Gamma_1, \dots, \Gamma_k\}$ is a fixing cosystem for M^* . By $\varrho_{\max}^*(M^*)$ we denote the maximal k for which there exists a primitive fixing cosystem $\{\Gamma_1, \dots, \Gamma_k\}$ for M^* (if there are primitive fixing cosystems for M^* with any large integer cardinality, we write $\varrho_{\max}^*(M^*) = \infty$).

Lemma 1. *For any compact, convex body $M \subset R^n$ containing the origin in its interior the equality $\varrho_{\max}^*(M^*) = \varrho_{\max}(M)$ holds.*

Proof. Let $F = \{x_1, \dots, x_k\} \subset \text{bd } M$ be a fixing system for the body M . Denote by $\Gamma_1, \dots, \Gamma_k$ the polar hyperplanes for the points $x_1, \dots, x_k$, i.e.,

$$\Gamma_i = x_i^* = \{y \in R^n: \langle y, x_i \rangle = 1\}, \quad i = 1, \dots, k.$$

Then $\Gamma_1, \dots, \Gamma_k$ are support hyperplanes of the polar body M^* . It is easily shown that $\{\Gamma_1, \dots, \Gamma_k\}$ is a fixing cosystem for M^* . Indeed, let $P \subset R^n$ be an open half-space with the boundary hyperplane through the origin. Denote by l the direction defined by the outward normal p of the half-space P . Since F is a fixing system for the body M , there exists a point $x_i \in F$ that is *illuminated by the direction l* (see Theorem 17 in [4]), i.e., there is $\lambda > 0$ such that the point $x_\lambda = x_i + \lambda p$ belongs to the interior of M . We may suppose (diminishing λ if necessary) that x_λ is distinct from the origin. Let z be a point of the set $B_i = M^* \cap \Gamma_i$. Since $z \in \Gamma_i$, we have

$$\langle z, x_i \rangle = 1. \quad (9)$$

Furthermore, since x_λ is an *interior* point of the body M , the polar hyperplane x_λ^* has no points in common with M^* , i.e., the body M^* is situated in the half-space $\{y: \langle y, x_\lambda \rangle < 1\}$. In particular,

$$\langle z, x_\lambda \rangle < 1. \quad (10)$$

From (9) and (10) we obtain $\langle z, x_\lambda - x_i \rangle < 0$, i.e., $\langle z, \lambda p \rangle < 0$, and hence $\langle z, p \rangle < 0$ (since $\lambda > 0$). This means that $z \in P$. The obtained inclusion holds for any point $z \in B_i$, i.e., $B_i \subset P$. Thus $\{\Gamma_1, \dots, \Gamma_k\}$ is a fixing cosystem for M^* .

Conducting the above reasoning in the opposite direction, we find that, conversely, if $\{\Gamma_1, \dots, \Gamma_k\}$ is a fixing cosystem for M^* , then the corresponding polar points $x_1, \dots, x_k$ (i.e., the points with $x_i^* = \Gamma_i$, $i = 1, \dots, k$) form a fixing system for the body M . This implies that the fixing cosystem $\{\Gamma_1, \dots, \Gamma_k\}$ is primitive if and only if the fixing system $\{x_1, \dots, x_k\}$ is primitive. $\square$

In the following two lemmas we conserve the terminology introduced in Theorems 29.1 and 29.7 of [7].

Lemma 2. *Let $M \subset R^n$ be an n -dimensional stack, outcat, or stack-outcat. Then $\varrho_{\max}(M) \leq 2n$.*

Proof. We investigate in detail the case when M is an n -dimensional stack (the cases of outcat and stack-outcat being similar).

A compact, convex body $M \subset R^n$ is said to be an n -dimensional *stack* if the corresponding polar body M^* is described in the following way (see case (B) of Theorem 29.1 in [7]). There exists a basis $e_1, \dots, e_n$ in R^n and closed, two-dimensional, convex sets

$C_2, \dots, C_n$ such that

- (i) $e_l \in C_l \subset \text{lin}(e_1, e_l)$ and $\text{ext } C_l \subset \{e_1\} \cup \Pi_{l,1}$ for $l = 2, \dots, n$, where $\Pi_{l,1}$ is the half-plane $\{v: v = x_1 e_1 + x_l e_l, x_1 \leq 0\}$;
- (ii) $M^* = \text{conv}(C_2 \cup \dots \cup C_n)$.

