Face Numbers of Scarf Complexes*

A. Björner

Department of Mathematics, Royal Institute of Technology,
S-100 44 Stockholm, Sweden
bjorner@math.kth.se

Abstract. Let A be a $(d + 1) \times d$ real matrix whose row vectors positively span $\mathbb{R}^d$ and which is generic in the sense of Bárány and Scarf [BS1]. Such a matrix determines a certain infinite d -dimensional simplicial complex Σ , as described by Bárány et al. [BHS]. The group $\mathbb{Z}^d$ acts on Σ with finitely many orbits. Let f_i be the number of orbits of $(i + 1)$ -simplices of Σ . The sequence $f = (f_0, f_1, \dots, f_{d-1})$ is the f -vector of a certain triangulated $(d - 1)$ -ball T embedded in Σ . When A has integer entries it is also, as shown by the work of Peeva and Sturmfels [PS], the sequence of Betti numbers of the minimal free resolution of $k[x_1, \dots, x_{d+1}]/I$, where I is the lattice ideal determined by A .

In this paper we study relations among the numbers f_i . It is shown that $f_0, f_1, \dots, f_{\lfloor (d-3)/2 \rfloor}$ determine the other numbers via linear relations, and that there are additional nonlinear relations. In more precise (and more technical) terms, our analysis shows that f is linearly determined by a certain M -sequence $(g_0, g_1, \dots, g_{\lfloor (d-1)/2 \rfloor})$, namely, the g -vector of the $(d - 2)$ -sphere bounding T . Although T is in general not a cone over its boundary, it turns out that its f -vector behaves as if it were.

1. Introduction

A construction appearing in the work of Scarf and coauthors [Sc1], [BHS], [BS1] shows a way to associate with sufficiently generic real $(d + 1) \times d$ matrices A a certain abstract simplicial complex Σ . This “big Scarf complex” is infinite, with $\mathbb{Z}^d$ as its set of vertices and with the group $\mathbb{Z}^d$ acting on it. It is however locally finite, and the choice of one of A ’s row vectors as privileged determines a certain finite subcomplex T of the link of Σ at the origin. This “small Scarf complex” is a triangulation of a $(d - 1)$ -dimensional ball, and its faces are in bijection with the orbits of the $\mathbb{Z}^d$ -action on Σ .

* Anders Björner was partially supported by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine, and by a KTH–Yale Collaboration Grant.

Let f_i be the number of i -dimensional simplices of the complex T . It was empirically observed by Scarf that all the face numbers f_i seem to be determined by f_0 alone for $d = 3$ and $d = 4$, and by f_0 and f_1 when $d = 5$. This led him to ask [Sc2] whether it is true that in general $f_0, f_1, \dots, f_{\lfloor (d-3)/2 \rfloor}$ determine the other f_i -numbers. The main purpose of this paper is to prove that this is true. The proof will show that the face numbers of T are determined by the face numbers of its boundary. This boundary is a $(d - 2)$ -dimensional sphere, and relying on the Dehn–Sommerville relations among the face numbers of spheres we reach the conclusion. From a realization of the big Scarf complex Σ as a polyhedral surface, due to Bárány et al. [BHS], one can glean the information that the boundary of T is isomorphic to the boundary complex of a convex polytope. Via the work of Stanley [St1] this introduces further relations of an algebraic nature on the face numbers of T , namely, nonlinear inequalities of Macaulay type.

The construction of the Scarf complexes Σ and T is reviewed in Sections 2 and 3. Here for motivation we briefly mention the reasons for their study.

The complexes Σ were introduced by Scarf [Sc1] for purposes to do with integer programming. In fact, he defined such complexes for $n \times d$ real matrices A that are sufficiently generic. In this paper only the $n = d + 1$ case is considered. The relevance for integer programming is that the 1-skeleton of Σ provides a complete test set for integer programs of the form

$$\begin{cases} \text{minimize} & a_0 \cdot x \\ \text{subject to} & a_i \cdot x \leq b_i, \quad i = 1, \dots, d, \end{cases}$$

where $a_0, \dots, a_d$ are the row vectors of the matrix A . Namely, if a point $x_0 \in \mathbb{Z}^d$ is in the feasible region and if a local minimum is achieved at x_0 (meaning that no improvement of the objective function can be attained at any Σ -neighbor of x_0), then x_0 is a global minimum. Furthermore, the Σ -neighbors of x_0 in the direction of decreasing objective function are determined by the vertices of the small Scarf complex T . The higher-dimensional structure of Σ is very interesting mathematically; its meaning for integer programming is however more elusive, see, e.g., [SS] for a result in this direction.