For each $l = 2, \dots, n$ denote by F_l^+ the angle $\{v: v = x_1 e_1 + x_l e_l, x_1 \leq 0, x_l > 0\}$, and by F_l^- the angle $\{v: v = x_1 e_1 + x_l e_l, x_1 \leq 0, x_l < 0\}$.

Let $\{\Gamma_1, \dots, \Gamma_k\}$ be a fixing cosystem for M^* and $B_1 = M^* \cap \Gamma_1, \dots, B_k = M^* \cap \Gamma_k$. For arbitrary $l = 2, \dots, n$ denote by $P_l^+ \subset R^n$ the open half-space consisting of all vectors $x_1 e_1 + \dots + x_n e_n$ with $x_l > 0$. Then at least one of the sets $B_1, \dots, B_k$ is contained in P_l^+ . Denote such a set by B_l^+ . Similarly, for arbitrary $l = 2, \dots, n$ denote by $P_l^- \subset R^n$ the open half-space consisting of all vectors $x_1 e_1 + \dots + x_n e_n$ with $x_l < 0$. Then at least one of the sets $B_1, \dots, B_k$ is contained in P_l^- . Denote such a set by B_l^- . We remark that B_l^+ (respectively, B_l^-) is a point or a segment contained in $\text{bd } M^* \cap F_l^+$ (respectively, in $\text{bd } M^* \cap F_l^-$).

Finally, denote by P_1^+ (respectively, P_1^-) the open half-space consisting of all vectors $x_1 e_1 + \dots + x_n e_n$ with $x_1 > 0$ (respectively, $x_1 < 0$). Then at least one of the sets $B_1, \dots, B_k$ is contained in P_1^+ , and it is easily shown that this set B_1^+ coincides with $\{e_1\}$.

Thus we already have $2n - 1$ different sets $B_1^+, B_l^+, B_l^-, l = 2, \dots, n$, taken from $B_1, \dots, B_k$ and hence $k \geq 2n - 1$. Furthermore, let $P \subset R^n$ be an open half-space with the boundary hyperplane through the origin. If $e_1 \notin P$ and $P \neq P_1^-$, then P contains at least one of the angles $F_2^+, \dots, F_n^+, F_2^-, \dots, F_n^-$. Consequently it contains at least one of the sets $B_l^+, B_l^-, l = 2, \dots, n$. If even $e_1 \in P$, then P contains the set $B_1^+ = \{e_1\}$. Thus every open half-space P with $0 \in \text{bd } P$ distinct from P_1^- contains at least one of the sets $B_1^+, B_l^+, B_l^-, l = 2, \dots, n$.

If now the half-space P_1^- contains at least one of the sets $B_l^+, B_l^-, l = 2, \dots, n$, then any open half-space of R^n with the boundary hyperplane through the origin contains at least one of the sets $B_1^+, B_l^+, B_l^-, l = 2, \dots, n$, i.e., we have a fixing cosystem for M^* taken from $\{\Gamma_1, \dots, \Gamma_k\}$ that contains $2n - 1$ hyperplanes. In this case $k = 2n - 1$. Even if the half-space P_1^- does not contain any one of the sets $B_l^+, B_l^-, l = 2, \dots, n$, then it is necessary to add one more of the sets $B_1, \dots, B_k$, i.e., $k = 2n$. Thus in any case $\varrho_{\max}^*(M^*) \leq 2n$.

By Lemma 1, we conclude that $\varrho_{\max}(M) \leq 2n$. □

Remark. In the condition of Lemma 2, the equality $\varrho_{\max}(M) = 2n$ is possible. For example, let M_1 be an n -dimensional cube and let M_2 be the n -dimensional right pyramid whose basis coincides with a facet F of the cube M_1 and the polytopes M_1, M_2 have no interior point in common. Then $\varrho_{\max}(M_1 \cup M_2) = 2n$. Indeed, taking a point in the relative interior of every facet of the cube M_1 except F (having thus $2n - 1$ points) and adding the vertex of the pyramid M_2 that is opposite to its basis, we obtain a primitive fixing system for the stack $M_1 \cup M_2$ that consists of $2n$ points.

Lemma 3. Let $M \subset R^4$ be the particular four-dimensional polytope with $\text{md } M = 2$. Then $\varrho_{\max}(M) = 18$.

Proof. A polytope $M \in R^4$ is said to be a *particular one* if the corresponding polar polytope M^* is described in the following way (see case (D) of Theorem 29.1 in [7]). There is a basis e_1, e_2, e_3, e_4 of the space R^4 such that the polar polytope M^* is the convex hull of the set consisting of the following nine points:

$$\begin{aligned} &e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4, \\ &-e_1 - e_2, -e_1 - e_3, -e_2 - e_4, -e_3 - e_4. \end{aligned}$$

In the coordinate system with the basis e_1, e_2, e_3, e_4 , consider the following nine hyperplanes in R^4 :

$$\begin{aligned} \Delta_1 &= \{x: x_1 = 0\}, & \Delta_2 &= \{x: x_2 = 0\}, \\ \Delta_3 &= \{x: x_3 = 0\}, & \Delta_4 &= \{x: x_4 = 0\}, \\ \Delta_5 &= \{x: x_1 - x_2 = 0\}, & \Delta_6 &= \{x: x_1 - x_3 = 0\}, & \Delta_7 &= \{x: x_2 - x_4 = 0\}, \\ \Delta_8 &= \{x: x_3 - x_4 = 0\}, & \Delta_9 &= \{x: x_1 - x_2 - x_3 + x_4 = 0\}. \end{aligned}$$

Each of these hyperplanes defines two half-spaces. Thus we obtain the following 18 open half-spaces:

$$\begin{aligned} P_1^+ &= \{x: x_1 > 0\}, & P_2^+ &= \{x: x_2 > 0\}, \\ P_3^+ &= \{x: x_3 > 0\}, & P_4^+ &= \{x: x_4 > 0\}, \\ P_5^+ &= \{x: x_1 - x_2 > 0\}, & P_6^+ &= \{x: x_1 - x_3 > 0\}, & P_7^+ &= \{x: x_2 - x_4 > 0\}, \\ P_8^+ &= \{x: x_3 - x_4 > 0\}, & P_9^+ &= \{x: x_1 - x_2 - x_3 + x_4 > 0\}, \\ P_1^- &= \{x: x_1 < 0\}, & P_2^- &= \{x: x_2 < 0\}, \\ P_3^- &= \{x: x_3 < 0\}, & P_4^- &= \{x: x_4 < 0\}, \\ P_5^- &= \{x: x_1 - x_2 < 0\}, & P_6^- &= \{x: x_1 - x_3 < 0\}, & P_7^- &= \{x: x_2 - x_4 < 0\}, \\ P_8^- &= \{x: x_3 - x_4 < 0\}, & P_9^- &= \{x: x_1 - x_2 - x_3 + x_4 < 0\}. \end{aligned}$$

Every hyperplane Δ_j , $j = 1, \dots, 9$, contains five vertices of the polytope M^* . For example, the hyperplane Δ_1 contains the vertices $e_2, e_3, e_4, -e_2 - e_4, -e_3 - e_4$. Furthermore, each of the half-spaces P_j^+ and P_j^- , $j = 1, \dots, 9$, contains exactly two vertices of M^* . For example, the half-space P_1^+ contains the vertices e_1 and $e_1 + e_2 + e_3 + e_4$ which are situated in the support hyperplane $\{x: x_1 = 1\}$ of the polytope M^* . Consequently, P_1^+ contains the edge I_1^+ of M^* with the endpoints e_1 and $e_1 + e_2 + e_3 + e_4$.

Similarly, every half-space P_j^+ (respectively, P_j^-), $j = 1, \dots, 9$, contains one edge I_j^+ (respectively, I_j^-) of the polytope M^* . Thus we obtain the following 18 edges of the polytope M^* :

$$\begin{aligned} I_1^+ &= [e_1, e_1 + e_2 + e_3 + e_4], & I_1^- &= [-e_1 - e_2, -e_1 - e_3], \\ I_2^+ &= [e_2, e_1 + e_2 + e_3 + e_4], & I_2^- &= [-e_1 - e_2, -e_2 - e_4], \\ I_3^+ &= [e_3, e_1 + e_2 + e_3 + e_4], & I_3^- &= [-e_1 - e_3, -e_3 - e_4], \\ I_4^+ &= [e_4, e_1 + e_2 + e_3 + e_4], & I_4^- &= [-e_2 - e_4, -e_3 - e_4], \\ I_5^+ &= [e_1, -e_2 - e_4], & I_5^- &= [e_2, -e_1 - e_3], \\ I_6^+ &= [e_1, -e_3 - e_4], & I_6^- &= [e_3, -e_1 - e_2], \\ I_7^+ &= [e_2, -e_3 - e_4], & I_7^- &= [e_4, -e_1 - e_2], \\ I_8^+ &= [e_3, -e_2 - e_4], & I_8^- &= [e_4, -e_1 - e_3], \\ I_9^+ &= [e_1, e_4], & I_9^- &= [e_2, e_3]. \end{aligned}$$