Scarf complexes have recently become of interest also in commutative algebra, due to the work of Bayer, Peeva, and Sturmfels [BPS], [BS2], [PS] on free resolutions. Since the numbers f_i studied in this paper have algebraic meaning in that setting we want to outline the connection. This is done only for the case of a sufficiently generic $(d + 1) \times d$ integer matrix A , although their work is more general.

For such a matrix A , let $\mathcal{L} \stackrel{\text{def}}{=} \{A \cdot y \mid y \in \mathbb{Z}^d\}$. Then $\mathcal{L}$ is a sublattice of $\mathbb{Z}^{d+1}$ with which we associate the ideal

$$I_{\mathcal{L}} \stackrel{\text{def}}{=} \langle \mathbf{x}^a - \mathbf{x}^b \mid a, b \in \mathbb{N}^{d+1} \text{ and } a - b \in \mathcal{L} \rangle$$

in the polynomial ring $S \stackrel{\text{def}}{=} k[\mathbf{x}] = k[x_1, \dots, x_{d+1}]$. In [PS] a minimal free resolution

$$0 \rightarrow S^{f_{d-1}} \rightarrow S^{f_{d-2}} \rightarrow \dots \rightarrow S^{f_0} \rightarrow S \rightarrow S/I_{\mathcal{L}} \rightarrow 0$$

is constructed, where f_i denotes the face numbers of the small Scarf complex T , as

previously discussed. Thus, the face numbers of T give the Betti numbers of the ring $S/I_{\mathcal{L}}$ in the generic case. In the nongeneric case it is shown in [PS] that f_i gives a lower bound for the i th Betti number of the ring.

2. The Big Complex Σ

In this section we review some definitions and general properties that are needed. Full details about this material can be found in [BHS] and [BS1].

Let A be a $(d+1) \times d$ real matrix. We require that A is of full rank, and that there exists a strictly positive vector $c \in \mathbb{R}^{d+1}$ such that $cA = 0$. For $b \in \mathbb{R}^{d+1}$ let

$$K_b \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid Ax \leq b\}.$$

If nonempty, K_b is either a point or a full-dimensional simplex in $\mathbb{R}^d$. K_b is said to be *lattice free* if $\text{int } K_b \cap \mathbb{Z}^d = \emptyset$. A point $n \in \mathbb{R}^d$ is called a *neighbor* (or *neighbor of the origin*) if the smallest set of the form K_b containing n and 0 is lattice free. Let N denote the set of neighbors. It is proved in [BS1] that the set N is nonempty and finite.

Denote by $a_0, \dots, a_d$ the row vectors of A . The matrix A is said to be *generic* if

$$n \in N \quad \Rightarrow \quad a_i \cdot n \neq 0, \quad \text{for all } 0 \leq i \leq d.$$

We assume that A is generic. Also, since only the directions provided by the row vectors a_i are important, not their magnitudes, we can without loss of generality normalize the vector c to be the unit vector $\mathbf{1} = (1, \dots, 1)$. In summary, from now on we require of the matrix $A \in \mathbb{R}^{(d+1) \times d}$ that

(A1) A is of rank d ,

(A2) $\mathbf{1} \cdot A = 0$,

(A3) A is generic.

Call a simplex K_b *maximal lattice free* if K_b is lattice free but every convex body strictly containing K_b has some point from $\mathbb{Z}^d$ in its interior. It can be shown that if K_b is maximal lattice free, then its boundary intersects $\mathbb{Z}^d$ in exactly $d+1$ points (one in the relative interior of each of its facets). The *Scarf complex* Σ (or Σ_A) is the abstract simplicial complex whose vertex set is $\mathbb{Z}^d$ and whose facets (maximal faces) are the intersections $K_b \cap \mathbb{Z}^d$, for all maximal lattice free K_b . Thus Σ is a pure d -dimensional complex. Its faces are the sets of the form $K_b \cap \mathbb{Z}^d$, for lattice free simplices K_b .

Although the convex hulls of Σ 's facets in general intersect in complicated ways, it turns out that the geometric realization of Σ is homeomorphic to real d -space. This was shown by Bárány et al. [BHS]. Some details of their method of proof are needed later, so we review them here.

For a fixed positive real number t , consider the injective mapping $\varphi_t: \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ defined by

$$\varphi_t: x \mapsto (e^{ta_0x}, \dots, e^{ta_dx}).$$

Let $M \stackrel{\text{def}}{=} \{(y_0, \dots, y_d) \in \mathbb{R}^{d+1} \mid y_i > 0 \text{ for all } i, \text{ and } \prod_{i=0}^d y_i = 1\}$. Because of assumption (A2) we have that $\varphi_t(\mathbb{R}^d) \subseteq M$, and, in fact [BHS, Lemma 1], $\varphi_t(\mathbb{R}^d) = M$. Let $V \stackrel{\text{def}}{=} \varphi_t(\mathbb{Z}^d)$, and let C be the convex hull of V (which is a discrete set). Define $F \subseteq C$ to be a *face* of C if there exists a closed half-space H in $\mathbb{R}^{d+1}$ with bounding hyperplane H^0 such that $C \subseteq H$ and $C \cap H^0 = F$. The zero-dimensional faces are the points of V (and thus they correspond bijectively to the lattice points $\mathbb{Z}^d$). The higher-dimensional faces of C are described as follows, for large enough t .

Theorem 1 [BHS, Theorem 3]. *There exists t_0 such that for $t > t_0$:*

- (i) *the maximal faces of C are d -simplices;*
- (ii) *suppose $x^0, \dots, x^d \in \mathbb{Z}^d$, then $\{x^0, \dots, x^d\} \in \Sigma$ if and only if $\varphi_t(x^0), \dots, \varphi_t(x^d)$ are the vertices of a maximal face of C .*

Remark. The proofs in [BHS] are based on a stricter definition of genericity than the one given here. However, they go through unchanged in the greater generality, see Remark 4 of [BS1].

Theorem 1 shows that the boundary complex of C provides a geometric realization of the Scarf complex Σ . In particular, since the boundary of C is obviously homeomorphic to $\mathbb{R}^d$ we obtain:

Corollary 2 [BHS]. $\|\Sigma\| \cong \mathbb{R}^d$.

3. The Small Complex T

It is clear from the definition that the group $\mathbb{Z}^d$ acts on Σ :

$$\sigma \in \Sigma \quad \text{and} \quad t \in \mathbb{Z}^d \quad \Rightarrow \quad \sigma + t \in \Sigma. \quad (1)$$

This action is transitive on vertices. Hence, to study the local structure of the complex it suffices to confine attention to the neighborhood of the origin. Let

$$L \stackrel{\text{def}}{=} \{\sigma \in \Sigma \mid 0 \notin \sigma, \sigma \cup \{0\} \in \Sigma\}$$

define the *link* of Σ at the origin. The vertices of the subcomplex L are precisely the neighbors. Because of condition (A3) the set N splits into *negative neighbors*

$$N^- \stackrel{\text{def}}{=} \{n \in N \mid a_0 \cdot n < 0\}$$

and *positive neighbors* $N^+ \stackrel{\text{def}}{=} -N^-$. The subcomplex

$$T \stackrel{\text{def}}{=} \{\sigma \subseteq N^- \mid \sigma \cup \{0\} \in \Sigma\}$$

of L is called the *top complex*, a terminology suggested by Scarf [Sc2]. Its dependence on the choice of a_0 , via the definition of N^- , is discussed at the end of this section.

The top complex T has been studied by Scarf for many years and its properties are discussed in a forthcoming treatise [Sc3]. To make this paper self-contained we give proofs of the key technical properties of T that are needed here. We refer to [Sc3] for a fuller treatment.

Proposition 3. *The mapping sending $\{x^1, \dots, x^j\}$ to the orbit of $\{0, x^1, \dots, x^j\}$ is a bijection between the $(j-1)$ -faces of T and the $\mathbb{Z}^d$ -orbits of j -faces of Σ .*

Proof. Let σ be a j -face of Σ . Choose $u \in \mathbb{R}$ such that the half-space $a_0x \leq u$ contains σ while the plane $a_0x = u$ intersects σ . Let x_0 be the unique intersection point (unique due to genericity). Then the translate $\sigma - x_0 = \{0, x^1, \dots, x^j\}$ has the property that $a_0x^i < 0$ for $i = 1, \dots, j$, i.e., $\{x^1, \dots, x^j\} \in T$. Hence, the mapping is surjective. Injectivity is clear. $\square$

Example 4. The matrix

$$A = \begin{pmatrix} -21 & -29 & 20 \\ 6 & -9 & 26 \\ -8 & 35 & -13 \\ 23 & 3 & -33 \end{pmatrix}$$

is generic and has 20 neighbors. Its top complex T is shown (unlabeled and up to combinatorial isomorphism) in Fig. 1. (Remark: This example was computed by a Maple program provided by B. Sturmfels.)