It is easily shown that every open half-space P with the boundary hyperplane through the origin contains at least one of the indicated 18 segments. Indeed, let $P = \{x: a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 > 0\}$ be an open half-space in R^4 (at least one of the numbers a_1, a_2, a_3, a_4 being distinct from 0). If $a_1 + a_2 + a_3 + a_4 > 0$, then at least one of the coefficients a_1, a_2, a_3, a_4 is positive. This implies that at least one of the segments $I_1^+, I_2^+, I_3^+, I_4^+$ is contained in P . Let now $a_1 + a_2 + a_3 + a_4 \leq 0$. If a_1 is the only positive coefficient, then at least one of the segments I_5^+, I_6^+ is contained in P . In general, if only one of the coefficients a_1, a_2, a_3, a_4 is positive, then at least one of the segments $I_j^+, I_j^-, j = 5, 6, 7, 8$, is situated in P . Furthermore, if a_1 and a_2 are the only positive coefficients, then $a_3 + a_4 < 0$ and hence $I_6^+ \subset P$. More general, if there are two or three positive coefficients (under the condition $a_1 + a_2 + a_3 + a_4 \leq 0$), then at least one of the segments $I_j^+, I_j^-, j = 5, \dots, 9$, is contained in P . If now all the coefficients are nonpositive, then at least one of the segments $I_j^-, j = 1, 2, 3, 4$, is contained in P . Thus in any case at least one of the segments $I_j^+, I_j^-, j = 1, \dots, 9$, is situated in P .

Let Σ_j^+ (respectively, Σ_j^-) be a support hyperplane of the body M^* with $\Sigma_j^+ \cap M^* = I_j^+$ (respectively, $\Sigma_j^- \cap M^* = I_j^-$), $j = 1, \dots, 9$. Then the hyperplanes $\Sigma_j^+, \Sigma_j^-, j = 1, \dots, 9$, form a fixing cosystem for M^* (since every open half-space P with the boundary hyperplane through the origin contains at least one of the indicated 18 segments). Moreover, this fixing cosystem is primitive (since for each one of the 18 segments there is a half-space that contains only the chosen segment). Thus $\varrho_{\max}^*(M^*) \geq 18$.

Now we establish the opposite inequality. Let $\Gamma_1, \dots, \Gamma_k$ be a fixing cosystem for M^* and let $B_i = M^* \cap \Gamma_i$, $i = 1, \dots, k$. Then every open half-space P with the boundary hyperplane through the origin contains at least one of the sets $B_1, \dots, B_k$. Hence at least one of the sets $B_1, \dots, B_k$ is contained in the half-space P_j^+ and at least one in P_j^- , $j = 1, \dots, 9$. Denote such a set by B_j^+ (respectively, B_j^-). Since the segment I_j^+ (respectively I_j^-) is the unique face of the polytope M^* in the half-space P_j^+ (respectively, P_j^-), we have $B_j^+ \subset I_j^+$ (respectively, $B_j^- \subset I_j^-$), $j = 1, \dots, 9$. The corresponding hyperplane (taken from $\Gamma_1, \dots, \Gamma_k$) denote by Γ_j^+ (respectively, Γ_j^-), $j = 1, \dots, 9$.

Thus we have 18 sets $B_j^+, B_j^-, j = 1, \dots, 9$, taken from $B_1, \dots, B_k$ (it is possible that some of them are coincident, and hence we have less than 18 hyperplanes Γ_j^+, Γ_j^-). The hyperplanes $\Gamma_j^+, \Gamma_j^-, j = 1, \dots, 9$, form a fixing cosystem for the body M^* (since every open half-space P with the boundary hyperplane through the origin contains at least one of the above 18 segments and hence contains at least one of the sets B_j^+, B_j^- , $j = 1, \dots, 9$). Thus no more than 18 hyperplanes taken from $\Gamma_1, \dots, \Gamma_k$ form a fixing cosystem for M^* , and hence $k \leq 18$ (by primitivity of the cosystem $\{\Gamma_1, \dots, \Gamma_k\}$), i.e., $\varrho_{\max}^*(M^*) \leq 18$. Consequently, $\varrho_{\max}^*(M^*) = 18$.