We say that a simplicial complex is a *polytopal $(d-1)$ -sphere* if it is combinatorially isomorphic to the boundary complex of some convex d -polytope. See [G] or [Z] for notions relating to polytopes and convex geometry. A simplicial complex is called a *regular d -ball* if it is combinatorially isomorphic to some regular triangulation of a convex d -polytope. A triangulation Δ of a d -polytope is regular if it is the projection of the lower boundary of a convex $(d+1)$ -polytope, see the discussion in Section 5.1 of [Z].

Fig. 1. The top complex of a 4×3 matrix.

The following result is important for our work with face numbers in the next section.

Theorem 5.

- (i) L is a polytopal $(d - 1)$ -sphere,
- (ii) T is a regular $(d - 1)$ -ball,
- (iii) ∂T is a polytopal $(d - 2)$ -sphere.

The proof follows a sequence of lemmas. The notation $[i, j] \stackrel{\text{def}}{=} \{z \in \mathbb{Z} \mid i \leq z \leq j\}$ is used.

Lemma 6. *Suppose $\{x^1, \dots, x^d\} \in L$. Then there exists a unique $k \in [0, d]$ such that $a_k x^j < 0$ for all $j \in [1, d]$.*

Proof. Since $\sigma = \{0, x^1, \dots, x^d\}$ is a facet of Σ there is a maximal lattice free simplex K_b such that $\sigma = K_b \cap \mathbb{Z}^d$. K_b has a unique facet F_0 containing 0 in its relative interior. Let $a_k x = 0$ be the equation of F_0 's supporting hyperplane. Then, by definition of K_b , $a_k x^j < 0$ for all $j \in [1, d]$.

If $i \neq k$, then K_b has a supporting hyperplane of the form $a_i x = u$ containing some x^j in the interior of its intersection with K_b . Since $0 \in K_b$ it follows that $u > 0$, and hence $a_i x^j > 0$. $\square$

For $x \in N$ let

$$I^+(x) \stackrel{\text{def}}{=} \{i \in [0, d] \mid a_i x > 0\},$$

$$I^-(x) \stackrel{\text{def}}{=} \{i \in [0, d] \mid a_i x < 0\}.$$

Both these sets are nonempty for all $x \in N$.

Lemma 7. *For every $j \in [0, d]$ there exists a unique $x \in N$ and a unique $x' \in N$ such that $I^+(x) = \{j\}$ and $I^-(x') = \{j\}$.*

Proof. Let $b = (0, \dots, 0, b_j, 0, \dots, 0)$ and choose $b_j > 0$ minimal such that K_b intersects $\mathbb{Z}^d$ in at least one point other than 0. This point x will be unique (due to genericity), it will be a neighbor, and it will satisfy $I^+(x) = \{j\}$. As a consequence, $-x \in N$ and $I^-(-x) = \{j\}$. $\square$

Let $\{x^1, \dots, x^d\} \in L$. By Theorem 1 the set $\{\mathbf{1}, \varphi_t(x^1), \dots, \varphi_t(x^d)\}$ spans a facet of C . Let

$$\lambda(y) \stackrel{\text{def}}{=} \lambda_0(y_0 - 1) + \dots + \lambda_d(y_d - 1) = 0$$

be the equation of the supporting hyperplane of this facet of C , oriented so that $\lambda(y) \geq 0$ for all $y \in C$. Then $\lambda_i > 0$ for all i , by Lemma 3 of [BHS].

Lemma 8. Suppose $\{x^1, \dots, x^d\} \in L$ and choose “ k ” as in Lemma 6. Then, for sufficiently large t ,

$$\lambda_k > (d+1)\lambda_j, \quad \text{for all } j \neq k.$$

Proof. Choose a number M such that $0 < M < a_i x^j$ for all $j \in [1, d]$ and $i \in I^+(x^j)$. Then choose $t > t_0$ large enough that $e^{Mt} - 1 > 2d$.

Choose $p \neq k$ such that $\lambda_p \geq \lambda_j$ for all $j \neq k$. Suppose (to reach a contradiction) that $\lambda_k \leq (d+1)\lambda_p$. By Lemma 6 (the uniqueness part) there exists $g \in [1, d]$ such that $p \in I^+(x^g)$. Using that $\lambda(\varphi_t(x^g)) = 0$ and the various inequalities we get

$$\begin{aligned} \sum_{i \in I^-(x^g)} \lambda_i &< 2d\lambda_p < (e^{Mt} - 1)\lambda_p \\ &\leq (e^{Mt} - 1) \sum_{i \in I^+(x^g)} \lambda_i \leq \sum_{i \in I^+(x^g)} (e^{ta_i x^g} - 1)\lambda_i \\ &= \sum_{i \in I^-(x^g)} (1 - e^{ta_i x^g})\lambda_i \leq \sum_{i \in I^-(x^g)} \lambda_i. \end{aligned}$$

This contradiction shows that $\lambda_k > (d+1)\lambda_p$, and since $\lambda_p \geq \lambda_j$ for all $j \neq k$ we are done. $\square$

Lemma 9. Let $\{x^1, \dots, x^d\} \in L$, and let $\lambda(y) = 0$ be the equation of the supporting hyperplane of the corresponding facet of C as before. Then, for sufficiently large t ,

$$\lambda((d+2, 0, \dots, 0)) \text{ is } \begin{cases} > 0, & \text{if } \{x^1, \dots, x^d\} \in T, \\ < 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\Phi \stackrel{\text{def}}{=} \lambda((d+2, 0, \dots, 0)) = (d+1)\lambda_0 - \sum_{j=1}^d \lambda_j$.

If $\{x^1, \dots, x^d\} \in T$, then $k = 0$ in Lemma 8 and we get $\sum_{j=1}^d \lambda_j < d(\lambda_0/(d+1)) < (d+1)\lambda_0$; hence $\Phi > 0$.

If $\{x^1, \dots, x^d\} \notin T$, then $k \neq 0$ in Lemma 8 and $(d+1)\lambda_0 < \lambda_k < \sum_{j=1}^d \lambda_j$; hence $\Phi < 0$. $\square$

Proof of Theorem 5. Choose t sufficiently large and let $P = C \cap H$, where H is the hyperplane $y_0 + \dots + y_d = d+2$ in $\mathbb{R}^{d+1}$. The vertex $\mathbf{1}$ of C is separated from its neighbors by H , so P is a bounded convex d -polytope whose boundary complex is isomorphic to L . This proves part (i).

Let $q = (d+2, 0, \dots, 0) \in \mathbb{R}^{d+1}$. Then $q \in H$, and Lemma 9 shows that (in the d -space H) q is beneath those facets of P that correspond to the facets of T , and beyond the remaining ones. (Being “beneath” means being on the same side as the interior of P , and the opposite for “beyond.”) Let T' be the part of P ’s boundary complex that realizes T , and let $Q \stackrel{\text{def}}{=} \text{conv}(P \cup \{q\})$. The facets of the polytope Q are the facets in T' and faces of the form $\text{conv}(F \cup \{q\})$ for facets F of the boundary $\partial T'$. It follows that T' is a $(d-1)$ -ball, and its boundary complex $\partial T'$ is isomorphic to the vertex figure of Q at the vertex q , which shows it is a polytopal $(d-2)$ -sphere. See Section 5 of [BL1]

for more about this kind of argument. Finally, a projective transformation that moves q (but no other vertex of Q) to infinity will move T' into a position showing that T is regular. $\square$

We return for a moment to the definition of the complex T and its dependence on the choice of a specific row of A . Generalizing our earlier definitions, let

$$N_i^- \stackrel{\text{def}}{=} \{n \in N \mid a_i \cdot n < 0\}$$

and

$$T_i \stackrel{\text{def}}{=} \{\sigma \subseteq N_i^- \mid \sigma \cup \{0\} \in \Sigma\},$$

for $i = 0, \dots, d$. This creates $d + 1$ “top” complexes, all satisfying the enumerative property of Proposition 3, as well as all the other properties we have derived. Examples show, however, that these complexes T_i are in general not isomorphic. Using Theorem 5 and Lemma 6 we can conclude the following:

Proposition 10. *The $(d - 1)$ -balls $T_0, T_1, \dots, T_d$ have pairwise disjoint interiors, and their union is the link L .*

4. Face Numbers of Scarf Complexes

We begin with a quick review of some definitions and results from the general theory of face numbers. For more about this topic, see, e.g., [G], [Z], or the survey [BB].