By Lemma 1, we conclude that $\varrho_{\max}(M) = 18$. □

Now we are able to prove the main theorem. For $\text{md } M = 1$ (i.e., for the case when M is an n -dimensional parallelotope) any primitive fixing system consists of $2n$ points situated in relative interiors of the facets of the parallelotope (each facet containing in its relative interior only one point of the fixing system).

Consider now the case $\text{md } M = 3$ and denote by $M \subset R^n$ the body $M_3 \oplus I \oplus \cdots \oplus I$, where M_3 is a three-dimensional body that is the convex hull of a ball and one of its external points, whereas each summand I is a segment (and the number of the summands I is equal to $n - 3$). Then M is a compact, convex body with $\text{md } M = 3$ and $\dim M = n$. At the same time $\varrho_{\max}(M) = \infty$ (see [5] or [12]), i.e., the upper bound indicated in the main theorem is attained. A similar example may be constructed for any $\text{md } M \geq 3$.

To finish the proof, it remains to establish that the estimates, indicated in the main theorem for the case $\text{md } M = 2$, indeed, are the exact upper bounds. Let $M \subset R^n$ be a compact, convex body with $\text{md } M = 2$ and let $M = M_1 \oplus \cdots \oplus M_k$ be its decomposition into the direct vector sum of indecomposable compact, convex sets. Then

$$\varrho_{\max}(M) = \varrho_{\max}(M_1) + \cdots + \varrho_{\max}(M_k) \quad (11)$$

(see the proof of Theorem 21 in [4]). Moreover, for each set M_i , $i = 1, \dots, k$, the following alternative opportunities are possible:

- (a) $\dim M_i = 1$, i.e., M_i is a segment and hence $\varrho_{\max}(M_i) = 2$.
- (b) $\dim M_i = 2$, i.e., M_i is a two-dimensional convex set distinct from a parallelogram; in this case $\varrho_{\max}(M_i) \leq 6$ (see (1)).
- (c) $\dim M_i \geq 3$, and M_i is a stack, outcut, or stack-outcut. In this case $\varrho_{\max}(M_i) \leq 2 \dim M_i$ (see Lemma 2).
- (d) M_i is the special four-dimensional polytope (see Theorem 29.7 in [7]). In this case $\varrho_{\max}(M_i) = 18$ (by Lemma 3).

If for an index i case (b) is realized with $\varrho_{\max}(M_i) \leq 5$, then we may replace M_i by a hexagon H with pairwise parallel opposite sides, increasing the number (11). Thus to obtain the maximal value of (11), we have to suppose that every M_i with $\dim M_i = 2$ is a hexagon with pairwise parallel opposite sides, and hence $\varrho_{\max}(M_i) = 6$.

By (11), we now have

$$\varrho_{\max}(M) = 2q_a + 6q_b + 2D + 18q_d, \quad (12)$$

where q_a, q_b, q_d are the numbers of the summands in (11) for which, respectively, cases (a), (b), (d) are realized and D is the sum of dimensions of the summands M_i for which case (c) holds.

Assume that for an index i the summand M_i is an n_i -dimensional stack, outcut, or stack-outcut ($n_i \geq 3$). Then we may replace it by $M'_i = H \oplus I \oplus \cdots \oplus I$ where H is a hexagon with pairwise parallel opposite sides and the number of the segments I is equal to $n_i - 2$. By this replacement, the number (12) increases, since $\varrho_{\max}(M'_i) = 2n_i + 2 > \varrho_{\max}(M_i)$. This means that to obtain the upper bound it is necessary to suppose that there is no summand M_i for which case (c) holds, i.e., $D = 0$.

We see that to obtain the upper bound the body M has to be the direct sum of the following form:

$$M = I \oplus \cdots \oplus I \oplus H \oplus \cdots \oplus H \oplus \Pi \oplus \cdots \oplus \Pi,$$

where Π is the special four-dimensional polytope. Now we notice that if the number of the summands H is greater than 1, then we may replace $H \oplus H$ by Π , increasing the

number (12). Similarly, if the number of the summands I is greater than 1, then we may replace $I \oplus I$ by H , again increasing the number (12).