Let Δ be a $(d - 1)$ -dimensional simplicial complex, and let f_i be the number of i -dimensional faces of Δ . The sequence $f = (f_0, \dots, f_{d-1})$ is called the f -vector of Δ . We put $f_{-1} = 1$. The h -vector $h = (h_0, \dots, h_d)$ of Δ is defined by the equation

$$\sum_{i=0}^d f_{i-1} x^{d-i} = \sum_{i=0}^d h_i (x + 1)^{d-i}. \quad (2)$$

Note that $h_0 = 1$, $h_1 = n - d$, and $h_d = (-1)^{d-1} \tilde{\chi}(\Delta)$, where $\tilde{\chi}(\Delta)$ is the reduced Euler characteristic of Δ . In particular,

$$h_d = \begin{cases} 1, & \text{if } \Delta \text{ is a sphere,} \\ 0, & \text{if } \Delta \text{ is a ball,} \end{cases}$$

where the conditions are shorthand for saying that Δ 's geometric realization is homeomorphic to a sphere, resp. a ball.

The following are called the *Dehn–Sommerville relations*:

$$\text{If } \Delta \text{ is a sphere, then } h_i = h_{d-i}, \text{ for all } 0 \leq i \leq d. \quad (3)$$

Hence, for spheres all f -vector information is encoded in the much shorter g -vector $g = (g_0, \dots, g_{\lfloor d/2 \rfloor})$, defined by $g_i = h_i - h_{i-1}$. We have that $g_0 = 1$, $g_1 = n - d - 1$.

A sequence of integers $(a_0, \dots, a_k)$ is called an M -sequence if there exists a nonempty family $\mathcal{M}$ of monomials in a_1 variables such that

- (i) if m divides m' and $m' \in \mathcal{M}$, then $m \in \mathcal{M}$, and
- (ii) $\mathcal{M}$ contains exactly a_i monomials of degree i , for all $0 \leq i \leq k$.

M -sequences have a number of algebraic and combinatorial characterizations, see, e.g., [St2] and [Z]. Their relevance for this paper is the following result due to Stanley [St1]:

If Δ is a polytopal sphere, then $(g_0, \dots, g_{\lfloor d/2 \rfloor})$ is an M -sequence. (4)

If Δ is a $(d-1)$ -ball, its boundary complex $\partial\Delta$ is a $(d-2)$ -sphere. Furthermore, $\partial\Delta$'s f -vector is determined by that of Δ , as shown by the following consequence of the Dehn–Sommerville relations, due to McMullen and Walkup [MW], see also Corollary 3.9 of [BL2]:

If Δ is a ball with boundary $\partial\Delta$, then $h_i^\Delta - h_{d-i}^\Delta = g_i^{\partial\Delta}$. (5)

After this review, we now turn our attention back to Scarf complexes. The f -vectors of L , T , and ∂T are denoted f^L , f^T , and $f^{\partial T}$, and similarly for their h - and g -vectors.

Proposition 11.

- (i) $f_i^L = (i+2)f_i^T$;
- (ii) $h_i^L = (i+1)h_i^T + (d-i+1)h_{i-1}^T$.

Proof. Let σ be an i -simplex of T . Then $\sigma \cup \{0\}$ is an $(i+1)$ -simplex of Σ . Each one of the $i+2$ vertices of $\sigma \cup \{0\}$ can be translated to the origin, and its $i+2$ maximal faces thus contribute distinct i -simplices to L . The proof of part (i) is concluded with the observation that every i -simplex of L is obtained from a unique i -simplex of T in this fashion.

To simplify notation, for the rest of this proof put $f \stackrel{\text{def}}{=} f^T$ and $h \stackrel{\text{def}}{=} h^T$. For part (ii) we begin by differentiating (2) and multiplying by $(x+1)$:

$$\sum_{i=0}^{d-1} (d-i)f_{i-1}(x+1)x^{d-i-1} = \sum_{i=0}^{d-1} (d-i)h_i(x+1)^{d-i}.$$

This relation gives

$$\begin{aligned} d \sum_{i=0}^d f_{i-1}x^{d-i} - \sum_{i=0}^d i f_{i-1}x^{d-i} + \sum_{i=0}^{d-1} (d-i)f_{i-1}x^{d-i-1} \\ = d \sum_{i=0}^d h_i(x+1)^{d-i} - \sum_{i=0}^d i h_i(x+1)^{d-i}. \end{aligned}$$

Using (2) and its derivative this simplifies to

$$\begin{aligned} \sum_{i=0}^d i f_{i-1}x^{d-i} &= \sum_{i=0}^d i h_i(x+1)^{d-i} + \sum_{i=0}^{d-1} (d-i)h_i(x+1)^{d-i-1} \\ &= \sum_{i=0}^d [i h_i + (d+1-i)h_{i-1}](x+1)^{d-i}. \end{aligned}$$

Hence, using part (i) we get

$$\begin{aligned}
 \sum_{i=0}^d h_i^L (x+1)^{d-i} &= \sum_{i=0}^d (i+1) f_{i-1} x^{d-i} \\
 &= \sum_{i=0}^d [i h_i + (d+1-i) h_{i-1}] (x+1)^{d-i} + \sum_{i=0}^d f_{i-1} x^{d-i} \\
 &= \sum_{i=0}^d [(i+1) h_i + (d+1-i) h_{i-1}] (x+1)^{d-i},
 \end{aligned}$$

which proves part (ii). $\square$

Proposition 12. *The h -vector of T satisfies*

$$h_i^T = h_{d-1-i}^T,$$

for all $0 \leq i \leq \lfloor (d-1)/2 \rfloor$.

Proof. The Dehn–Sommerville relations $h_i^L = h_{d-i}^L$ together with Proposition 11(ii) show that

$$(i+1)(h_i^T - h_{d-i-1}^T) = (d-i+1)(h_{d-i}^T - h_{i-1}^T). \quad (6)$$

We have that $h_d^T = 0$ and $h_d^L = 1$ (T being a ball and L a sphere), so $h_{d-1}^T = 1$ follows from Proposition 11(ii). Hence, since $h_0^T = 1$, (6) gives

$$h_0^T = h_{d-1}^T \Rightarrow h_1^T = h_{d-2}^T \Rightarrow h_2^T = h_{d-3}^T \Rightarrow \cdots. \quad \square$$

The following is our main result.

Theorem 13. *Let ∂T be the boundary complex of T . Then*

$$h_i^T = h_i^{\partial T},$$

for all $0 \leq i \leq d-1$.

Proof. Using Proposition 12 and (5) we get that

$$h_i^T - h_{i-1}^T = h_i^T - h_{d-i}^T = g_i^{\partial T} = h_i^{\partial T} - h_{i-1}^{\partial T},$$

for all $1 \leq i \leq \lfloor (d-1)/2 \rfloor$. Since $h_0^T = h_0^{\partial T} = 1$ it follows that $h_i^T = h_i^{\partial T}$ for all $0 \leq i \leq \lfloor (d-1)/2 \rfloor$. This extends to all $0 \leq i \leq d-1$ via Proposition 12 and the Dehn–Sommerville relations $h_i^{\partial T} = h_{d-1-i}^{\partial T}$. $\square$

Corollary 14. *T has the same f -vector as the cone over its boundary, namely, $f_i^T = f_i^{\partial T} + f_{i-1}^{\partial T}$.*

Proof. If $(h_0, \dots, h_{d-1})$ is the h -vector of some $(d-2)$ -complex Δ , straightforward computation (multiply (2) by $x+1$) shows that $(h_0, \dots, h_{d-1}, 0)$ is the h -vector of the cone over Δ . $\square$

One particular consequence, already known to Scarf [Sc2], [Sc3], is that T has a unique interior vertex, although (as illustrated by Fig. 1) T is in general *not* the cone over its boundary. The unique interior vertex of T is the neighbor x that in Lemma 7 was characterized by the property $I^-(x) = \{0\}$.

Corollary 15. *The f -vector of T determines, and is determined by, the M -sequence $(1, g_1^{\partial T}, \dots, g_{\lfloor (d-1)/2 \rfloor}^{\partial T})$.*

Proof. The g -vector of ∂T is an M -sequence, since ∂T is polytopal (Theorem 5). The rest follows from the theorem. $\square$

The theorem also implies a direct relationship between the g -vectors of the two spheres L and ∂T .

Corollary 16. $g_i^L = (i+1)g_i^{\partial T} + (d-i+2)g_{i-1}^{\partial T}$.