Consequently, for attaining the upper bound only the following four cases are possible:

$$\begin{aligned} M &= \Pi \oplus \cdots \oplus \Pi, \\ M &= I \oplus \Pi \oplus \cdots \oplus \Pi, \\ M &= H \oplus \Pi \oplus \cdots \oplus \Pi, \\ M &= H \oplus I \oplus \Pi \oplus \cdots \oplus \Pi. \end{aligned} \tag{13}$$

These cases correspond, respectively, to the possibilities

$$\begin{aligned} \dim M &\equiv 0 \pmod{4}, & \dim M &\equiv 1 \pmod{4}, \\ \dim M &\equiv 2 \pmod{4}, & \dim M &\equiv 3 \pmod{4} \end{aligned}$$

and give the equalities (1)–(3). For example, in the case $n = \dim M = 4p + 3$ (see (13)), we have

$$\varrho_{\max}(M) = 18p + 8 = 18 \left(\frac{n-3}{4} \right) + 8 = \frac{9}{2}n - \frac{11}{2},$$

accordingly the second formula (7). The other cases are considered analogously. $\square$

Remark. Combining the main theorem with inequality (3), we obtain the following corollary: *Let $M \subset \mathbb{R}^n$ be a compact, convex body with $\text{md } M = 2$. Then the cardinality of any primitive fixing system $F \subset \text{bd } M$ satisfies the inequalities*

$$\frac{3n}{2} \leq \text{card } F \leq \frac{9}{2}n + \alpha_n,$$

α_n being defined by (6) and (7). We remark that not every integer satisfying these inequalities can be the cardinality of a primitive fixing system. For example, if $\dim M = 4$ and $\text{md } M = 2$, then only the integers 3, ..., 13, 15, and 18 can be the cardinalities of primitive fixing systems, since from the proof of Lemma 3 it is possible to conclude that in the case $M = \Pi$ there is no primitive fixing system of cardinality 14, 16, or 17. Indeed, if instead of a segment I_j^+ or I_j^- we consider one of its endpoints (as the set B_j^+ or B_j^-), then the cardinality of the corresponding primitive fixing system is no greater than 15; for example, if we replace the segment I_1^+ by its endpoint e_1 , then, by primitivity, the segments I_5^+ , I_6^+ , and I_9^+ can be thrown out, i.e., we obtain the cardinality 15 (instead of 18).

Acknowledgment

The authors thank Professor Branko Grünbaum for kindness, attention, scientific conversations and, in connection with his birthday, wish him health, happiness, and success during many years.

References

1. Bollobás, B. Fixing systems for convex bodies. *Studia Sci. Math. Hungar.* **2** (1967), 351–354.
2. Boltyanski, V.G. Helly's theorem for H -convex sets (in Russian). *Dokl. Akad. Nauk SSSR* **226**(2) (1976), 249–252. (See also *Soviet. Math. Dokl.* **17**(1) (1976), 78–81).
3. Boltyanski, V.G. A new step in the solution of the Szökefalvi-Nagy problem. *Discrete Comput. Geom.* **8** (1992), 27–49.
4. Boltyanski, V., and Martini, H. Combinatorial geometry of belt bodies. *Results Math.* **28** (1995), 224–249.
5. Boltyanski, V., and Martini, H. On maximal primitive fixing systems. *Beiträge Algebra Geom.* **37**(1) (1996), 199–207.
6. Boltyanski, V., and Martini, H. Fixing and hindering systems in combinatorial geometry. *Beiträge Algebra Geom.* **40**(2) (1999), 551–563.
7. Boltyanski, V., Martini, H., and Soltan, P. *Excursions into Combinatorial Geometry*. Springer-Verlag, New York, 1997. xiii + 419 pp.
8. Boltyanski, V.G., and Morales Amaya, E. Minimal fixing systems for convex bodies. *J. Appl. Anal.* **1**(1) (1995), 1–11.
9. Danzer, L. *Math. Reviews* **2942** (1963), 569–570.
10. Fejes Tóth, L. On primitive polyhedra. *Acta Math. Acad. Sci. Hungar.* **13** (1952), 379–382.
11. Grünbaum, B. Fixing systems and inner illumination. *Acta Math. Acad. Sci. Hungar.* **15** (1964), 161–168.
12. Morales Amaya, E. A note on maximal primitive fixing systems for convex bodies. *Note Mat.*, to appear.
13. Szökefalvi-Nagy, B. Ein Satz über Parallelverschiebungen konvexer Körper. *Acta Sci. Math.* **15**(3-4) (1954), 169–177.

Received October 14, 1998, and in revised form May 5, 1999. Online publication May 3, 2000.