Proof. We have from Proposition 11 and the theorem that

$$\begin{aligned} g_i^L &= h_i^L - h_{i-1}^L = (i+1)h_i^T + (d-i+1)h_{i-1}^T - (ih_{i-1}^T + (d-i+2)h_{i-2}^T) \\ &= (i+1)(h_i^T - h_{i-1}^T) + (d-i+2)(h_{i-1}^T - h_{i-2}^T) \\ &= (i+1)g_i^{\partial T} + (d-i+2)g_{i-1}^{\partial T}. \end{aligned} \quad \square$$

5. Remarks

1. The property of being a ball with a unique interior vertex does not by itself imply any special relationship between the f -vector of T and that of its boundary, such as that of Corollary 14. For example, take two tetrahedra glued together along one triangle and then perform a stellar subdivision of one of them, thus introducing an interior vertex. The resulting ball has five facets, whereas its boundary has six. One can also construct a unique-interior-vertex triangulation of the 3-ball with seven facets, whose boundary is the same six-facet 2-sphere.

2. Let $\mathcal{F}_d$ be the set of all f -vectors of Scarf top complexes T coming from generic $(d+1) \times d$ -matrices. What is the dimension of the affine span of $\mathcal{F}_d$ in $\mathbb{R}^d$? We have shown that

$$\dim \text{aff } \mathcal{F}_d \leq \left\lfloor \frac{d-1}{2} \right\rfloor.$$

Is this upper bound sharp, or are other linear relations satisfied by these f -vectors?

3. A more ambitious question is to ask which M -sequences $(1, g_1, \dots, g_{\lfloor (d-1)/2 \rfloor})$ are “Scarf” in the sense of Corollary 15, i.e., correspond to the elements of $\mathcal{F}_d$.

4. The Stanley–Reisner rings of top complexes T may be interesting to investigate further from an algebraic point of view. For instance, Proposition 12 indicates that they might have a “Lefschetz element,” similar to the ones provided by toric geometry in [St1].

Acknowledgments

My understanding of this subject has greatly benefited from conversations with László Lovász, Herbert Scarf, and Bernd Sturmfels. In particular, I thank Herbert Scarf for generously sharing unpublished material on the top complex [Sc2].

References

- [BHS] I. Bárány, R. Howe, and H. E. Scarf, The complex of maximal lattice free simplices, *Math. Programming* **66** (1994), 273–281.
- [BS1] I. Bárány and H. E. Scarf, Matrices with identical sets of neighbors, *Math. Oper. Res.* **23** (1998), 863–873.
- [BPS] D. Bayer, I. Peeva, and B. Sturmfels, Monomial resolutions, *Math. Res. Lett.* **5** (1998), 31–46.
- [BS2] D. Bayer and B. Sturmfels, Cellular resolutions of monomial modules, *J. Reine Angew. Math.* **502** (1998), 123–140.
- [BB] L. J. Billera and A. Björner, Face numbers of polytopes and complexes, in *Handbook of Discrete and Computational Geometry* (ed. J. E. Goodman and J. O’Rourke), CRC Press, Boca Raton, FL, 1997, pp. 291–310.
- [BL1] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen’s conditions for f -vectors of simplicial convex polytopes, *J. Combin. Theory Ser. A* **31** (1981), 237–255.
- [BL2] L. J. Billera and C. W. Lee, The numbers of faces of polytope pairs and unbounded polyhedra, *European J. Combin.* **2** (1981), 307–322.
- [G] B. Grünbaum, *Convex Polytopes*, Wiley–Interscience, London, 1967.
- [MW] P. McMullen and D. W. Walkup, A generalized lower bound conjecture for simplicial polytopes, *Mathematika* **18** (1971), 264–273.
- [PS] I. Peeva and B. Sturmfels, Generic lattice ideals, *J. Amer. Math. Soc.* **11** (1998), 363–373.
- [Sc1] H. E. Scarf, Neighborhood systems for production sets with indivisibilities, *Econometrica* **54** (1986), 507–532.
- [Sc2] H. E. Scarf, Private communication, March 1995.
- [Sc3] H. E. Scarf, Work in preparation. (An outline appears in *Test Sets for Integer Programs*, extended abstract (6 pp.) for talks delivered at Lunteren, The Netherlands, Jan. 12–14, 1999.)
- [SS] H. E. Scarf and D. F. Shallcross, The Frobenius problem and maximal lattice free bodies, *Math. Oper. Res.* **18** (1993), 511–515.
- [St1] R. P. Stanley, The number of faces of simplicial convex polytopes, *Adv. in Math.* **35** (1980), 236–238.
- [St2] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston, 1983. (Second edition, 1996.)
- [Z] G. M. Ziegler, *Lectures on Polytopes*, GTM series, Springer-Verlag, Berlin, 1995.

Received January 20, 1999. Online publication May 3, 2000.